Classifying Second Order PDE's with n Independent Variables

For generality we write the PDE in the form:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\frac{\partial^2\phi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial\phi}{\partial x_i} + c\phi = d$$

Classification is based on the matrix formed by the coefficients $a_{i,j}$. However, for off diagonal elements you've got to know one mathematical trick to make the matrix A symmetric. If I have an equation,

$$\frac{\partial^2\phi}{\partial x^2} - 2\frac{\partial^2\phi}{\partial x \partial y} + \frac{\partial^2\phi}{\partial y^2} = f(x,y),$$

then I start by renaming $x=x_1$ and $y=x_2$ to get into the general subscript notation:

$$\frac{\partial^2\phi}{\partial x_1^2} - 2\frac{\partial^2\phi}{\partial x_1 \partial x_2} + \frac{\partial^2\phi}{\partial x_2^2} = f(x_1,x_2).$$

I now remember my basic calculus and noting

$$\frac{\partial^2\phi}{\partial x_1 \partial x_2} = \frac{\partial^2\phi}{\partial x_2 \partial x_1},$$

I rewrite the original equation as:

$$\frac{\partial^2\phi}{\partial x_1^2} - \frac{\partial^2\phi}{\partial x_1 \partial x_2} - \frac{\partial^2\phi}{\partial x_2 \partial x_1} + \frac{\partial^2\phi}{\partial x_2^2} = f(x_1,x_2).$$

Comparing to the general equation form at the top of the page, I can see that:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Once you've got a matrix A through this sort of trickery, you follow the prescription laid down by some mathematician to classify the equation.

First solve the eigenvalue problem $\det(A-\lambda I) = 0$, and do two different counts based on the solution values. Count the number of solutions to the eigenvalue polynomial that have a value of zero (call that count "Z"). Actually as you will see below, you don't have to count beyond 1 to get a useful answer. Count the number of solutions that are positive (call that count "P"). Now you can assign a category to the PDE.

The simple one is "Parabolic". All you've got to do is look to see if any of the eigenvalues have a value of zero. If so, the matrix A is singular, and the second order PDE is parabolic.

If none of the eigenvalues are zero (Z=0), and there are either zero or n positive eigenvalues (P=0 or P=n), then you have an Elliptic PDE.

If none of the eigenvalues are zero (Z=0), and there are either one or n-1 positive eigenvalues (P=1 or P=n-1), then you have a Hyperbolic PDE.

If none of the eigenvalues are zero (Z=0), and there are from 2 through n-2 positive eigenvalues (1<P<n-1), then you have an Ultra-Hyperbolic PDE.

Classification Examples

(1)
Start with the equation used to illustrate generation of the symmetric equation matrix. The eigenvalue equation is:

$$\det\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} = 0$$

or

$$(1-\lambda)^2 - 1 = 0$$

or

$$\lambda^2 - 2\lambda = 0.$$

Hence the two eigenvalues are $\lambda=0$ and $\lambda=2$. The equation is **parabolic**.

(2)

Consider a Cartesian Steady State Conduction equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0.$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Hence there are three positive eigenvalues (all one), and the equation is **elliptic**.

(3)

The equation for a vibrating membrane

$$\frac{\partial^2 T}{\partial t^2} - c^2\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = 0.$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -c^2 & 0 \\ 0 & 0 & -c^2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -c^2 - \lambda & 0 \\ 0 & 0 & -c^2 - \lambda \end{vmatrix} = 0$$

Hence there is only one positive eigenvalue (one), and the equation is **hyperbolic**.