

# PREDICTOR-CORRECTOR METHOD

## INTRODUCTION

Basic Principle – Uses Two Steps

Predictor (Implicit)

- Form the finite difference equation and the error term at each node using the Taylor Series Method (DEM)
- Solve the set of finite difference equations to find  $T^*$  the predicted temperature

Corrector (Explicit)

- Approximate the derivatives which appear in the error expressions  $\varepsilon^*$ , at each node using the  $T^*$  distributions.
- Correct  $T^*$ , i.e.  $T = T^* + \delta$

**Example** – Steady State 2-D Heat Conduction With Internal Heat Generation

$$\nabla^2 T + \frac{\dot{q}'''}{k} = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}'''}{k} = 0$$

Use Differential Equation Method (DEM) and the Taylor Series Method

Let  $\Delta x = \Delta y = \Delta$

Finite Difference Equation

$$\frac{T_E + T_W - 2T_O}{\Delta x^2} + \frac{T_N + T_S - 2T_O}{\Delta y^2} + \frac{\dot{q}'''}{k} + ERROR = 0$$

where

$$ERROR = \varepsilon = -\frac{2\Delta^2}{4!} \left[ \frac{\partial^4 T}{\partial x^4} \Big|_{\xi_1} + \frac{\partial^4 T}{\partial y^4} \Big|_{\xi_2} \right]$$

with

$$x_w < \xi_1 < x_E$$

$$y_S < \xi_2 < y_N$$

### Predictor step

For this step assume  $\varepsilon=0$  and solve the set of finite difference equations to obtain values of  $T^*$ 's. The equation solved for interior points is:

$$T_E^* + T_W^* - 4T_O^* + T_N^* + T_S^* = -\frac{\dot{q}''' \Delta^2}{k}$$

### Corrector step

Determine the approximation for

$$\left. \frac{\partial^4 T}{\partial x^4} \right|_O \quad \text{and} \quad \left. \frac{\partial^4 T}{\partial y^4} \right|_O$$

**How do we find an approximation for the derivatives at O?**

Use Taylor Series expansions.

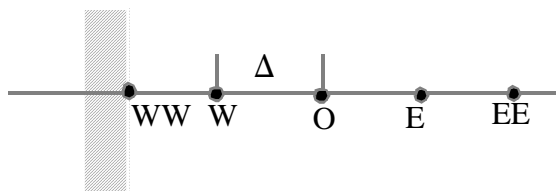
**How many expansions do we need?**

Introduce weighting coefficients to find out.

**What conditions do we use to evaluate the weighting coefficients?**

**For the grid shown, is any special treatment needed near the boundaries?**

### Interior node



Approximation for  $\left. \frac{\partial^4 T}{\partial x^4} \right|_o$

Weighting

Coefficients

Taylor Series Expansions

$$\begin{array}{l}
 \text{[A]} \quad T_E = T_o + \Delta \left. \frac{\partial T}{\partial x} \right|_o + \frac{\Delta^2}{2} \left. \frac{\partial^2 T}{\partial x^2} \right|_o + \frac{\Delta^3}{3!} \left. \frac{\partial^3 T}{\partial x^3} \right|_o + \frac{\Delta^4}{4!} \left. \frac{\partial^4 T}{\partial x^4} \right|_o + \dots \\
 \text{[B]} \quad T_{EE} = T_o + 2\Delta + \frac{(2\Delta)^2}{2!} + \frac{(2\Delta)^3}{3!} + \frac{(2\Delta)^4}{4!} \\
 \text{[C]} \quad T_w = T_o - \Delta + \frac{\Delta^2}{2!} - \frac{\Delta^3}{3!} + \frac{\Delta^4}{4!} \\
 \text{[D]} \quad T_{ww} = T_o - 2\Delta + \frac{(2\Delta)^2}{2!} - \frac{(2\Delta)^3}{3!} + \frac{(2\Delta)^4}{4!}
 \end{array}$$

After summing the above equations requirements on coefficients of derivatives are:

$$\sum [\text{COEF}] \left. \frac{\partial T}{\partial x} \right|_o = 0 = A\Delta + 2\Delta B - \Delta C - 2\Delta D$$

$$A + 2B - C - 2D = 0$$

$$\sum [\text{COEF}] \left. \frac{\partial^2 T}{\partial x^2} \right|_o = 0 = A \frac{\Delta^2}{2} + B \frac{(2\Delta)^2}{2} + C \frac{\Delta^2}{2} + D \frac{(2\Delta)^2}{2}$$

$$A + 4B + C + 4D = 0$$

$$\sum [\text{COEF}] \left. \frac{\partial^3 T}{\partial x^3} \right|_o = 0 = A \frac{\Delta^3}{3!} + B \frac{(2\Delta)^3}{3!} - C \frac{\Delta^3}{3!} - D \frac{(2\Delta)^3}{3!}$$

$$A + 8B - C - 8D = 0$$

$$\sum [\text{COEF}] \frac{\partial^4 T}{\partial x^4} \Big|_o = 1 = A \frac{\Delta^4}{4!} + B \frac{(2\Delta)^4}{4!} + C \frac{\Delta^4}{4!} + D \frac{(2\Delta)^4}{4!}$$

$$A + 16B + C + 16D = \frac{4!}{\Delta^4}$$

Solution of this simultaneous set of equations is:

$$A = C = -\frac{4}{\Delta^4}, \quad B = D = \frac{1}{\Delta^4}$$

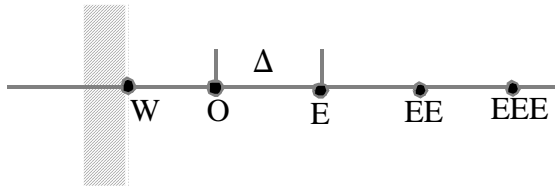
Approximation becomes

$$\frac{\partial^4 T}{\partial x^4} \Big|_o = \frac{T_{EE}^* - 4T_E^* + 6T_O^* - 4T_W^* + T_{WW}^*}{\Delta^4} + O(\Delta^2)$$

Error based upon predictor temperatures is:

$$\varepsilon^* = -\frac{2\Delta^4}{(4!)} \frac{1}{\Delta^4} \left[ (T_{EE}^* - 4T_E^* + 6T_O^* - 4T_W^* + T_{WW}^*) + (T_{SS}^* - 4T_S^* + 6T_O^* - 4T_N^* + T_{NN}^*) \right]$$

**At near wall nodes**



What is the appropriate expression for  $\frac{\partial^4 T}{\partial x^4} \Big|_o = ?$

Weighting  
Coefficients

Taylor Series Expansions

$$[A] \quad T_w = T_o - \Delta \frac{\partial T}{\partial x} \Big|_o + \frac{\Delta^2}{2!} \frac{\partial^2 T}{\partial x^2} \Big|_o - \frac{\Delta^3}{3!} \frac{\partial^3 T}{\partial x^3} \Big|_o + \frac{\Delta^4}{4!} \frac{\partial^4 T}{\partial x^4} \Big|_o$$

$$\begin{aligned}
\text{[B]} \quad T_E &= T_O + \Delta + \frac{\Delta^2}{2!} + \frac{\Delta^3}{3!} + \frac{\Delta^4}{4!} \\
\text{[C]} \quad T_{EE} &= T_O + 2\Delta + \frac{(2\Delta)^2}{2} + \frac{(2\Delta)^3}{3!} + \frac{(2\Delta)^4}{4!} \\
\text{[D]} \quad T_{EEE} &= T_O + 3\Delta + \frac{(3\Delta)^2}{2} + \frac{(3\Delta)^3}{3!} + \frac{(3\Delta)^4}{4!}
\end{aligned}$$

Requirements

$$\begin{aligned}
\sum [\text{COEF}] \frac{\partial T}{\partial x} \Big|_0 &= 0 = -A\Delta + B\Delta + 2\Delta C + 3\Delta D \\
-A + B + 2C + 3D &= 0
\end{aligned}$$

$$\begin{aligned}
\sum [\text{COEF}] \frac{\partial^2 T}{\partial x^2} \Big|_0 &= 0 = A \frac{\Delta^2}{2!} + B \frac{\Delta^2}{2!} + C \frac{(2\Delta)^2}{2!} + D \frac{(3\Delta)^2}{2!} \\
A + B + 4C + 9D &= 0
\end{aligned}$$

$$\begin{aligned}
\sum [\text{COEF}] \frac{\partial^3 T}{\partial x^3} \Big|_0 &= 0 = -A \frac{\Delta^3}{3!} + B \frac{\Delta^3}{3!} + C \frac{(2\Delta)^3}{3!} + D \frac{(3\Delta)^3}{3!} \\
-A + B + 8C + 27D &= 0
\end{aligned}$$

$$\begin{aligned}
\sum [\text{COEF}] \frac{\partial^4 T}{\partial x^4} \Big|_0 &= 1 = A \frac{\Delta^4}{4!} + B \frac{\Delta^4}{4!} + C \frac{(2\Delta)^4}{4!} + D \frac{(3\Delta)^4}{4!} \\
A + B + 16C + 81D &= \frac{4!}{\Delta^4}
\end{aligned}$$

Simultaneous solution of the set of equations

$$A = \frac{1}{\Delta^4}, \quad B = \frac{6}{\Delta^4}, \quad C = -\frac{4}{\Delta^4}, \quad D = \frac{1}{\Delta^4}$$

So

$$\frac{\partial^4 T}{\partial x^4} \Big|_0 = \frac{T_W^* - 4T_O^* + 6T_E^* - 4T_{EE}^* + T_{EEE}^*}{\Delta^4} + O(\Delta)$$

The error term is:

$$\varepsilon^* = -\frac{2\Delta^4}{(4!)} \frac{1}{\Delta^4} \left[ (T_W^* - 4T_O^* + 6T_E^* - 4T_{EE}^* + T_{EEE}^*) + \right. \\ \left. (T_{SS}^* - 4T_S^* + 6T_O^* - 4T_N^* + T_N^*) \right]$$

To generate the corrected solution, we now introduce the error terms into the equation and solve.

$$\frac{T_E + T_W - 4T_O + T_N + T_S}{4} = -\frac{\dot{q}''' \Delta^2}{4k} - \varepsilon^*$$

Here we are solving a linear system with the same coefficient matrix used for the predictor step, but with a modified right hand side. With the right choice of solution method (e.g. LU decomposition), this can be substantially less time consuming than the original predictor solution. If A is the coefficient matrix then the correction equation can be written as:

$$\vec{T} = \vec{T}^* + \vec{\delta} = \vec{T}^* - A^{-1} \vec{\varepsilon}^*$$

Here I've added arrows over symbols to emphasize that we are dealing with the vectors of all mesh point temperatures and vector of all mesh point errors.