Explicit Method

$$T_O^{\ m+1} = \alpha\Delta t\left[\frac{T_N^{\ m} + T_S^{\ m}}{\Delta y^2}\right] + \alpha\Delta t\left[\frac{T_E^{\ m} + T_W^{\ m}}{\Delta x^2}\right] + \alpha\Delta t\,\frac{\dot{q}'''}{k} + \left[1 - 2\alpha\Delta t\left[\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right]\right] T_O^{\ m}$$

Assume a uniform mesh $\Delta x = \Delta y = \Delta$, Constant temperature B.C.s, and no heat source term.

Write the collection of all difference equations as a linear system (E is a matrix).

$T^{m+1} = E\ T^m + b$

Let A $\equiv \dfrac{\alpha\Delta t}{\Delta^2}$

$$\begin{matrix} T_1^{m+1} \\ T_2^{m+1} \\ T_3^{m+1} \\ T_4^{m+1} \\ T_5^{m+1} \\ T_6^{m+1} \end{matrix} \quad = \quad \begin{matrix} (1-4A) & A & O & A & O & O \\ A & (1-4A) & A & O & A & O \\ O & A & (1-4A) & O & O & A \\ A & O & O & (1-4A) & A & O \\ O & A & O & A & (1-4A) & A \\ O & O & A & O & A & (1-4A) \end{matrix} \quad \begin{matrix} T_1^m \\ T_2^m \\ T_{m3}^m \\ T_4^m \\ T_5^m \\ T_6^m \end{matrix} \quad + \quad \begin{matrix} 200A \\ 100A \\ 100A \\ 100A \\ 0 \\ 0 \end{matrix}$$

Example of Error Propagation

FDE $$\frac{1}{\alpha\Delta t}[\in_1^{m+1} - \in_1^{m}] = \frac{1}{\Delta^2}[\in_O^{m} + \in_2^{m} - 2\in_{1m}]$$

$$\in_1^{m+1} = \underbrace{\frac{\alpha\Delta t}{\Delta^2}}_{A}[\in_O^{m} + \in_2^{m}] + [1 - \frac{2\alpha\Delta t}{\Delta^2}]\in_1^{m} = A[\in_O^{m} + \in_2^{m}] + [1 - 2A]\in_1^{m}$$

Start stepping through time with an initial error of 1 at mesh point 3 and no error at other points

A = 0.25

m	$\in_0$	$\in_1$	$\in_2$	$\in_3$	$\in_4$	$\in_5$	Σ
0	0	0	0	1	0	0	1
1	0	0	.25	.5	.25	0	1
2	0	.063	.25	.375	.25	0	.9325
3	0	.094	.234	.312	.203	0	.843
4	0	.105	.218	.265	.179	0	.767
5	0	.107	.201	.232	.156	0	.696

The sum of errors decreases.

Example of Error Propagation

A = 1

$$\in_1^{m+1} = \underbrace{\frac{\alpha\Delta t}{\Delta^2}}_{A}[\in_O^{\,m} + \in_2^{\,m}] + [1 - \frac{2\alpha\Delta t}{\Delta^2}]\in_1^{\,m} = A[\in_O^{\,m} + \in_2^{\,m}] + [1-2A]\in_1^{\,m} \text{ so } \in_i^{m+1} = [\in_{i+1}^{\,m} + \in_{i-1}^{\,m} - \in_1^{\,m}]$$

m	$\in_0$	$\in_1$	$\in_2$	$\in_3$	$\in_4$	$\in_5$	Σ
0	0	0	0	1	0	0	1
1	0	0	1	-1	1	0	1
2	0	1	-2	3	-2	0	0
3	0	-3	6	-7	5	0	1
4	0	9	-16	18	-12	0	-1
5	0	25	43	-46	30	0	2

The solution is unstable

Exact Determination of Stability

For an Explicit (Forward) difference method

$T^{m+1} = ET^{m} + b$

$$T_i^{\,m+1} = A[T_{i+1}^{\,m} + T_{i-1}^{\,m}] - (1-2A)\,T_i^{\,m}$$

Set boundary conditions: $T_0 = 100$ $T_5 = 5$

$$\begin{pmatrix} T_1^{\,m+1} \\ T_2^{\,m+1} \\ T_3^{\,m+1} \\ T_4^{\,m+1} \end{pmatrix} = \begin{pmatrix} (1-2A) & A & 0 & 0 \\ A & (1-2A) & A & 0 \\ 0 & A & (1-2A) & A \\ 0 & 0 & A & (1-2A) \end{pmatrix} \begin{pmatrix} T_1^{\,m} \\ T_2^{\,m} \\ T_3^{\,m} \\ T_4^{\,m} \end{pmatrix} + A \begin{pmatrix} 100 \\ 0 \\ 0 \\ 5 \end{pmatrix}$$

Stability requires that the spectral radius is less than 1.

$$\left| \left| \lambda_{SR} \right|_E \right| \leq 1$$

Exact Determination of Stability

To calculate the spectral radius, look at details of the matrix

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + A \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

E = I + A F

The eigenvalues of F are: $\lambda\big|_F = -4\,SIN^2[\frac{s\pi}{2(N+1)}], \quad s = 1, 2, 3, \cdots N$

where N is the order of matrix. Hence, the spectral radius of E is:

$$\lambda\big|_E = 1 - 4A\,SIN_2[\frac{s\pi}{2(N+1)}]$$

Exact Determination of Stability

For stability $\left| \lambda_{SR} \big|_E \right| \leq 1$

$$\left| 1 - 4A\,SIN^2[\frac{s\pi}{2(N+1)}] \right| \leq 1$$

$$-1 \leq 1 - 4A\,SIN^2[\frac{s\pi}{2(N+1)}] \leq 1$$

so

$$0 \leq 2 - 4A\,SIN^2[\frac{s\pi}{2(N+1)}] \leq 2$$

Upper limit satisfied automatically

Exact Determination of Stability

Lower limit

$$0 \le 2 - 4\,A\,SIN^{2}\left[\frac{s\pi}{2(N+1)}\right]$$

$$4\,A\,SIN_{2}\left[\frac{s\pi}{2(N+1)}\right] \le 2$$

or

$$A \le \frac{2}{4\,SIN^{2}\left[\frac{s\pi}{2(N+1)}\right]}$$

Spectral radius (largest eigenvalue) corresponds to s = N

Examples

N = 4

$$A \leq \frac{1}{2SIN^2[\frac{4\pi}{2(5)}]} \leq 0.553$$

so $$\Delta t \leq \frac{0.553\Delta^2}{\alpha}$$

N = 100

$$A \leq \frac{1}{2SIN^2[\frac{100\pi}{2(101)}]} \leq 0.50012$$

so $$\Delta t \leq \frac{0.50012\,\Delta^2}{\alpha}$$

Gerschgorin's Theorem

Given a time advancement method in the form $T^{n+1} = ET^{n} + b$,

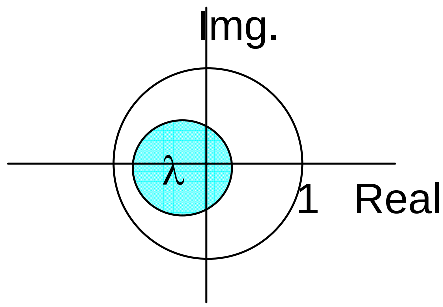
and writing the individual elements of the matrix E as $e_{i,j}$

then the eigenvalues of E fall within circles in the complex plane described by the expression

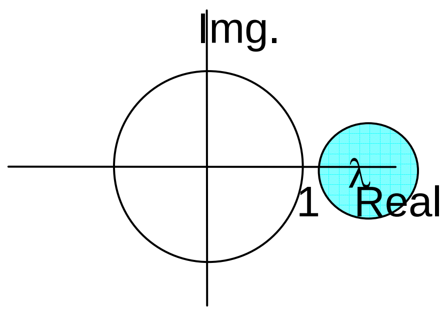
$$\left|\lambda - e_{i,i}\right| \leq \left|e_{i,1}\right| + \left|e_{i,2}\right| + \ldots \left|e_{i,i-1}\right| + \left|e_{i,i+1}\right| + \ldots + \left|e_{i,N}\right|$$

All rows (values of i) must be evaluated to determine bounds on stability.

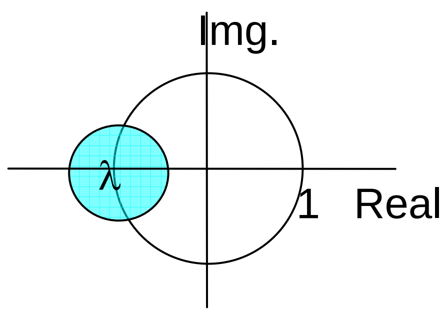
What This Can Tell Us



Magnitude of Associated Eigenvalues is less than one.



Magnitude of Associated Eigenvalues is greater than one



No useful information

Example 1. Eigenvalues associated with an interior volume.

$$\frac{T_O^{n+1} - T_O^n}{\Delta t} = \alpha \frac{T_E^n - 2T_O^n + T_W^n}{\Delta x^2}$$

or

$$T_O^{n+1} = \left(1 - \frac{2\alpha\,\Delta t}{\Delta x^2}\right) T_O^n + \frac{\alpha\,\Delta t}{\Delta x^2} T_E^n + \frac{\alpha\,\Delta t}{\Delta x^2} T_W^n$$

Hence Gerschgorin's Theorem becomes:

$$\left|\lambda - \left(1 - \frac{2\alpha\,\Delta t}{\Delta x^2}\right)\right| \le \left|\frac{\alpha\,\Delta t}{\Delta x^2}\right| + \left|\frac{\alpha\,\Delta t}{\Delta x^2}\right|$$

Because the center of this circle bounding eigenvalues is on the Real axis, we only need to worry about the two points were the circle intersects the real axis. Call them λ_1 (high side) and λ_2 (low) side.

$$\lambda_1 = \left(1 - \frac{2\alpha\,\Delta t}{\Delta x^2}\right) + \left|\frac{\alpha\,\Delta t}{\Delta x^2}\right| + \left|\frac{\alpha\,\Delta t}{\Delta x^2}\right| = 1$$

$$\lambda_2 = \left(1 - \frac{2\alpha\,\Delta t}{\Delta x^2}\right) - \left|\frac{\alpha\,\Delta t}{\Delta x^2}\right| - \left|\frac{\alpha\,\Delta t}{\Delta x^2}\right| = 1 - \frac{4\alpha\,\Delta t}{\Delta x^2}$$

Example 1. Continued

For stability, the lower bound on eignenvalues must be greater than -1, so:

$$\lambda_2 = 1 - \frac{4\alpha \Delta t}{\Delta x^2} \geq -1$$

or

$$\frac{4\alpha \Delta t}{\Delta x^2} \leq 2$$

or

$$\Delta t \leq \frac{\Delta x^2}{2\alpha}$$

Example 2. Eigenvalues associated with a boundary volume.

$$BC \quad x=0: \quad h_\infty (T_\infty - T_o) = -k \frac{\partial T}{\partial X}\Big|_O$$

A simple finite difference equation is

$$\frac{T_o^{\,n+1} - T_o^{\,n}}{\alpha\, \Delta t} = \frac{2}{\Delta x^2}\left[T_E^{\,n} - T_o^{\,n}\right] + \frac{2h}{k\Delta}\left[T_\infty - T_o^{\,n}\right]$$

or

$$T_0^{\,n+1} = \frac{2\alpha\Delta t}{\Delta x^2} T_W^{\,n} + \left[1 - \frac{2\alpha\Delta t}{\Delta x^2} - \frac{2h\alpha\Delta t}{k\Delta x}\right] T_0^{\,m} + \frac{2h\alpha\Delta t T_\infty}{k\Delta x}$$

Apply Gerschgorin's Theorem:

$$\left|\lambda - (1 - \frac{2\alpha\Delta t}{\Delta x^2} - \frac{2h\alpha\Delta t}{k\Delta x})\right| \le \frac{2\alpha\Delta t}{\Delta x^2}$$

so

$$\lambda_1 = 1 - \frac{2\alpha\Delta t}{\Delta x^2} - \frac{2h\alpha\Delta t}{k\Delta x} + \frac{2\alpha\Delta t}{\Delta x^2} = 1 - \frac{2h\alpha\Delta t}{k\Delta x}$$

$$\lambda_2 = 1 - \frac{2\alpha\Delta t}{\Delta x^2} - \frac{2h\alpha\Delta t}{k\Delta x} - \frac{2\alpha\Delta t}{\Delta x^2} = 1 - \frac{4\alpha\Delta t}{\Delta x^2} - \frac{2h\alpha\Delta t}{k\Delta x}$$

Example 2. Continued

The upper bound on the real axis must be less than one.

$\lambda_1 = 1 - \frac{2h\alpha\Delta t}{k\Delta x} \le 1$ which can is always true since h, k, α, Δx, and Δt are all positive.

The lower bound on the real axis must be greater than minus one.

$$\lambda_2 = 1 - \frac{4\alpha\Delta t}{\Delta x^2} - \frac{2h\alpha\Delta t}{k\Delta x} \ge -1$$

Hence the requirement on time step size of:

$$\Delta t \le \frac{\Delta x^2}{\alpha\left(2 + \frac{h\Delta x}{k}\right)}$$

Stability of an Implicit Method

$$\frac{T_0^{\;n+1} - T_0^{\;n}}{\Delta t} = \alpha \frac{T_E^{\;n+1} - 2T_0^{\;n+1} + T_W^{\;n+1}}{\Delta x^2} + \alpha \frac{T_N^{\;n+1} - 2T_0^{\;n+1} + T_S^{\;n+1}}{\Delta y^2}$$

or

$$\left[1 + 2\alpha\Delta t\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)\right] T_0^{\;n+1} - \frac{\alpha\Delta t}{\Delta x^2}\left(T_E^{n+1} + T_W^{n+1}\right) - \frac{\alpha\Delta t}{\Delta y^2}\left(T_N^{n+1} + T_S^{n+1}\right) = T_0^{\;n}$$

which has the form $HT^{n+1}=T^n+b$ but for analysis we need to think in terms of a matrix equation in the form $T^{n+1}=ET^n+b$. The transformation from one to the other means that $E=H^{-1}$. Since the eigenvalues of E and the inverse of the eigenvalues of H, stability will be guaranteed if all of the eigenvalues of H are greater than or equal to one. We can show this via Gerschgorin’s Theorem.

$$\left|\lambda - \left[1 + 2\alpha\Delta t\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)\right]\right| \le \left|-\frac{\alpha\Delta t}{\Delta y^2}\right| + \left|-\frac{\alpha\Delta t}{\Delta x^2}\right| + \left|-\frac{\alpha\Delta t}{\Delta y^2}\right| + \left|-\frac{\alpha\Delta t}{\Delta x^2}\right|$$

Implicit Stability Continued

The upper intersection of the bounding circle with the real axis is:

$$\lambda_1 = 1 + 2\Delta t\alpha\left[\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right] + 2\Delta t\alpha\left[\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right]$$

$$= 1 + 4\Delta t\alpha\left[\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right]$$

Which is always greater than 1.

The lower intersection of the bounding circle with the real axis is:

$$\lambda_2 = 1 + 2\Delta t\alpha\left[\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right] - 2\Delta t\alpha\left[\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right] = 1$$

So the eigenvalues of the H matrix associated with the interior implicit equations are always greater than or equal to one. This means that the eigenvalues associated with the inverse of H are always less than or equal to one, producing unconditional stability.

Crank-Nicholson

For a simple conduction problem:

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right] + q$$

The Crank-Nicholson method is:

$$\frac{T_0{}^{n+1} - T_0{}^{n}}{\Delta t} = \frac{\alpha}{2} \left[\frac{T_E{}^{n+1} - 2T_0{}^{n+1} + T_W{}^{n+1}}{\Delta x^2} + \frac{T_N{}^{n+1} - 2T_0{}^{n+1} + T_S{}^{n+1}}{\Delta y^2} \right]$$

$$+ \frac{\alpha}{2} \left[\frac{T_E{}^{n} - 2T_0{}^{n} + T_W{}^{n}}{\Delta x^2} + \frac{T_N{}^{n} - 2T_0{}^{n} + T_S{}^{n}}{\Delta y^2} \right] + q^{n+1/2}$$

For error analysis the difference equation is taken to be an approximation to the differential equation at point “O” and time level m+1/2.

Crank-Nicholson Error Analysis

To simplify the exercise, I will just work with a 1-D problem. In that case I need six Taylor series expansions.

$$T_i^{n+1} = T_i^{n+1/2} + \frac{\Delta t}{2}\frac{\partial T}{\partial t}\bigg|_i^{n+1/2} + \frac{\Delta t^2}{2^2 2!}\frac{\partial^2 T}{\partial t^2}\bigg|_i^{n+1/2} + \frac{\Delta t^3}{2^3 3!}\frac{\partial^3 T}{\partial t^3}\bigg|_i^{n+1/2} + \frac{\Delta t^4}{2^4 4!}\frac{\partial^4 T}{\partial t^4}\bigg|_i^{n+1/2} + \ldots$$

$$T_i^{n} = T_i^{n+1/2} - \frac{\Delta t}{2}\frac{\partial T}{\partial t}\bigg|_i^{n+1/2} + \frac{\Delta t^2}{2^2 2!}\frac{\partial^2 T}{\partial t^2}\bigg|_i^{n+1/2} - \frac{\Delta t^3}{2^3 3!}\frac{\partial^3 T}{\partial t^3}\bigg|_i^{n+1/2} + \frac{\Delta t^4}{2^4 4!}\frac{\partial^4 T}{\partial t^4}\bigg|_i^{n+1/2} + \ldots$$

$$\begin{aligned} T_{i+1}^{n+1} = T_i^{n+1/2} &+ \frac{\Delta t}{2}\frac{\partial T}{\partial t}\bigg|_i^{n+1/2} + \frac{\Delta t^2}{2^2 2!}\frac{\partial^2 T}{\partial t^2}\bigg|_i^{n+1/2} + \frac{\Delta t^3}{2^3 3!}\frac{\partial^3 T}{\partial t^3}\bigg|_i^{n+1/2} + \frac{\Delta t^4}{2^4 4!}\frac{\partial^4 T}{\partial t^4}\bigg|_i^{n+1/2} \\ &+ \Delta x\frac{\partial T}{\partial x}\bigg|_i^{n+1/2} + \frac{\Delta x^2}{2!}\frac{\partial^2 T}{\partial x^2}\bigg|_i^{n+1/2} + \frac{\Delta x^3}{3!}\frac{\partial^3 T}{\partial x^3}\bigg|_i^{n+1/2} + \frac{\Delta x^4}{4!}\frac{\partial^4 T}{\partial x^4}\bigg|_i^{n+1/2} \\ &+ \Delta x\frac{\Delta t}{2}\frac{\partial^2 T}{\partial x \partial t}\bigg|_i^{n+1/2} + 3\frac{\Delta x}{2!}\left(\frac{\Delta t}{2}\right)^2\frac{\partial^3 T}{\partial x \partial t^2}\bigg|_i^{n+1/2} + 3\frac{\Delta x^2}{2!}\left(\frac{\Delta t}{2}\right)\frac{\partial^3 T}{\partial x^2 \partial t}\bigg|_i^{n+1/2} + \ldots \end{aligned}$$

Error Analysis Continued

$$T_{i-1}{}^{n+1} = T_i{}^{n+1/2} + \frac{\Delta t}{2}\frac{\partial T}{\partial t}\bigg|_i^{n+1/2} + \frac{\Delta t^2}{2^2 2!}\frac{\partial^2 T}{\partial t^2}\bigg|_i^{n+1/2} + \frac{\Delta t^3}{2^3 3!}\frac{\partial^3 T}{\partial t^3}\bigg|_i^{n+1/2} + \frac{\Delta t^4}{2^4 4!}\frac{\partial^4 T}{\partial t^4}\bigg|_i^{n+1/2}$$

$$-\Delta x\frac{\partial T}{\partial x}\bigg|_i^{n+1/2} + \frac{\Delta x^2}{2!}\frac{\partial^2 T}{\partial x^2}\bigg|_i^{n+1/2} - \frac{\Delta x^3}{3!}\frac{\partial^3 T}{\partial x^3}\bigg|_i^{n+1/2} + \frac{\Delta x^4}{4!}\frac{\partial^4 T}{\partial x^4}\bigg|_i^{n+1/2}$$

$$-\Delta x\frac{\Delta t}{2}\frac{\partial^2 T}{\partial x \partial t}\bigg|_i^{n+1/2} - 3\frac{\Delta x}{2!}\left(\frac{\Delta t}{2}\right)^2\frac{\partial^3 T}{\partial x \partial t^2}\bigg|_i^{n+1/2} + 3\frac{\Delta x^2}{2!}\left(\frac{\Delta t}{2}\right)\frac{\partial^3 T}{\partial x^2 \partial t}\bigg|_i^{n+1/2} + \ldots$$

$$T_{i+1}{}^{n} = T_i{}^{n+1/2} - \frac{\Delta t}{2}\frac{\partial T}{\partial t}\bigg|_i^{n+1/2} + \frac{\Delta t^2}{2^2 2!}\frac{\partial^2 T}{\partial t^2}\bigg|_i^{n+1/2} - \frac{\Delta t^3}{2^3 3!}\frac{\partial^3 T}{\partial t^3}\bigg|_i^{n+1/2} + \frac{\Delta t^4}{2^4 4!}\frac{\partial^4 T}{\partial t^4}\bigg|_i^{n+1/2}$$

$$+\Delta x\frac{\partial T}{\partial x}\bigg|_i^{n+1/2} + \frac{\Delta x^2}{2!}\frac{\partial^2 T}{\partial x^2}\bigg|_i^{n+1/2} + \frac{\Delta x^3}{3!}\frac{\partial^3 T}{\partial x^3}\bigg|_i^{n+1/2} + \frac{\Delta x^4}{4!}\frac{\partial^4 T}{\partial x^4}\bigg|_i^{n+1/2}$$

$$-\Delta x\frac{\Delta t}{2}\frac{\partial^2 T}{\partial x \partial t}\bigg|_i^{n+1/2} + 3\frac{\Delta x}{2!}\left(\frac{\Delta t}{2}\right)^2\frac{\partial^3 T}{\partial x \partial t^2}\bigg|_i^{n+1/2} - 3\frac{\Delta x^2}{2!}\left(\frac{\Delta t}{2}\right)\frac{\partial^3 T}{\partial x^2 \partial t}\bigg|_i^{n+1/2} + \ldots$$

Error Analysis Continued

$$T_{i-1}^{\ n} = T_i^{n+1/2} - \frac{\Delta t}{2}\frac{\partial T}{\partial t}\bigg|_i^{n+1/2} + \frac{\Delta t^2}{2^2 2!}\frac{\partial^2 T}{\partial t^2}\bigg|_i^{n+1/2} - \frac{\Delta t^3}{2^3 3!}\frac{\partial^3 T}{\partial t^3}\bigg|_i^{n+1/2} + \frac{\Delta t^4}{2^4 4!}\frac{\partial^4 T}{\partial t^4}\bigg|_i^{n+1/2}$$

$$-\Delta x\frac{\partial T}{\partial x}\bigg|_i^{n+1/2} + \frac{\Delta x^2}{2!}\frac{\partial^2 T}{\partial x^2}\bigg|_i^{n+1/2} - \frac{\Delta x^3}{3!}\frac{\partial^3 T}{\partial x^3}\bigg|_i^{n+1/2} + \frac{\Delta x^4}{4!}\frac{\partial^4 T}{\partial x^4}\bigg|_i^{n+1/2}$$

$$+\Delta x\frac{\Delta t}{2}\frac{\partial^2 T}{\partial x \partial t}\bigg|_i^{n+1/2} - 3\frac{\Delta x}{2!}\left(\frac{\Delta t}{2}\right)^2\frac{\partial^3 T}{\partial x \partial t^2}\bigg|_i^{n+1/2} - 3\frac{\Delta x^2}{2!}\left(\frac{\Delta t}{2}\right)\frac{\partial^3 T}{\partial x^2 \partial t}\bigg|_i^{n+1/2} + \ldots$$

Things simplify quickly just by looking at parts of the sum:

$$T_{i-1}^n + T_{i+1}^n + T_{i-1}^{n+1} + T_{i+1}^{n+1} = 4T_i^{n+1/2} + \frac{\Delta t^2}{2!}\frac{\partial^2 T}{\partial t^2}\bigg|_i^{n+1/2} + \frac{\Delta t^4}{2^2 4!}\frac{\partial^4 T}{\partial t^4}\bigg|_i^{n+1/2} + 4\frac{\Delta x^2}{2!}\frac{\partial^2 T}{\partial x^2}\bigg|_i^{n+1/2} + 4\frac{\Delta x^4}{4!}\frac{\partial^4 T}{\partial x^4}\bigg|_i^{n+1/2} + \ldots$$

$$T_i^{n+1} + T_i^n = T_i^{n+1/2} + \frac{\Delta t^2}{2\cdot 2!}\frac{\partial^2 T}{\partial t^2}\bigg|_i^{n+1/2} + \frac{\Delta t^4}{2^3\cdot 4!}\frac{\partial^4 T}{\partial t^4}\bigg|_i^{n+1/2} + \ldots$$

So:

$$\frac{\left(T_{i-1}^n + T_{i+1}^n + T_{i-1}^{n+1} + T_{i+1}^{n+1} - 2T_i^n - 2T_i^{n+1}\right)}{2\cdot\Delta x^2} = \frac{\partial^2 T}{\partial x^2}\bigg|_i^{n+1/2} + 2\frac{\Delta x^2}{4!}\frac{\partial^4 T}{\partial x^4}\bigg|_i^{n+1/2} + \ldots$$

1-D Crank-Nicholson Stability

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{\alpha}{2}\left[\frac{T_{i+1}^{\;\;n+1} - 2T_i^{n+1} + T_{i-1}^{\;\;n+1}}{\Delta x^2}\right] + \frac{\alpha}{2}\left[\frac{T_{i+1}^{\;\;n} - 2T_i^n + T_{i-1}^{\;\;n}}{\Delta x^2}\right] + q^{n+1/2}$$

Rearrange to the form:

$$-\alpha\Delta t\left[\frac{T_{i+1}^{\;\;n+1} + T_{i-1}^{\;\;n+1}}{\Delta x^2}\right] + 2\left[1 + \frac{\alpha\Delta t}{\Delta x^2}\right]T_i^{n+1} = \alpha\Delta t\left[\frac{T_{i+1}^{n} + T_{i-1}^{\;\;n}}{\Delta x^2}\right] + 2\left[1 - \frac{\alpha\Delta t}{\Delta x^2}\right]T_i^n$$

Define a matrix H such that $h_{i,i} = -\frac{2\alpha\Delta t}{\Delta x^2}$ and $h_{i,i+1} = h_{i,i+1} = \frac{\alpha\Delta t}{\Delta x^2}$

Define matrix A = 2 I -H and B = 2 I +H = 4I-A, then the method can be written as

$$T^{n+1} = A^{-1}BT^n + A^{-1}d$$

$$= A^{-1}(4I - A)T^n + A^{-1}d$$

$$= (4IA^{-1} - 1)T^n + A^{-1}d$$

$$= (\frac{4I}{A} - 1)T^n + A^{-1}d$$

1-D Crank-Nicholson Stability

Stability requires that $\left(\frac{4I}{A}-1\right)$ have a spectral radius equal or less than 1, or

$\left|\frac{4}{\lambda_A}-1\right|\leq 1$ for any eigenvalue of A

$$-1\leq \frac{4}{\lambda_A}-1\leq 1$$

$$-\lambda_A \leq 4-\lambda_A \leq \lambda_A$$

$$0\leq 4\leq 2\lambda_A$$

or $\lambda_A \geq 2$

Time to use Gerschgorin's Theorem

Gerschgorin Analysis

$$\left| \lambda_A - 2\left[1 + \frac{\alpha\Delta t}{\Delta x^2}\right] \right| \le \left| \frac{2\alpha\Delta t}{\Delta x^2} \right|$$

The real interval containing these eigenvalues is:

$$\begin{array}{ccccc} \Big| \leftarrow & \dfrac{2\alpha\Delta t}{\Delta x^2} & \rightarrow \Big| \leftarrow & \dfrac{2\alpha\Delta t}{\Delta x^2} & \rightarrow \Big| \\ \lambda_2 & & 2\left[1 + \dfrac{\alpha\Delta t}{\Delta x^2}\right] & & \lambda_1 \end{array}$$

so

$$\lambda_1 = 2 + 4\frac{\alpha\Delta t}{\Delta x^2} \quad \text{and} \quad \lambda_2 = 2$$

Therefore $\lambda_A \ge 2$, always and the Crank-Nicholson finite difference equations are unconditionally stable