Unimodality of the Distribution of Record Statistics

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Abstract

Let $X_1, X_2, \cdots,$ be a sequence of independent and identically distributed random variables with absolutely continuous distribution function F. For $n \geq 1$, we denote the order statistics of X_1, X_2, $\cdots$, X_n by $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$. Define

$L(1) = 1, L(n+1) = \min\{j : j > L(n), X_j > X_{j-1,j-1}\}$, and

$X(n) = X_{L(n),L(n)}, n \geq 1$.

The sequence $\{X(n)\}$ ($\{L(n)\}$) is called upper record statistics (times). In this article, we deal with unimodality of record statistics. We also deal with the strong unimodality of record statistics.

Key Words: Record statistics.

1 Introduction

Let $X_1, X_2, \cdots$ be independent and identically distributed random variables with absolutely continuous cumulative distribution function F and corresponding probability density function (pdf) f. For $n \geq 1$, we denote the order statistics of X_1, X_2, $\cdots$, X_n by $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$. Define

$$\left.\begin{array}{l} L(1) = 1, L(n+1) = \min\{j : j > L(n), X_j > X_{j-1,j-1}\}, \text{ and} \\ X(n) = X_{L(n),L(n)}, n \geq 1. \end{array}\right\} \quad (1.1)$$

The sequence $\{X(n)\}$ ($\{L(n)\}$) is called upper record statistics (times). Good references for record values are Ahsanullah [1] and Arnold et al. [3]. Alam [2] studied unimodality of order statistics, and Huang and Ghosh [4] studied strong unimodality of order statistics. In this article, we deal with unimodality and strong unimodality of record statistics.

Throughout the article, we will use the following notations.

$$\begin{array}{rcl} S(x) & = & 1 - F(x), \text{the survival function of } X, \\ f(x) & = & F'(x) = -S'(x), \text{the pdf of } X, \\ R(x) & = & -\ln S(x), \text{the cumulative hazard rate of } X, \\ r(x) & = & R'(x) = f(x)/S(x), \text{the hazard rate of } X. \end{array}$$

2 Unimodality of Records

A random variable X or its distribution function $F(x)$, on the real line, is called unimodal if there exists a value $x = a$, such that, $F(x)$ is convex for $x < a$ and concave for $x > a$. Let $n \geq 1$ be a positive integer. In this section, we look at the unimodality of record statistics, $X(n)$, given that $F(x)$ is unimodal. We have the following theorem.

Theorem 1 *Suppose the pdf f of F is such that $\frac{1}{f}$ is convex. Then each record statistic $X(n), n \geq 1$ has a unimodal distribution.*

Proof. The density function $f_n(x)$ of $X(n)$ is given by

$$f_n(x) = \frac{R^{n-1}(x)}{\Gamma(n)} f(x). \tag{2.1}$$

So,

$$\begin{aligned} f_n'(x) &= \frac{1}{\Gamma(n)} \left[(n-1)R^{n-2}(x)r(x)f(x) + R^{n-1}(x)f'(x)\right] \\ &= \frac{R^{n-1}(x)f(x)r(x)}{\Gamma(n)} \left[\frac{(n-1)}{R(x)} + \frac{f'(x)}{f(x)r(x)}\right] \\ &= \frac{R^{n-1}(x)f(x)r(x)}{\Gamma(n)} \left[\frac{(n-1)}{R(x)} + \frac{f'(x)}{f^2(x)} S(x)\right]. \end{aligned} \tag{2.2}$$

By hypothesis, $\frac{f'(x)}{f^2(x)}$ is non-increasing. Also, $S(x)$ is non-increasing and $R(x)$ is non-decreasing, so the quantity inside the brackets on the right-hand side of equation (2.2) is non-increasing in x. The quantity outside the brackets on the right-hand side is positive. Thus we see that $f_n'(x)$ changes sign at most once as x varies from $-\infty$ to ∞, and any change of sign must be from positive to negative. Thus, $f_n(x)$ is unimodal.

Remark: Similar result holds for lower record statistics, which is obtained by switching the inequality in equation (1.1).

The next theorem deals with relaxed condition on F for unimodality of $X(n)$.

Theorem 2 *Let the distribution function F be unimodal with mode at a, and the probability density function f. Let f be continuous at a, and $\frac{1}{f}$ be convex in $x > a$. Then $X(n)$, $n \geq 1$ is unimodal.*

Proof. The pdf of $X(n)$ is given by $f_n(x) = \frac{R^{n-1}(x)}{\Gamma(n)} f(x)$. Clearly, $f_n(x)$ is non-decreasing for $x \leq a$. Using the same argument as in the previous theorem, in $x > a$, the condition that $\frac{1}{f}$ is convex implies that $f_n(x)$ is either decreasing or it is first increasing then decreasing. Therefore, $X(n)$ is unimodal.

Remark: Similar result holds good for lower record statistics, when we consider the convexity of $\frac{1}{f}$ in $x < a$.

The next theorem deals with strong unimodality of record statistics $X(n)$.

Theorem 3 *Suppose the pdf f and the hazard function $R(x)$ are both logconcanve. Then each record statistics $X(n)$, $n \geq 1$ has a logconcave density, i.e., $X(n), n \geq 1$ is strongly unimodal.*

Proof. The density function $f_n(x)$ of $X(n)$ is given by equation(2.1). So, the result follows immediately by looking at the logarithm of the pdf of $f_n(x)$, which is given by

$$\ln f_n(x) = (n-1)\ln R(x) + \ln f(x) - \ln \Gamma(n). \tag{2.3}$$

Remarks: 1. The logconcavity of f means the order statistics are strongly unimodal (see Huang and Ghosh [4]), but logconcavity of both the f and R are necessary for the record statistics to be stongly unimodal for all values of $n \geq 1$. There are number of cases of standard statistical distributions, where logconcavity of f is not enough – Beta, Triangular and Uniform distributions have logconcave density but R is not logconcave for any of these distributions. So, the record statistics could be strongly unimodal (i.e., have logconcave density) or fail to be strongly unimodal depending on the value of n in the equation (2.3) (except in the case of uniform, where records are never strongly unimodal). On the other hand, there are distributions, where the R is logconcave but f is not – F, Lognormal, Pareto distributions. So, the same conclusions as above can be made about record statistics. Of course, if f is logconcave then the record statistics will be unimodal because logconcavity of f implies convexity of $1/f$, and theorem 1 would give the result.

It can easily be seen that if $X(n)$ is strongly unimodal, then $X_{r,n}, 1 \leq r \leq n$, will be strongly unimodal but not vice versa.

Next we give an example, which shows that like order statistics, unimodality of parent F is not sufficient to give unimodality for record statistics.

Example. Consider the pdf

$$f(x) = \begin{cases} 1 & \text{if} \quad -\frac{1}{2} < x < 0 \\ \frac{1}{2} & \text{if} \quad 0 \leq x < 1 \end{cases}$$

with the cumulative distribution function

$$F(x) = \begin{cases} \frac{1}{2} + x & \text{if} \quad -\frac{1}{2} < x < 0 \\ \frac{1}{2} + \frac{x}{2} & \text{if} \quad 0 \leq x < 1 \\ 1 & \text{if} \quad x \geq 1. \end{cases}$$

$f(x)$ is unimodal. $R(x)$ is given by

$$R(x) = \begin{cases} -\ln(\frac{1}{2} - x) & \text{if} \quad -1/2 < x < 0 \\ -\ln(\frac{1}{2} - \frac{x}{2}) & \text{if} \quad 0 \leq x < 1. \end{cases}$$

Since F is strictly increasing on $[-1/2, 1]$, R is strictly increasing on $[-1/2, 1]$ and f has a downward jump at 0, we see that f_n has two local modes at 0 and at 1, implying f_n is not unimodal. Thus unimodality of F is not sufficient to give unimodality for record values.

References

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Logconcavity of f(x), R(x), and nth Record

Distribution	f logconcave	R logconcave	Rec Unimodal	Rec logconcave
Chisquare	Yes	Yes	Yes	Yes
Exponential	Yes	Yes	Yes	Yes
Extremevalue	Yes	Yes	Yes	Yes
Gamma	Yes	Yes	Yes	Yes
Normal	Yes	Yes	Yes	Yes
Laplace	Yes	Yes	Yes	Yes
Logistic	Yes	Yes	Yes	Yes
Rayleigh	Yes	Yes	Yes	Yes
Weibull	Yes	Yes	Yes	Yes
Beta	Yes	No	Yes	Yes/No*
Triangular	Yes	No	Yes	Yes/No*
Uniform	Yes	No	Yes	No
FRatio	No	Yes	Yes	No/Yes*
Lognormal	No	Yes	Yes	No/Yes*
Pareto	No	Yes	Yes	No/Yes*
Cauchy	No	No	Yes	No
Student t	No	No	Yes	No
$x^2 + 11/12$	No	No	Yes	No
$x^2 + 1/6$	No	No	Yes	No
$1/\sqrt{x}$	No	No	Yes	No

*With higher values of n.