TRANSIENT HEAT CONDUCTION

A. INTRODUCTION

Remember that the transient energy conservation law is:

$$\begin{Bmatrix} \textit{Net rate of} \\ \textit{accumulation of} \\ \textit{energy by system} \end{Bmatrix} = \begin{Bmatrix} \textit{Net rate of} \\ \textit{heat transfer} \\ \textit{across surfaces} \end{Bmatrix} + \begin{Bmatrix} \textit{Net rate} \\ \textit{of} \\ \textit{heat generation} \end{Bmatrix}$$

$$\rho c_p \frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \dot{q}'''$$

The equation system is Parabolic, a Time Marching Solution

General

Expression $$\frac{\partial \phi}{\partial t} = \nabla^2 \phi + S$$

Note: It is common practice to use a “false” transient method to obtain a solution for a steady state problem.

One Dimensional

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad \text{where} \quad \alpha \equiv \frac{k}{\rho c_p} \quad \text{(Thermal diffusivity)}$$

$$\rightarrow|\Delta x|\leftarrow$$

$$\underset{i-1 \quad i \quad i1}{\longrightarrow} x$$

When writing difference equations we will continue to use subscripts to denote position on the spatial mesh. Now we have to think in terms of the mesh extending into time. Location in time will be marked with a superscript. Superscript of "m" or "n" will denote the "current" time level, at which we have full information on the state of the system. Superscript of "m+1" or "n+1" denotes the next time level, separated from the current one by and interval Δt. The name of the game is to solve for the full state of the system at level "m+1", and then move on to another time step, until the desired end time of the calculation is reached.

1D conduction difference equation at an Interior node:

$$\frac{T_i^{m+1} - T_i^m}{\Delta t} = \frac{\alpha}{\Delta x^2}\left[\overline{T}_{i+1} - 2\overline{T}_i = \overline{T}_{i-1}\right]$$

Where $\overline{T}$ is an abbreviation to hide the fact that we haven't decided on a time level for these terms. This is defined more directly in terms of a time weighting factor "f" as:

$$\overline{T}_i = \mathrm{f}\, T_i^{m+1} + (1\text{- f})\; T_i^m$$

So the longer definition of the difference equation

$$\frac{T_i^{m+1} - T_i^m}{\Delta t} = \frac{\alpha}{\Delta x^2}\left[\left(f\,T_{i+1}^{m+1} + (1-f)T_{i+1}{}^{m}\right) - 2\left(f\,T_i^{m+1} + (1-f)T_i^m\right) + \left(f\,T_{i-1}^{m+1} + (1-f)T_{i-1}{}^{m}\right)\right]$$

Three specific values of f provide the three main forms of transient time level methods.

f = 0 Forward Difference (Explicit Method)

f = 0.5 Crank-Nicolson

f = 1 Backward Difference (Implicit Method)

B. FORWARD DIFFERENCE METHOD

a) Introduction

Taylor series approximation of $\left.\frac{\partial T}{\partial t}\right|_{o}^{m}$

$$T_O{}^{m+1} = T_O{}^{m} + \Delta t \left.\frac{\partial T}{\partial t}\right|_O^m + \frac{\Delta t^2}{2}\left.\frac{\partial^2 T}{\partial t^2}\right|_O^m + \ldots$$

$$\frac{\partial T}{\partial t} = \frac{T_O{}^{m+1} - T_o{}^{m}}{\Delta t} - \underbrace{\frac{\Delta t}{2}\left.\frac{\partial^2 T}{\partial t^2}\right|_O^{\xi}}_{error} \quad m\Delta t < \zeta < (m+1)\Delta t$$

Using our compass notation for spatial grid points, the explicit finite difference equation is:

$$\frac{1}{\alpha}\left[\frac{T_O{}^{m+1} - T_O{}^{m}}{\Delta t}\right] = \left[\frac{T_N{}^{m} + T_S{}^{m} - 2T_O{}^{m}}{\Delta y^2} + \frac{T_E{}^{m} + T_W{}^{m} - 2T_O{}^{m}}{\Delta x^2}\right] \quad + \frac{\dot{q}'''}{k}$$

Rearrange

$$T_O{}^{m+1} = \alpha\Delta t\left[\frac{T_N{}^{m} + T_S{}^{m}}{\Delta y^2}\right] + \alpha\Delta t\left[\frac{T_E{}^{m} + T_W{}^{m}}{\Delta x^2}\right] + \alpha\Delta t\,\frac{\dot{q}'''}{k} + \left[1 - 2\alpha\Delta t\left[\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right]\right]T_O{}^{m}$$

Assume a uniform grid and let $\Delta x = \Delta y = \Delta$

For no heat generation:

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T$$

Write the collection of all difference equations as a linear system (E is a matrix).

$$T^{m+1} = E\ T^{m} + b$$

Let $A \equiv \dfrac{\alpha\Delta t}{\Delta^2}$

$$\begin{matrix} T_1^{m+1} \\ T_2^{m+1} \\ T_3^{m+1} \\ T_4^{m+1} \\ T_5^{m+1} \\ T_6^{m+1} \end{matrix} \quad = \quad \begin{matrix} (1-4A) & A & O & A & O & O \\ A & (1-4A) & A & O & A & O \\ O & A & (1-4A) & O & O & A \\ A & O & O & (1-4A) & A & O \\ O & A & O & A & (1-4A) & A \\ O & O & A & O & A & (1-4A) \end{matrix} \quad \begin{matrix} T_1^{m} \\ T_2^{m} \\ T_{m3}^{m} \\ T_4^{m} \\ T_5^{m} \\ T_6^{m} \end{matrix} \quad + \quad \begin{matrix} 200A \\ 100A \\ 100A \\ 100A \\ 0 \\ 0 \end{matrix}$$

Example

$$T_3{}^{m+1} = A\ T_2 + (1\text{-}4A)\ T_3{}^{m} + A\ T_6{}^{m} + 100A$$

b) Truncation error

$\in^T \Big|^m \equiv$ finite difference equation – differential equation

$$\in^T \Big|_1^m = \frac{1}{\alpha}\left[\frac{T_1^{m+1} - T_1^m}{\Delta t}\right] - \left[\frac{T_0^m - 2T_1^m + T_2^m}{\Delta^2}\right] - \left[\frac{1}{\alpha}\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2}\right]_1^m$$

Expansions

$$T_1^{m+1} = T_1^m + \Delta t \frac{\partial T}{\partial t}\Big|_1^m + \frac{\Delta t^2}{2}\frac{\partial^2 T}{\partial t^2}\Big|_1^m + \frac{\Delta t^3}{3!}\frac{\partial^3 T}{\partial t^3}\Big|_1^m + \frac{\Delta t^4}{4!}\frac{\partial^4 T}{\partial t^4}\Big|_1^m + \cdots$$

$$T_0^m = T_1^m - \Delta \frac{\partial T}{\partial x}\Big|_1^m + \frac{\Delta^2}{2}\frac{\partial^2 T}{\partial x^2}\Big|_1^m - \frac{\Delta^3}{3!}\frac{\partial^3 T}{\partial x^3}\Big|_1^m + \cdots$$

$$T_2^m = T_1^m + \Delta \frac{\partial T}{\partial x}\Big|_1^m + \frac{\Delta^2}{2}\frac{\partial^2 T}{\partial x^2}\Big|_1^m + \frac{\Delta^3}{3!}\frac{\partial^3 T}{\partial x^3}\Big|_1^m + \cdots$$

Substitute into expression for $\in^T \Big|^m$

$$\in^T \Big|_1^m = \frac{1}{\alpha}\left[\frac{\partial T}{\partial t}\Big|_1^m + \frac{\Delta t}{2}\frac{\partial^2 T}{\partial t^2}\Big|_1^m + \frac{\Delta t^3}{3!}\frac{\partial^3 T}{\partial t^3}\Big|_1^m + \cdots\right] - \left[\frac{\partial^2 T}{\partial x^2}\Big|_1^m + \frac{2\Delta^2}{4!}\frac{\partial^4 T}{\partial x^4}\Big|_1^m + \cdots\right]$$

$$- \left[\frac{1}{\alpha}\frac{\partial T}{\partial t}\Big|_1^m - \frac{\partial^2 T}{\partial x^2}\Big|_1^m\right]$$

$$= \frac{\Delta t}{2\alpha}\frac{\partial^2 T}{\partial t^2}\Big|_1^m + \frac{\Delta t^3}{6\alpha}\frac{\partial^3 T}{\partial t^3}\Big|_1^m + \cdots - \frac{\Delta^2}{12}\frac{\partial^4 T}{\partial x^4}\Big|_1^m - \frac{\Delta^4}{360}\frac{\partial^6 T}{\partial t^6}\Big|_1^m - \cdots$$

The forward difference method is first order accurate in time and second order in space.

Consistency

A difference method is said to be "consistent" or "compatible" with a differential equation if the truncation error approaches zero as the size of the increments of the independent variables approach 0. We are in luck here. Later I will show you examples where this situation does not hold.

$$\text{Limit} \quad \in^T\Big|^m = 0 \qquad \left.\begin{array}{l}\Delta \to 0\\ \Delta t \to 0\end{array}\right\} \textit{any order}$$