EQUATIONS OF MOTION II: BASICS

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"There is no more common error than to assume that, because prolonged and accurate mathematical calculations have been made the application of the results to some fact of nature is absolutely certain."
Alfred North Whitehead (1861-1947)

EQUATIONS OF MOTION

Equations of Motion – set of mathematical equations which describe the forces and movements of a body.

Inverse Dynamics – starting from the motion of the body determines the forces and moments causing the motion.
Process: measure joint displacements, differentiate to obtain velocities and accelerations, use Newton's Laws to compute forces and moments acting on body.

Direct Dynamics – starting from the forces and moments acting on a body determines the motion arising from these forces and moments. (Also called forward dynamics.)
Process: "obtain" moments and forces acting at joints, integrate to obtain joint displacements.

MECHANICAL APPROACHES

Although conservation laws can be invoked most common approach is to consider forces and moments acting on a body, and use these to determine equations of motion.

Two main approaches are

Newton-Euler

Newton $\sum F = m.a$

Euler $\sum T = I.\alpha = I.\ddot{\theta}$ (2D case only)

Lagrangian

Lagrangian Equation $L = K - P$

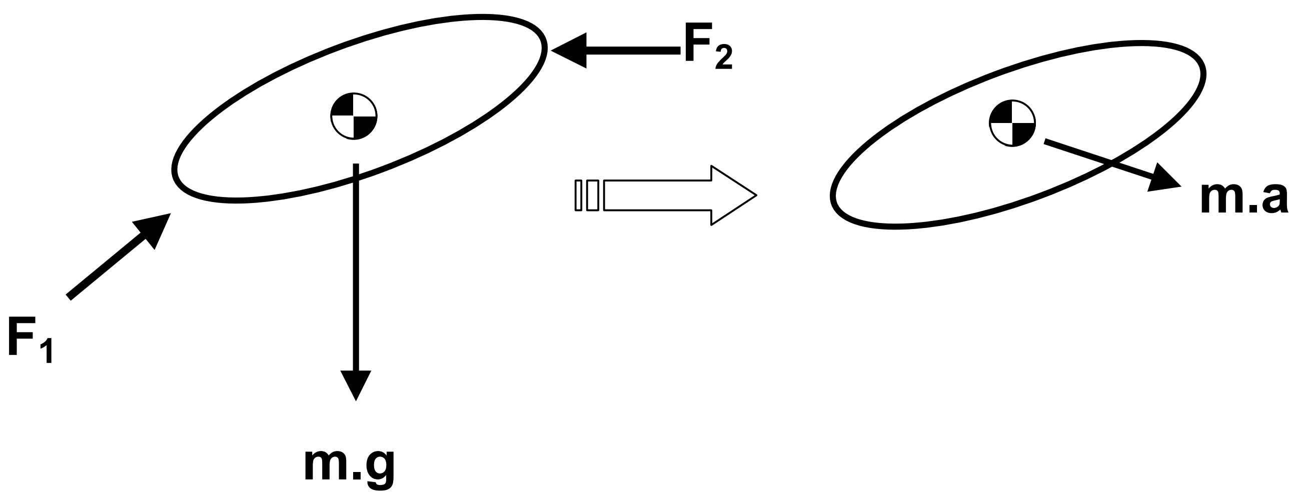
Equation of Motion $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = T$ (Non-conservative)

Others Methods

- Kane’s Method
- Gibbs-Appell
- Jourdain

EXAMPLE I – A SINGLE RIGID BODY

The net effect of all forces acting on a rigid body is equal to the product of the mass of the segment and the acceleration of the center of mass.



$$\sum F_i = m.a$$

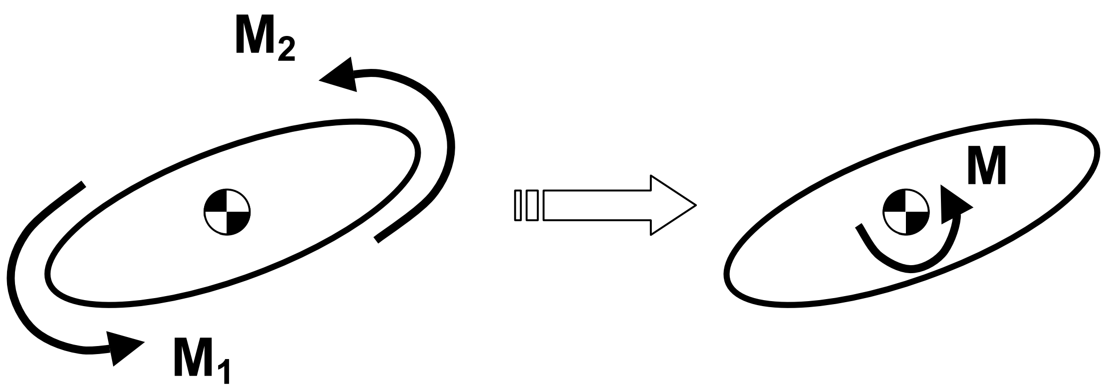
$$F_1 + F_2 + F_3 = m.a$$

Where

$$F_3 = m.g$$

EXAMPLE I – A SINGLE RIGID BODY

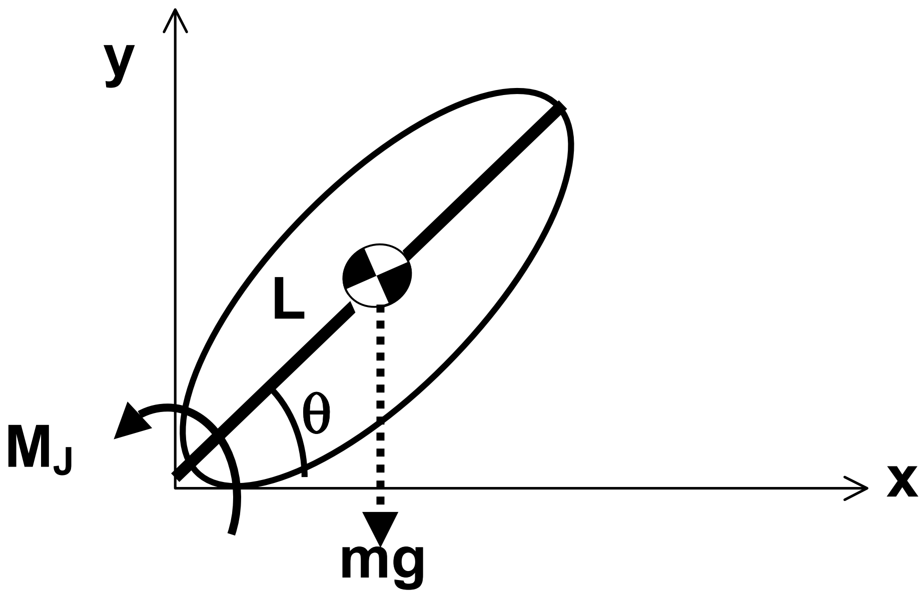
The net effect of all moments acting on a rigid body to cause rotation about the segments center of mass is equal to the product of the moment of inertia of the segment and the angular acceleration of the center of mass.



$$\sum M = I.\alpha$$

$$M_1 + M_2 = I.\alpha$$

EXAMPLE I – A SINGLE RIGID BODY



$$\sum M = M_J + m.g.L.\cos(\theta) = I.\alpha = I.\ddot{\theta}$$

$$\ddot{\theta} = [M_J + m.g.L.\cos(\theta)]/I$$

Where

M_J - applied moment at joint
m - mass of the segment
g - acceleration due to gravity
L - distance from joint center to segment center of mass
θ - angle of segment to horizontal
I - moment of inertia of segment
$\ddot{\theta}$ - angular acceleration

MECHANICAL BASICS

Mechanical Degrees of Freedom - refers to the number of independent coordinates required to specify the location of a body or system of bodies.

Mechanical degrees of freedom for the arm?

Shoulder	three degrees of freedom
Elbow	two degrees of freedom
Wrist	two degrees of freedom

Total Degrees of Freedom - 7

So in theory for the whole upper limb ***seven generalized coordinates*** should specify the location of the arm. Standard mechanical analyses start with specifying the x, y, z coordinates of each segments center of mass, and the three angles (6 x 3 = 18).

MECHANICAL BASICS

Generalized Coordinates - refers to the minimum number of number of independent coordinates needed to specify the motion of a body. There are often many sets of generalized coordinates for a given system, but a prudent choice can simplify the analysis significantly.

q - generalized coordinate

$\dot{q}$ - generalized velocity

$\ddot{q}$ - generalized acceleration

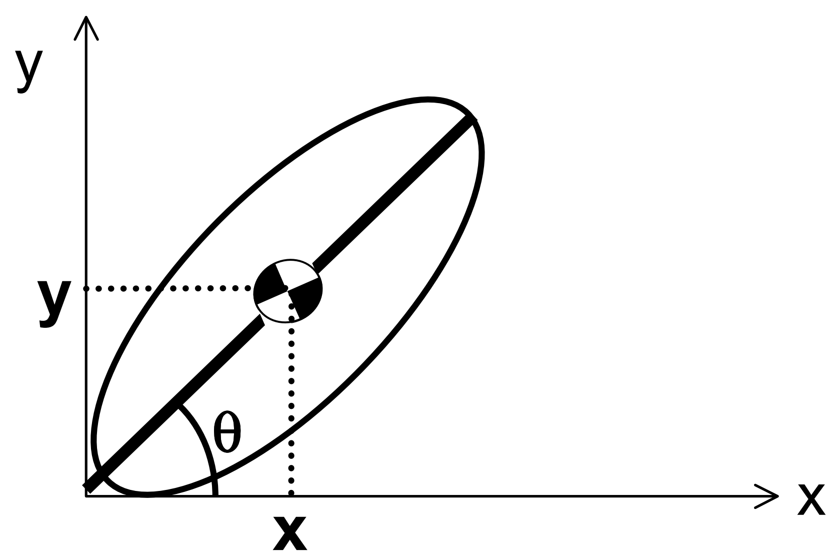
Example

$$x = L.\cos(\theta)$$

$$y = L.\sin(\theta)$$

$$\dot{x} = - L.\dot{\theta}.\cos(\theta)$$

$$\dot{y} = L.\dot{\theta}.\sin(\theta)$$



So we can reduce the number of variables required in the equations of motion by such substitutions.

MECHANICAL BASICS

Kinetic Energy – the energy possessed by a body due to its motion.

$K_L = \frac{1}{2} m.v^2$ linear kinetic energy

$K_A = \frac{1}{2} I.\omega^2$ rotational kinetic energy

Potential Energy – the energy posed by a body due to its position.

$$P = m.g.h$$

[For a conservative system $K_L + K_A + P =$ constant, so if potential energy increases then kinetic energy must decrease, and vice versa.]

MECHANICAL BASICS

Centripetal Force – a force causing a body to deviate from its motion in a straight line to motion along a curved path. For motion about a radius (r) the centripetal force acts towards the center of the circle

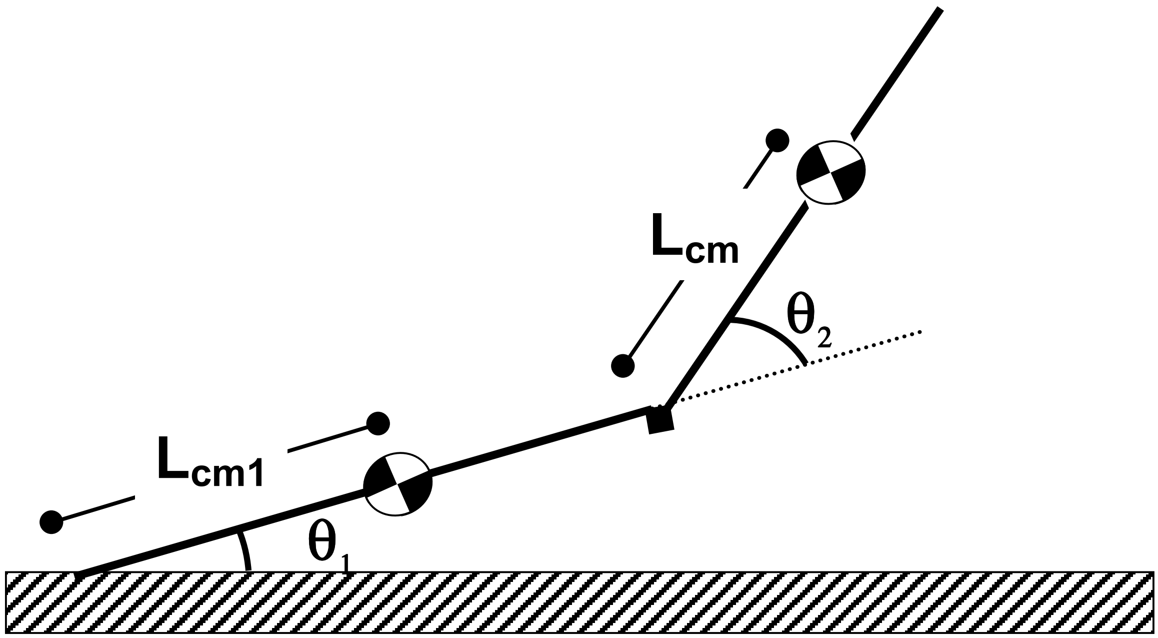
$$F_{Cent} = \frac{m.v^2}{r} = m.\omega^2.r = m.\dot{\theta}^2.r$$

[Centrifugal force – an inertial force acting in opposite direction to centripetal force.]

Coriolis Force – (after G.G de Coriolis, 1792-1843) is a force acting on a body perpendicular to the axis of rotation and the direction of motion of the body.

$$F_{Cor} = 2.m.v.\omega = 2.m.v.\dot{\theta}$$

EXAMPLE II - TWO RIGID BODIES



Where

θ_i - angle of i^{th} segment

L_i - length of i^{th} segment

m_i - mass of ith segment

L_{cmi} - distance from joint center to center of mass for i^{th} segment

I_i - moment of inertia of i^{th} segment

EXAMPLE II - TWO RIGID BODIES

$$T = M(\theta).\ddot{\theta} + v(\theta,\dot{\theta}) + G(\theta)$$

$$M(\theta)_{1,1} = m_1.L_{cm1}^2 + I_1 + m_2\left[L_1^2 + L_{cm2}^2 + 2.L_1.L_{cm2}.\cos(\theta_2)\right] + I_2$$

$$M(\theta)_{1,2} = m_2.L_1.L_{cm2}.\cos(\theta_2) + m_2.L_{cm2}^2 + I_2$$

$$M(\theta)_{2,1} = m_2.L_1.L_{cm2}.\cos(\theta_2) + m_2.L_{cm2}^2 + I_2$$

$$M(\theta)_{2,2} = m_2.L^2{}_{cm2} + I_2$$

$$v(\dot{\theta})_{1,1} = -m_2.L_1.L_{cm2}.\sin(\theta_2).\left(2.\dot{\theta}_1.\dot{\theta}_2 + \dot{\theta}_2^2\right)$$

$$v(\dot{\theta})_{1,2} = m_2.L_1.L_{cm2}.\sin(\theta_2).\dot{\theta}_1^2$$

$$G(\theta)_{1,1} = m_1.L_{cm1}.g.\cos(\theta_1) + m_2.g.\left[L_{cm2}.\cos(\theta_1 + \theta_2) + L_1.\cos(\theta_1)\right]$$

$$G(\theta)_{1,2} = m_2.g.L_{cm2}.\cos(\theta_1 + \theta_2)$$

EXAMPLE II - TWO RIGID BODIES

- For each link there is a second order non-linear differential equation describing the relationship between the moments and angular motion of the two link system.

- Terms from adjacent links occur in the equations for a link – the equations are coupled. For example

$$G(\theta)_{1,1} = m_1.L_{cm1}.g.\cos(\theta_1) + m_2.g.[L_{cm2}.\cos(\theta_1 + \theta_2) + L_1.\cos(\theta_1)]$$

- Group all the terms including moment of inertia – gives the inertia matrix.

- The inertia matrix is symmetric and positive definite. Matrices having these properties are normally invertible.

- Group all terms involving angular velocity, gives terms dealing with centripetal and Coriolis forces

- Group all terms relating to gravity – gives vector of gravity terms.

GENERAL FORMAT OF EQUATIONS OF MOTION

Equation for **Inverse Dynamics**

$$T = M(\theta).\ddot{\theta} + v(\theta,\dot{\theta}) + G(\theta)$$

Where

T - vector of joint moments (n x 1)
$M(\theta)$ - inertia matrix (n x n)
$v(\theta,\dot{\theta})$ - vector of centrifugal/Coriolis terms (n x 1)
$G(\theta)$ - vector of gravity terms
and n is the number of joints in the system

Re-arrange equation for **Direct Dynamics**

$$\ddot{\theta} = M(\theta)^{-1}.(T - v(\theta,\dot{\theta}) - G(\theta))$$

n = 1 no matrix inversion required, simple to perform

n > 1 then you require an inversion of the mass matrix

MATRIX INVERSION

A matrix is just a rectangular array of numbers or symbols organized into rows and columns, for example

$$[A] = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \qquad [A] = \begin{bmatrix} 0.25 & 0.33 \\ 0.125 & 0.666 \end{bmatrix}$$

Given the equation,

$$F = m.a$$

if we know the force and the acceleration we can work out the mass from

$$m = \frac{F}{a}$$

For matrices division does not exist so we use the inverse

$$m = a^{-1}.F$$

Properties of an Inverse Matrix

If $[A]$ is a matrix with an equal number of rows and columns then

$$[A]^{-1}.[A] = [A].[A]^{-1} = [I]$$

Where $[I]$ is the identity matrix (matrix equivalent of 1)

MATRIX INVERSION
ANALYTICAL SOLUTIONS

For certain matrices the inverse can be computed analytically, for example

$$[A] = \begin{bmatrix} cos(\theta) & -\ sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

$$[A]^{-1} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}$$

$$[A]^{-1}.[A] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Such calculations can be done by hand (tricky) or via a symbolic algebraic manipulation package (e.g. MAPLE).

MATRIX INVERSION
NUMERICAL SOLUTIONS

If an analytical solution does not exist then there are various numerical techniques. For example

$$[A] = \begin{bmatrix} 0.25 & 0.33 \\ 0.125 & 0.666 \end{bmatrix}$$

$$[A]^{-1} = \begin{bmatrix} 5.3174 & -2.6347 \\ -0.9980 & 1.9960 \end{bmatrix}$$

$$[A]^{-1}.[A] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I]$$

Question: What are the numerical techniques for numerically inverting a matrix?

MATRIX INVERSION
NUMERICAL SOLUTIONS

There are a variety of techniques for the numerical inversion of a matrix, including

- Gaussian Elimination (Gauss-Seidel, LU factorization)
- Cholesky Factorization (symmetric definite matrices)
- Singular Value Decomposition

Numerically more robust routines are more time consuming, numerically less robust may consider a matrix singular when this is not the case.

MATRIX INVERSION
NUMERICAL SOLUTIONS

Algorithms to invert a matrix are readily available the trade-off must be made between numerical robustness and time for computation.

In MATLAB for example to solve,

$$[A].[x] = [B]$$

where $[x]$ is unknown, you can write

>x=A\B

This perform the following matrix operation

$$[x] = [A]^{-1}.[B]$$

Sparse Matrices – these arise in certain applications where a large number of the matrix elements are zero. Special algorithms are available to solve these problems.

MATRIX INVERSION
SINGULAR SYSTEMS

If a matrix is not invertible it is said to be **singular** (it exists on its own).

Most packages will report that the matrix is singular by stating either

- the matrix is singular
- the determinant of the matrix is equal to zero

Determinant – this is a value associated with every square matrix, for example for a 2 x 2 matrix it is computed thus

$$det[A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a.d - b.c$$

If the determinant of a matrix is zero it is not invertable.

Singular Systems arise when

- the equations representing the rows in a matrix are closely inter-related
- the data in the matrix contains significant errors which makes it seem as if the rows in the matrix are closely inter-related.

MECHANICAL APPROACHES - SUMMARY

- Silver (1982) has shown that equations of the same form can be derived using Lagrange or Newton-Euler methods if constraints are imposed when using the Newton-Euler approach.

- Theoretically the same equivalence can be shown between equations derived from other formulations (e.g. Kane's method).

- Given the equivalence of formulations what becomes important is how easily the equations of motion can be formed if the researcher wants to examine a variety of different mechanical systems, and ease of computer implementation.

- Ease of formulation often depends on selection of appropriate generalized coordinates.

REVIEW QUESTIONS

1. Write the equations which describe the Newton-Euler approach to formulation of the equations of motion of a mechanical system?

2. Write the equations which describe the Lagrangian approach to formulation of the equations of motion of a mechanical system?

3. What is meant by the term mechanical degrees of freedom?

4. What is meant by the term generalized coordinates?

5. Describe the relationship between kinetic and potential energy for a conservative system.

6. Write the equation for direct dynamics in matrix and vector form.

7. List techniques which could be used to invert a numerical matrix.

8. What does it mean if a matrix is singular?