EQUATIONS OF MOTION I: BASICS

Overview

- Equations of Motion
- Approaches
- Classification of Mechanical Systems
- Selection Criteria
- Example I – Newton-Euler
- Example II – Lagrangian
- Example III – Conservation of Energy
- Example IV – Conservation of Angular Momentum
- Computer Packages

"…numbers and mathematics are the language of nature."
Galileo

EQUATIONS OF MOTION

Equations of Motion – set of mathematical equations which describe the forces and movements of a body.

Inverse Dynamics – starting from the motion of the body determines the forces and moments causing the motion.

Direct Dynamics – starting from the forces and moments acting on a body determines the motion arising from these forces and moments. (Also called forward dynamics.)

APPROACHES

If external forces and moments vary most popular methods for formulating the equations are:-

- Newton-Euler (e.g. Marshall et al., 1985)
- Lagrangian (e.g. Alexander, 1991)
- Kanes Method (e.g. Pandy and Zajac, 1991)

Under certain conditions conservation laws can be invoked

- Angular momentum (e.g. Yeadon, 1991)
- Energy

CLASSIFICATION OF MECHANICAL SYSTEMS

Holonomic – if the constraints on the system are a function of the coordinates describing the position and orientation of the system and time, then the system is said to be holonomic.

Conservative – if all the forces acting on a system are derivable from a potential function the system is said to be conservative.

SELECTION CRITERIA

- Task dependent
- What you know
- Computation efficiency
- Ease of implementation

Newton-Euler - generally considered most intuitive.

Kanes Method - generally considered to produce most efficient computer code.

Minimum Set of Equations - minimum number of degrees of freedom, equations are highly coupled and complicated.

Maximum Set of Equations - large number of equations including constraint equations. [*Matrices are often sparse producing increased computation demands.*]

EXAMPLE I - NEWTON-EULER

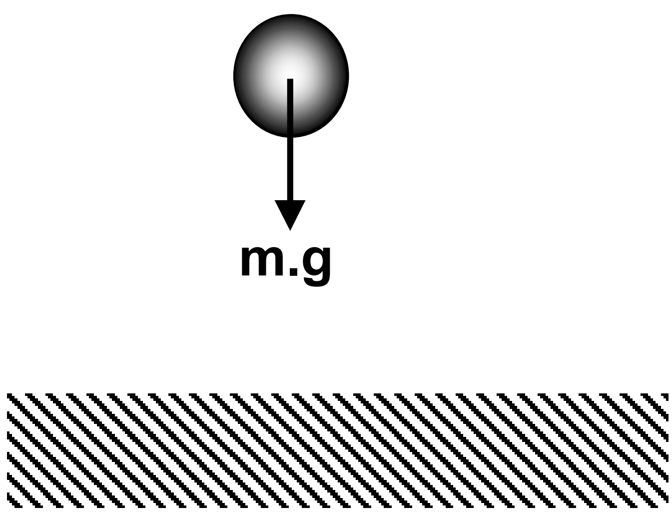
Principle: use Newtonian mechanics and Eulers equation.

Base Equations:

Newton $\sum F = m.a(t)$

Euler $\sum T = I.\alpha = I.\ddot{\theta}$ (2D case only)

Example: A ball is falling under the influence of gravity.



EXAMPLE I - NEWTON-EULER

Assumptions

- Ignore air resistance.
- No rotation is occurring.
- Look at aerial phase only

Initial conditions

$y(0) = 0$, $\dot{y}(0) = v_0$

Equations of Motion

$$\sum F = m.\ddot{y}(t) = m.g = \ddot{y}(t) = g$$

Integrate once

$$\dot{y}(t) = g.t + constant$$

substitute initial condition

$$\dot{y}(t) = v_0 + g.t = (v = u + a.t)$$

Integrate again

$$y(t) = v_0.t + \frac{1}{2}.g.t^2 \Rightarrow \left(s = u.t + \frac{1}{2}.a.t^2\right)$$

(remember initial condition was $y(0) = 0$.)

EXAMPLE II - LAGRANGIAN

Principle: uses the Lagrangian and the Lagrange's equation of motion.

Base Equations:

Lagrangian Equation $L = K - P$

Equation of Motion $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$ (Conservative)

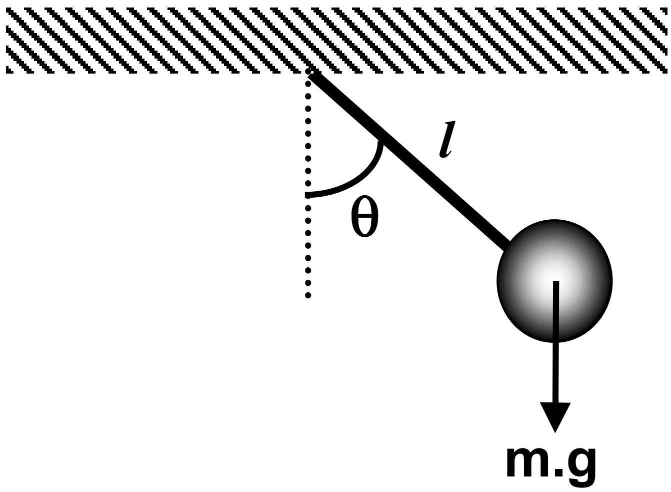
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = T$$ (Non-conservative)

Where

- K - kinetic energy
- P - potential energy
- T - moments/forces moments being applied to the system
- q - generalized coordinates (e.g. joint angles)

EXAMPLE II - LAGRANGIAN

Example: A simple pendulum.



Assumptions

- Ignore air resistance.
- No other resistive forces acting, other than gravity.
- String/wire is mass less

Initial conditions

$\theta(0)=\theta_0,\ \dot{\theta}(0)=0$

Useful Equations

$v = l.\dot{\theta}$

The height of the mass above lowest position is equal to

$H = l - l.\cos\theta$

EXAMPLE II - LAGRANGIAN

Form Lagrangian Equation

Kinetic Energy

$$K = \frac{1}{2}.m.v^2 = \frac{1}{2}m.(l.\dot{\theta})^2 = \frac{1}{2}.m.l^2.\theta^2$$

Potential Energy

$$P = m.g.h = m.g.(l - l.\cos\theta) = m.g.l(1 - \cos\theta)$$

Lagrangian Equation.

$$L = K - P = \frac{1}{2}.m.l^2.\dot{\theta}^2 - m.g.l.(1 - \cos\theta)$$

Lagrange's Equation of Motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

$$\frac{\partial L}{\partial q} = -m.g.l.\sin\theta$$

$$\frac{\partial L}{\partial \dot{q}} = m.l^2.\dot{\theta} \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = m.l^2.\ddot{\theta}$$

Equation of Motion

$$m.l^2.\ddot{\theta} - m.g.l.\sin\theta = 0$$

$$\ddot{\theta} - \frac{g}{l}.\sin\theta = 0$$

EXAMPLE III - CONSERVATION OF ENERGY

Example: A simple pendulum.

Assumptions

- Ignore air resistance.
- No other resistive forces acting other than gravity.
- String/wire is mass less.

Initial conditions

$\theta(0) = \theta_0,\ \dot{\theta}(0) = 0$

Equations of Motion

$$K + P = \frac{1}{2}.m.l^2.\dot{\theta}^2 + m.g.l.(1 - \cos\theta) = constant$$

Differentiate both sides with respect to t

$$m.l^2.\dot{\theta}.\ddot{\theta} - m.g.l.\sin\theta.\dot{\theta} = 0$$

$$\theta\ddot{\theta} - \frac{g}{l}.\sin\theta = 0$$

EXAMPLE IV - CONSERVATION OF ANGULAR MOMENTUM

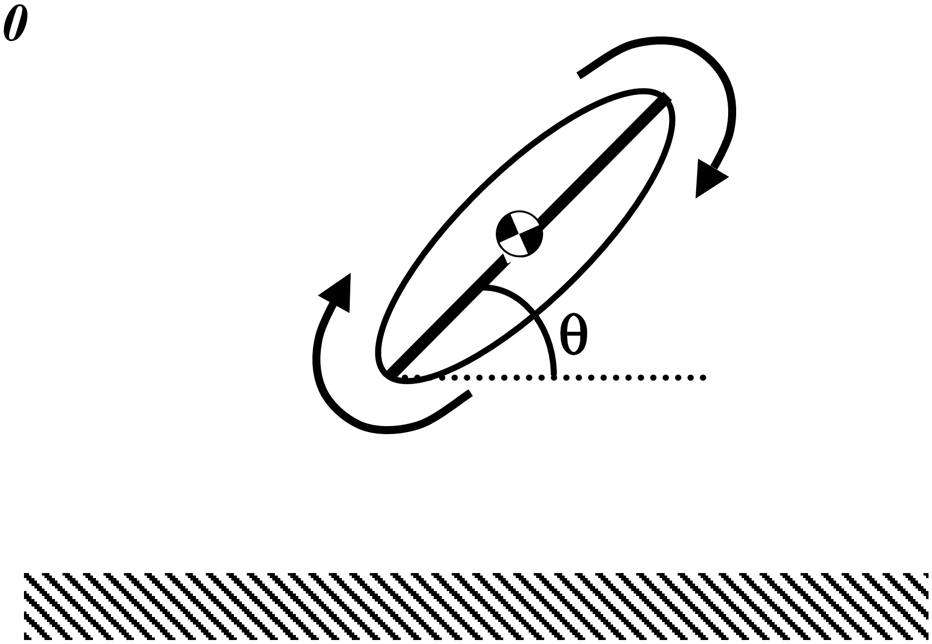
Example: A somersaulting gymnast.

Assumptions

- Ignore air resistance.
- No other resistive forces acting, other than gravity.
- Body is a single rigid body (cylinder)

Initial conditions

$\theta(0) = 0$



EXAMPLE IV - CONSERVATION OF ANGULAR MOMENTUM

Equations of Motion

$$H = I.\dot{\theta} = Constant$$

Where

H - angular momentum about transverse axis through center of mass

I - moment of inertia about transverse axis

Differentiate equation with respect to t

$$H.t = I.\theta \Rightarrow \theta = \frac{H.t}{I}$$

[*This weeks reading will give you more insight into how this principle can be used.]*

COMPUTER PACKAGES

Newton-Euler	Lagrangian	Kanes Method
NEWEUL	***ADAMS***	***AUTOLEV***
MULTI-BODY	***DADS***	***SD/FAST***

(Automatic Dynamic Analysis of Mechanical Systems)

(Dynamic Analysis and Design System)

Also consider

MAPLE
MATHEMATICA

COMPUTER PACKAGES

Differences between packages:-

- Mechanical principles employed
- Non-holonomic constraints implemented
- How body orientations are defined
- Options on how segments are connected
- Are closed loops permitted?
- Level of knowledge required for implementation
- How much it does for you

No package offers everything (Schiehlen, 1991, Richard and Gosselin, 1993).

REVIEW QUESTIONS

1) List different methods via which you could determine the equations of motion for a system.

2) Define a holonomic system. Define a conservative system.

3) What are the base equations from which the equations of motion can be determine using the Newton-Euler approach?

4) What are the base equations from which the equations of motion can be determined us Lagrangian method?

5) List five criteria via which you may compare computer packages for the automatic generation of the equations of motion for a system of rigid bodies.