Propagation of Computational Errors, Stability of the Numerical Method

General Background

Consider two solutions to the problem. The first the hypothetical result of exact arithmetic and the second the result of a finite precision computer.

$T^m = ET^{m-1} + b \qquad \hat{T}^m = E\hat{T}^{m-1} + b$

$$
\begin{aligned}
\in^m &= T^m - \hat{T}^m \\
&= T^m - E\hat{T}^{m-1} - b \\
&= (T^m - b) - E\hat{T}^{m-1} \\
&= E(T^{m-1} - \hat{T}^{m-1}) \\
&= E \in^{m-1}
\end{aligned}
$$

$$
\begin{aligned}
\in^{m-1} &= E \in^{m-2} \\
&\vdots \\
\in^1 &= E \in^o
\end{aligned}
$$

So with some back substitution we get

$$\in^m = E^m \in^o$$

We want the error to be bounded no matter how large m becomes, which requires:

The spectral radius of E must be equal to or less than 1

Example

FDE $$\frac{1}{\alpha\Delta t}[\in_1^{m+1} - \in_1^{m}] = \frac{1}{\Delta^2}[\in_O^{m} + \in_2^{m} - 2 \in_{1m}]$$

$$\in_1^{m+1} = \underbrace{\frac{\alpha\Delta t}{\Delta^2}}_{A}[\in_O^{m} + \in_2^{m}] + [1 - \frac{2\alpha\Delta t}{\Delta^2}]\in_1^{m}$$

$$= A[\in_O^{m} + \in_2^{m}] + [1 - 2A]\in_1^{m}$$

Start stepping through time with an initial error of 1 at mesh point 3 and no error at other points

A = 0.25

m	$\in_0$	$\in_1$	$\in_2$	$\in_3$	$\in_4$	$\in_5$	Σ
0	0	0	0	1	0	0	1
1	0	0	.25	.5	.25	0	1
2	0	.063	.25	.375	.25	0	.9325
3	0	.094	.234	.312	.203	0	.843
4	0	.105	.218	.265	.179	0	.767
5	0	.107	.201	.232	.156	0	.696

The sum of errors decreases.

A = 1

m	$\in_0$	$\in_1$	$\in_2$	$\in_3$	$\in_4$	$\in_5$	Σ
0	0	0	0	1	0	0	1
1	0	0	1	−1	1	0	1
2	0	1	−2	3	−2	0	0
3	0	−3	6	−7	5	0	1
4	0	9	−16	18	−12	0	−1
5	0	25	43	−46	30	0	2

$$\in_i^{m+1} = [\in_{i+1}^{m} + \in_{i-1}^{m} - \in_1^{m}]$$

The solution is unstable

An Exact method for predicting stability

For a Forward difference method

$T^{m+1} = ET^m + b$

$$T_i^{m+1} = A[T_{i+1}^{m} + T_{i-1}^{m}] - (1-2A)T_i^{m}$$

Set boundary conditions: $T_0 = 100$ $\qquad$ $T_5 = 5$

$$\begin{pmatrix} T_1^{m+1} \\ T_2^{m+1} \\ T_3^{m+1} \\ T_4^{m+1} \end{pmatrix} = \begin{pmatrix} (1-2A) & A & 0 & 0 \\ A & (1-2A) & A & 0 \\ 0 & A & (1-2A) & A \\ 0 & 0 & A & (1-2A) \end{pmatrix} \begin{pmatrix} T_1^{m} \\ T_2^{m} \\ T_3^{m} \\ T_4^{m} \end{pmatrix} + A \begin{pmatrix} 100 \\ 0 \\ 0 \\ 5 \end{pmatrix}$$

Stability requires that the spectral radius is less than 1.

$$\left| \lambda_{SR} \right|_E \le 1$$

To calculate the spectral radius, look at details of the matrix

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + A \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

E = l + A F

The eigenvalues of F are:

$$\lambda\big|_F = -4\, SIN^2[\frac{s\pi}{2(N+1)}], \quad s = 1, 2, 3, \cdots N$$

where N is the order of matrix. Hence, the eigenvalues of E are:

$$\lambda\big|_E = 1 - 4A\, SIN_2[\frac{s\pi}{2(N+1)}]$$

For stability $\left|\lambda_{SR}\big|_E\right| \leq 1$

$$\left|1 - 4A\, SIN^2[\frac{s\pi}{2(N+1)}]\right| \leq 1$$

$$-1 \leq 1 - 4\, A\, SIN^2[\frac{s\pi}{2(N+1)}] \leq 1$$

so

$$0 \leq 2 - 4\, A\, SIN^2[\frac{s\pi}{2(N+1)}] \leq 2$$

Upper limit satisfied automatically

Lower limit $$0 \le 2 - 4\,A\,SIN^2\left[\frac{s\pi}{2(N+1)}\right]$$

$$4A\,SIN_2\left[\frac{s\pi}{2(N+1)}\right] \le 2$$

or $$A \le \frac{2}{4\,SIN^2\left[\frac{s\pi}{2(N+1)}\right]}$$

Spectral radius (largest eigenvalue) corresponds to s = N

Example N = 4

$$A \le \frac{1}{2SIN^2[\frac{4\pi}{2(5)}]} \le 0.553$$

so $$\Delta t \le \frac{0.553\Delta^2}{\alpha}$$

If N = 100

$$A \le \frac{1}{2SIN^2[\frac{100\pi}{2(101)}]} \le 0.50012$$

so $$\Delta t \le \frac{0.50012\,\Delta^2}{\alpha}$$

Convergence

We would like the solution of our finite difference, or finite volume equations to converge to the actual solution of the differential equations as time steps and mesh spacing get very small. Convergence is usually demonstrated via the Lax Equivalence Theorem:

> Given a properly posed initial-value problem and a finite-difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence.

Be aware that this theorem can only be rigorously proved for linear PDE's but is a good bet for more complicated systems

Bounds that Insure Stability

We can't always do an exact calculation of the spectral radius associated with a method. However, there are ways of establishing bounds on the time step size that maintain stability of the solution.

Gerschgorin's theorem maps a set of disks in the complex plane that include all eigenvalues for a matrix. If all points contained in these disks are contained in the unit circle of the complex plane, we will have stability. Let $e_{i,j}$ be the coefficients of matrix E (order N), then for eigenvalues λ of the matrix, the following equation taken for all rows in the matrix defines the bounding disks:

$$\left|\lambda - e_{i,i}\right| \le \left|e_{i,1}\right| + \left|e_{i,2}\right| + \dots \left|e_{i,i-1}\right| + \left|e_{i,i+1}\right| + \dots + \left|e_{i,N}\right|$$

To determine stability limits, this equation must be applied for every nodal finite difference equation.

Example Finite difference equation for node 3

$$T_3^{m+1} = AT_2^{\ m} + (1-2A)T_3^{\ m} + AT_4^{\ m}$$

$$\left|\lambda - (1-2A)\right| \le \left|A\right| + \left|A\right| \quad \le 2\left|A\right|$$

$$\left| \leftarrow 2A \rightarrow \right| \leftarrow 2A \rightarrow \left|\right.$$

$$\lambda_2 \qquad (1-2A) \qquad \lambda_1$$

thus

$$\lambda_1 = 1 - 2A + 2A = 1$$
$$\lambda_2 = 1 - 2A - 2A = 1 - 4A$$

For stability $\left|\lambda\right| \le 1$

λ_1 ok since $\lambda_1 = 1$

λ_2 $\left|1-4A\right| \le 1$

$$-1 \le 1 - 4A \le 1$$
$$0 \le 2 - 4A \le 2$$

Since A is positive upper condition is satisfied for λ_2

i.e. $2-4A\leq 2$

Lower limit

$$0\leq 2-4A$$

$$A\leq\frac{1}{2} \;\Rightarrow\; \Delta t\leq\frac{\Delta^2}{2\alpha}$$

Second Example Boundary Node

$$DE.\ \frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}$$

$$BC\quad x=0 \qquad h_\infty(T_\infty-T_o)=-k\,\frac{\partial T}{\partial X}\Big|_O$$

On the finite difference grid mesh point 0 is at the surface and mesh point 1 is located distance Δ into the conducting material. The difference equation at the surface is:

$$\text{FDE}\qquad \frac{T_o^{\,m+1}-T_o^{\,m}}{\alpha\,\Delta t}=\frac{2}{\Delta^2}\left[T_1^{\,m}-T_o^{\,m}\right]+\frac{2h}{k\Delta}\left[T_\infty-T_o^{\,m}\right]$$

Rearrange FDE in form $T^{m+1} = ET^{m} + b$

$$T_0^{m+1} = \frac{2\alpha\Delta t}{\Delta^2} T_1^{m} + \left[1 - \frac{2\alpha\Delta t}{\Delta^2} - \frac{2h\alpha\Delta t}{k\Delta}\right] T_0^{m} + \frac{2h\alpha\Delta t T\infty}{k\Delta}$$

Use Gerschgorin's theorem

$$\left|\lambda - (1 - \frac{2\alpha\Delta t}{\Delta^2} - \frac{2h\alpha\Delta t}{k\Delta})\right| \le \frac{2\alpha\Delta t}{\Delta^2}$$

$$\left| \leftarrow \frac{2\alpha\Delta t}{\Delta^2} \rightarrow \right| \leftarrow \frac{2\alpha\Delta t}{\Delta^2} \rightarrow |$$

$$\lambda_2 \quad 1 - \frac{2a\Delta t}{\Delta^2} - \frac{2h\alpha\Delta t}{k\Delta} \quad \lambda_1$$

$$\lambda_1 = 1 - \frac{2\alpha\Delta t}{\Delta^2} - \frac{2h\alpha\Delta t}{k\Delta} + \frac{2\alpha\Delta t}{\Delta^2} = 1 - \frac{2h\alpha\Delta t}{k\Delta}$$

$$\lambda_2 = 1 - \frac{2\alpha\Delta t}{\Delta^2} - \frac{2h\alpha\Delta t}{k\Delta} - \frac{2\alpha\Delta t}{\Delta^2} = 1 - \frac{4\alpha\Delta t}{\Delta^2} - \frac{2h\alpha\Delta t}{k\Delta}$$

λ_1 $\qquad\qquad$ λ_2

$$-1 \le \lambda_1 \le 1 \qquad\qquad -1 \le 1 - \frac{4\alpha\Delta t}{\Delta^2} - \frac{2h\alpha\Delta t}{k\Delta} \le 1$$

$$-1 \le 1 - \frac{2h\alpha\Delta t}{k\Delta} \le 1 \qquad\qquad 0 \le \underbrace{2 - \frac{4\alpha\Delta t}{\Delta^2} - \frac{2h\alpha\Delta t}{k\Delta}}_{Ok} \le 2$$

$$0 \le \underbrace{2 - \frac{2h\alpha\Delta t}{k\Delta}}_{Ok} \le 2$$

Lower limit

$$0 \le 2 - \frac{2h\alpha\Delta t}{k\Delta}$$

$$\frac{h\alpha\Delta t}{k\Delta} \le 1$$

$$\Delta t \le \frac{k\Delta}{h\alpha}$$

Lower Limit

$$0 \le 2 - \frac{4\alpha\Delta t}{\Delta^2} - \frac{2h\alpha\Delta t}{k\Delta}$$

$$\frac{2\alpha\Delta t}{\Delta^2} + \frac{h\alpha\Delta t}{k\Delta} \le 1$$

$$\frac{\alpha\Delta t}{\Delta^2}\left[2 + \frac{h\Delta}{k}\right] \le 1$$

$$\Delta t \le \frac{k\Delta^2}{\alpha(2k + h\Delta)}$$

$$\Delta t \le \frac{k\Delta}{h\alpha}\left[\frac{1}{1 + \frac{2k}{h\Delta}}\right] \leftarrow \textit{most severe}$$

We will return to stability when we discuss von Neumann stability analysis