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Estimation for the three-parameter inverse Gaussian distribution under progressive Type-II censoring

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Inverse Gaussian distribution has been used widely as a model in analysing lifetime data. In this regard, estimation of parameters of two-parameter (IG2) and three-parameter inverse Gaussian (IG3) distributions based on complete and censored samples has been discussed in the literature. In this paper, we develop estimation methods based on progressively Type-II censored samples from IG3 distribution. In particular, we use the EM-algorithm, as well as some other numerical methods for determining the maximum-likelihood estimates (MLEs) of the parameters. The asymptotic variances and covariances of the MLEs from the EM-algorithm are derived by using the missing information principle. We also consider some simplified alternative estimators. The inferential methods developed are then illustrated with some numerical examples. We also discuss the interval estimation of the parameters based on the large-sample theory and examine the true coverage probabilities of these confidence intervals in case of small samples by means of Monte Carlo simulations.

Keywords: progressive Type-II censoring; EM-algorithm; maximum-likelihood estimates; asymptotic variances; Newton–Raphson algorithm; confidence intervals; coverage probabilities

1. Introduction

In lifetime data, when we study the reliability, the choice of distribution is based on the failure mechanism. Inverse Gaussian distribution is one of the distributions commonly used for modelling lifetimes or reaction-times, and is especially useful for modelling data that are long-tailed and are positively skewed. A number of authors have discussed the inverse Gaussian distribution and inferential methods for it; see, for example, [1–18]. Patel [19] discussed the maximum-likelihood estimates (MLEs) of the parameters of the IG2 distribution based on progressively Type-I censored data. Interested readers may refer to the books by Johnson \textit{et al.} [20] and Seshadri [21] for further details.
The probability density function (pdf) of IG3 distribution, with parameters $-\infty < \gamma < \infty$, $\mu > 0$ and $\lambda > 0$, is given by

$$f(x; \gamma, \mu, \lambda) = \left[ \frac{\lambda}{2\pi(x-\gamma)^3} \right]^{1/2} \exp \left\{ -\frac{\lambda(x-\gamma-\mu)^2}{2\mu^2(x-\gamma)} \right\}, \quad \gamma < x < \infty. \tag{1}$$

In Equation (1), $\gamma$ is a threshold parameter, $\lambda$ is a scale parameter, and $(\gamma + \mu)$ is the mean. It should be noted here that the cumulative distribution function (cdf) of IG3 random variable can be written as a simple function in terms of normal probability integrals as [8,20]

$$F(x; \gamma, \mu, \lambda) = \Phi \left[ \sqrt{\beta} \left( \frac{x-\gamma}{\beta} - \alpha \right) \right] + e^{2\alpha/\mu} \Phi \left[ -\sqrt{\beta} \left( \frac{x-\gamma}{\beta} + \alpha \right) \right], \tag{2}$$

where $\Phi$ denotes the cdf of a standard normal distribution. Letting $\mu = \beta\alpha$ and $\lambda = \beta\alpha^2$, we get a convenient reparametrization, denoted here by $\text{IG3}(\gamma, \beta, \alpha)$, where $-\infty < \gamma < \infty$, $\beta > 0$ and $\alpha > 0$ are location, scale and shape parameters, respectively. In this form, if $X \sim \text{IG3}(\gamma, \beta, \alpha)$ with pdf

$$f(x; \gamma, \beta, \alpha) = \left[ \frac{\beta\alpha^2}{2\pi(x-\gamma)^3} \right]^{1/2} \exp \left\{ -\frac{(x-\gamma-\beta\alpha)^2}{2\beta(x-\gamma)} \right\}, \quad \gamma < x < \infty, \tag{3}$$

then $(X - \gamma)/\beta \sim \text{IG3}(0, 1, \alpha)$. Then, Equation (2) yields

$$F(x; \gamma, \beta, \alpha) = \Phi \left[ \sqrt{\beta} \left( \frac{x-\gamma}{\beta} - \alpha \right) \right] + e^{2\alpha/\beta} \Phi \left[ -\sqrt{\beta} \left( \frac{x-\gamma}{\beta} + \alpha \right) \right]. \tag{4}$$

The moment-generating function of $\text{IG3}(\gamma, \beta, \alpha)$ is given by

$$E[e^{tX}] = \exp[\gamma t + \alpha(1 - \sqrt{1 - 2\beta t})] \quad \text{for } t < \frac{1}{2\beta}. \tag{5}$$

Equation (5) gives, respectively, the mean, variance and the coefficients of skewness and kurtosis to be

$$\gamma + \beta\alpha, \quad \beta^2\alpha, \quad \frac{3}{\sqrt{\alpha}} \quad \text{and} \quad 3 + \frac{15}{\alpha}.$$
of the \( n - 1 \) surviving units are randomly withdrawn (censored) from the experiment, \( R_2 \) of the \( n - 2 - R_1 \) surviving units are censored at the time of the second failure (the second stage), and so on. Finally, at the time of the \( m \)th failure (the \( m \)th stage), all the remaining \( R_m = n - m - R_1 - \cdots - R_{m-1} \) surviving units are censored. This scheme is referred to as the progressive Type-II right censoring scheme \((R_1, R_2, \ldots, R_m)\). It is clear that this scheme includes the conventional Type-II right censoring scheme when \( R_1 = R_2 = \cdots = R_m = 0 \) and \( R_n = n - m \), and the complete sampling scheme when \( n = m \) and \( R_1 = R_2 = \cdots = R_m = 0 \). The ordered lifetime data which arise from such a progressive Type-II right censoring scheme are called progressively Type-II right censored order statistics. For theory, methods and applications of progressive censoring, readers are referred to the book by Balakrishnan and Aggarwala [23] and the recent survey article by Balakrishnan [24].

Suppose \( n \) independent units with IG3 distributed lifetimes are placed on a life-testing experiment. Let \( Y_{1:m:n} \leq \cdots \leq Y_{m:n:m} \) denote the \( m \) progressively Type-II right censored order statistics observed from the above-described experimental scheme. For brevity, we will use \( Y_j (j = 1, \ldots, m) \) to denote these \( Y_{j:m:n} \)'s. Note that we observe only \( Y = (Y_1, \ldots, Y_m) \). In this article, we discuss different estimation procedures for the parameter \( \theta = (\gamma, \beta, \alpha) \) based on progressively Type-II right censored order statistics \( Y \). It should be mentioned that some work has been done in the past on estimation methods for the three-parameter log-normal and some other lifetime distributions based on complete and censored samples; see, for example, the books by Cohen and Whitten [25], Cohen [17] and Balakrishnan and Cohen [16].

The rest of this paper is organized as follows. In Section 2, we provide the Newton–Raphson algorithm for determining the MLEs of the parameter \( \theta \) based on a progressively Type-II censored sample. The second derivatives of the log-likelihood are required in order to use the algorithm, and these are quite complicated when data are progressively Type-II censored. Another viable alternative to the Newton–Raphson algorithm is the well-known EM-algorithm and in Section 3 we discuss how it can be used to determine the MLEs in this case. Asymptotic variances and covariances of the MLEs determined by the EM-algorithm are given in Section 4. Section 5 describes some simplified estimation methods which yield simple alternative estimators. All these methods of estimation are then illustrated with some numerical examples in Section 6. Using these examples, we also discuss the interval estimation of parameters based on \( c \) and then examine the actual coverage probabilities in case of small samples by means of Monte Carlo simulations.

2. Newton–Raphson algorithm

One of the common numerical methods used to determine MLEs is the Newton–Raphson algorithm. In this section, we describe the Newton–Raphson algorithm for finding the MLEs numerically when lifetimes are distributed as IG3 with parameter \( \theta \). These MLEs are local MLEs corresponding to the Newton–Raphson algorithm, and would be denoted here by LMLE1.

For IG3 distribution in Equation (3), the log-likelihood function \( l(\theta) = \log L(\theta) \) based on the progressively Type-II right censored order statistics \( Y \) is given by

\[
l(\theta) = \log L(\theta) = \log \left[ \text{const.} \prod_{i=1}^{m} f(y_i)(1 - F(y_i))^{R_i} \right]
\]

\[
= \text{const.} + \frac{m}{2} \log \beta + m \log \alpha - \frac{3}{2} \sum_{j=1}^{m} \log(y_j - \gamma) - \frac{1}{2\beta} \sum_{j=1}^{m} (y_j - \gamma)
\]
+ mα - \frac{βα²}{2} \sum_{j=1}^{m} \frac{1}{y_j - γ} + \sum_{j=1}^{m} R_j \log[1 - \Phi(ψ_{1j}) - e^{2α} \Phi(ψ_{2j})],

where \( ψ_{1j} = \sqrt{β/(y_j - γ)} ((y_j - γ)/β - α) \), \( ψ_{2j} = -\sqrt{β/(y_j - γ)} ((y_j - γ)/β + α) \), and \( F(x; γ, β, α) \) is as given in Equation (4). The likelihood equations that need to be solved simultaneously for the required estimate \( \hat{θ} = (\hat{γ}, \hat{β}, \hat{α}) \) are as follows:

\[
\frac{∂l}{∂γ} = \frac{m}{2β} + \sum_{j=1}^{m} \frac{y_j - γ}{2β²} - \sum_{j=1}^{m} \frac{α²}{2(y_j - γ)} + \sum_{j=1}^{m} \frac{y_j - γ}{β} R_j L_{1j} = 0
\]

\[
\frac{∂l}{∂β} = \frac{m}{2β} + \sum_{j=1}^{m} \frac{y_j - γ}{2β²} - \sum_{j=1}^{m} \frac{α²}{2(y_j - γ)} + \sum_{j=1}^{m} \frac{y_j - γ}{β} R_j L_{1j} = 0
\]

\[
\frac{∂l}{∂α} = \frac{m}{α} + m - \sum_{j=1}^{m} \frac{βα}{y_j - γ} + \sum_{j=1}^{m} 2R_j \left( \frac{y_j - γ}{α} L_{1j} - e^{2α} M_{2j} \right) = 0
\]

where

\[
L_{1j} = \frac{f(y_j; γ, β, α)}{1 - F(y_j; γ, β, α)} = \sqrt{\frac{βα²}{(y_j - γ)³}} \frac{φ(ψ_{1j})}{1 - \Phi(ψ_{1j}) - e^{2α} \Phi(ψ_{2j})},
\]

\[
M_{2j} = \frac{φ(ψ_{2j})}{1 - F(y_j; γ, β, α)} = \frac{φ(ψ_{2j})}{1 - \Phi(ψ_{1j}) - e^{2α} \Phi(ψ_{2j})}.
\]

In the Newton–Raphson algorithm, the simultaneous solution is obtained through an iterative procedure. In each iterative step, the corrections \( a, b, c \) to the previous estimates \( γ_0, β_0, α_0 \) produce new estimates \( \hat{γ}, \hat{β}, \hat{α} \) as

\[
\hat{γ} = γ_0 + a, \quad \hat{β} = β_0 + b \quad \text{and} \quad \hat{α} = α_0 + c.
\]

The iteration method is based on Taylor series expansions of the estimating equations in (6) in the neighbourhood of the previous simultaneous estimates. Neglecting powers of \( a, b \) and \( c \) above the first-order and using Taylor’s theorem, we get the following equations which need to be solved for \( a, b, c \) (with all the partial derivatives evaluated at \( γ = γ_0, β = β_0, α = α_0 \)):

\[
\frac{a}{∂γ} \left( \frac{∂² \log L}{∂γ²} + b \frac{∂² \log L}{∂γ∂β} + c \frac{∂² \log L}{∂γ∂α} \right) = -\frac{∂ \log L}{∂γ} \]

\[
\frac{a}{∂β} \left( \frac{∂² \log L}{∂β²} + b \frac{∂² \log L}{∂β∂γ} + c \frac{∂² \log L}{∂β∂α} \right) = -\frac{∂ \log L}{∂β} \]

\[
\frac{a}{∂α} \left( \frac{∂² \log L}{∂α²} + b \frac{∂² \log L}{∂α∂β} + c \frac{∂² \log L}{∂α∂γ} \right) = -\frac{∂ \log L}{∂α} \]
The required second derivatives in Equation (7) are as follows:

\[
\frac{\partial^2 l}{\partial \gamma^2} = \sum_{j=1}^{m} \frac{3}{2(y_j - \gamma)^2} - \sum_{j=1}^{m} \frac{\beta \alpha^2}{(y_j - \gamma)^3} + \sum_{j=1}^{m} R_j L_{1j} (\eta_{1j} - L_{1j}) \\
\frac{\partial^2 l}{\partial \beta^2} = -m \frac{\alpha^2}{2\beta^2} - \sum_{j=1}^{m} \frac{y_j - \gamma}{\beta^3} + \sum_{j=1}^{m} R_j L_{1j} (y_j - \gamma) \left( \frac{(y_j - \gamma)(n_{1j} - L_{1j}) - 2}{\beta^2} \right) \\
\frac{\partial^2 l}{\partial \alpha^2} = -m \frac{\beta}{\alpha^2} - \sum_{j=1}^{m} \frac{\beta}{y_j - \gamma} + \sum_{j=1}^{m} 2R_j \left[ \frac{y_j - \gamma}{\alpha} - \eta_{2j} L_{1j} - \frac{2(y_j - \gamma)^2}{\alpha^2} L_{1j} \right] \\
+ \frac{4(y_j - \gamma) e^{2\alpha} L_{1j} M_{2j} - (y_j - \gamma)^2 L_{1j}}{\alpha^2} + \frac{\beta e^{2\alpha}}{\sqrt{\beta(y_j - \gamma)}} L_{2j} - 2e^{4\alpha} M_{2j} \right], \\
\frac{\partial^2 l}{\partial \gamma \partial \beta} = -m \frac{\alpha^2}{2\beta^2} - \sum_{j=1}^{m} \frac{2(y_j - \gamma)^2}{\beta^3} + \sum_{j=1}^{m} R_j L_{1j} \left( \frac{(y_j - \gamma)(n_{1j} - L_{1j}) - 1}{\beta} \right) \\
\frac{\partial^2 l}{\partial \gamma \partial \alpha} = -m \frac{\beta \alpha}{\alpha} - \sum_{j=1}^{m} \frac{\beta}{y_j - \gamma} + \sum_{j=1}^{m} R_j L_{1j} \left[ \frac{\alpha \eta_{2j} - 2(y_j - \gamma) L_{1j} + 2\alpha e^{2\alpha} M_{2j}}{\alpha} \right] \\
\frac{\partial^2 l}{\partial \beta \partial \alpha} = -m \frac{\alpha}{y_j - \gamma} + \sum_{j=1}^{m} R_j L_{1j} (y_j - \gamma) \left[ \frac{\alpha \eta_{2j} - 2\beta L_{1j} + 2\beta \alpha e^{2\alpha} M_{2j}}{\beta \alpha} \right]
\]

where

\[
\eta_{1j} = \frac{1}{2\beta} + \frac{3}{2(y_j - \gamma)} - \frac{\beta \alpha^2}{2(y_j - \gamma)^2} \quad \text{and} \quad \eta_{2j} = 1 + \frac{1}{\alpha} - \frac{\beta \alpha}{y_j - \gamma}.
\]

Ng et al. [26] used a similar Newton–Raphson method for finding MLEs for lognormal and Weibull lifetime distributions based on progressively Type-II censored samples, but these cases are easier as they are two-parameter cases without the threshold parameter.

3. EM-algorithm

For progressively censored samples, the computation of second derivatives become quite complicated. So, it is useful to explore the use of the EM-algorithm for the numerical determination of the MLEs. The EM-algorithm, introduced by Dempster et al. [27], is a very popular tool to handle any missing or incomplete data situation; readers are referred to the book by McLachlan and Krishnan [28] for a detailed discussion on the EM-algorithm and its applications. The algorithm is an iterative method which has two steps: (i) in the E-step, any missing data are replaced by their expected values, and (ii) in the M-step, the log-likelihood function is maximized with the observed data and expected value of the incomplete data producing an update of the parameter estimates. The MLEs of the parameters are obtained by repeating the E- and M-steps until convergence occurs. Since the progressive censoring model can be viewed as a missing data problem, the EM-algorithm can be applied to obtain the MLEs of the parameters, as illustrated previously by Ng et al. [26].

Let us denote the censored data vector by \( Z = (Z_1, Z_2, \ldots, Z_m) \), where the \( j \)th stage censored data vector \( Z_j \) is a \( 1 \times R_j \) vector, \( Z_j = (Z_{j1}, Z_{j2}, \ldots, Z_{JR_j}) \) for \( j = 1, 2, \ldots, m \). The complete data set is then obtained by combining the observed data \( Y \) and the censored data \( Z \).
E-step of the algorithm requires the computation of the conditional expectation of functions of censored data $Z$, conditional on the observed data $Y$ and the current value of the parameters. Specifically, one computes the conditional expectation of the log-likelihood, $E[\ell(Y, Z, \theta)|Y = y]$, as

$$E[\ell(Y, Z, \theta)|Y = y] = \text{const.} - \frac{n}{2} \log \gamma - \frac{3}{2} \sum_{j=1}^{m} \log(y_j - \gamma)$$

$$- \frac{1}{2} \sum_{j=1}^{m} \frac{(y_j - \gamma - \beta \alpha)^2}{\beta(y_j - \gamma)} - \frac{3}{2} \sum_{j=1}^{m} \sum_{k=1}^{R_j} E[\log(Z_{jk} - \gamma) | Z_{jk} > y_j]$$

$$- \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{R_j} \left[ \frac{(Z_{jk} - \gamma - \beta \alpha)^2}{\beta(Z_{jk} - \gamma)} | Z_{jk} > y_j \right].$$

The above conditional expectations are obtained by using the fact that given $Y_j = y_j, Z_j$’s have a left-truncated distribution $F$, truncated at $y_j$. More specifically, the conditional probability density of $Z$, given $Y$, is given by (see [23])

$$f_{Z|Y}(z|\gamma; \theta) = \prod_{j=1}^{m} \prod_{k=1}^{R_j} f_{Z_{jk}|Y_j}(z_{jk}|y_j; \theta) = \prod_{j=1}^{m} \prod_{k=1}^{R_j} \frac{f(z_{jk}; \theta)}{1 - F(y_j; \theta)},$$

where $f(z_{jk}; \theta)$ is given by Equation (3) and $F$ denotes the corresponding cdf given by Equation (4).

In the M-step of the $(h + 1)$th iteration, we will denote the updated estimate of the parameter $\theta$ by $\theta_{(h+1)}$. This $\theta_{(h+1)}$ maximizes the log-likelihood function based on the observed data $Y$ and the conditional expectation of functions of censored data $Z$ given the observed data $Y$, and the $h$th iterate value of the parameter $\theta_{(h)}$. As a starting value $\theta_{(0)}$, one can use a $\gamma_{(0)} < y_1$ and $\beta_{(0)}$ and $\alpha_{(0)}$ computed on the basis of the so-called ‘pseudo-complete’ sample which involves observed $s Y$ and the censored observations at the $j$th step $Z_j$ all taken to be $y_j$. Thus, $\beta_{(0)} = \beta(\gamma_{(0)})$ and $\alpha_{(0)} = \alpha(\gamma_{(0)})$ are then given by

$$\beta_{(0)} = \frac{m_2'}{m_1' - \hat{\gamma}_{(0)}} = \left\{ \frac{1/n \sum_{j=1}^{m}(R_j + 1)(y_j - \hat{\gamma}_{(0)})^2 - \left[1/n \sum_{j=1}^{m}(R_j + 1)(y_j - \hat{\gamma}_{(0)})\right]^2}{1/n \sum_{j=1}^{m}(R_j + 1)(y_j - \hat{\gamma}_{(0)})} \right\}.$$  

$$\alpha_{(0)} = \frac{(m_1' - \hat{\gamma}_{(0)})^2}{m_2'} = \left\{ \frac{1/n \sum_{j=1}^{m}(R_j + 1)(y_j - \hat{\gamma}_{(0)})^2}{1/n \sum_{j=1}^{m}(R_j + 1)(y_j - \hat{\gamma}_{(0)})} \right\}.$$  

Starting with the initial estimates in Equation (10), the $(h + 1)$th iteration value $\theta_{(h+1)}$ is obtained, using $h$th iteration value $\theta_{(h)}$, as follows:

$$\beta_{(h+1)} = \frac{m_2}{m_1}$$

$$\alpha_{(h+1)} = \frac{m_1^2}{m_2}.$$  

(11)
where
\[
m_1 = \frac{1}{n} \left[ \sum_{j=1}^{m} (y_j - \gamma(h)) + \sum_{j=1}^{m} R_j E[(Z - \gamma(h))|Z > y_j; \theta(h)] \right].
\]
\[
m_2 = \frac{1}{n} \left[ \sum_{j=1}^{m} (y_j - \gamma(h))^2 + \sum_{j=1}^{m} R_j E[(Z - \gamma(h))^2|Z > y_j; \theta(h)] \right] - m_1^2,
\]
and \(\gamma(h+1)\) is obtained by solving the following equation for \(\gamma\):
\[
\frac{m}{2\beta(h+1)} + \sum_{j=1}^{m} \frac{3}{2(y_j - \gamma)^2} - \sum_{j=1}^{m} \frac{\beta(h+1)\alpha^2}{2(y_j - \gamma)^2} + \sum_{j=1}^{m} \frac{3R_j}{2} E\left[ \frac{1}{Z - \gamma} | Z > y_j; \gamma, \beta(h+1), \alpha(h+1) \right]
\]
\[
- \sum_{j=1}^{m} \frac{\beta(h+1)\alpha^2}{2} R_j E\left[ \frac{1}{(Z - \gamma)^2} | Z > y_j; \gamma, \beta(h+1), \alpha(h+1) \right] = 0.
\] (12)

Patel [29] obtained a recurrence relation for the moments of singly as well as doubly truncated inverse Gaussian distribution with two parameters. Here, we write the recurrence relation for the moments in our case, viz., the left-truncated IG3 distribution, truncated at \(x_0 = y_j - \gamma\):
\[
a \mu_r(x_0) - \lambda \mu_{r-2}(x_0) - 2p_0x_0^r = (2r - 3)\mu_{r-1}(x_0), \quad r = 0, \pm 1, \pm 2, \ldots,
\] (13)

where
\[
a = \frac{1}{\beta},
\]
\[
p_0 = \frac{f(x_0; 0, \beta, \alpha)}{1 - F(x_0; 0, \beta, \alpha)},
\]
\[
\mu_r(x_0) = \frac{\int_{x_0}^{\infty} t^r f(t; 0, \beta, \alpha) dt}{1 - F(x_0; 0, \beta, \alpha)}.
\]

It can be shown that [8, p. 154]
\[
\mu_1(x_0) = \frac{\int_{x_0}^{\infty} tf(t; 0, \beta, \alpha) dt}{1 - F(x_0; 0, \beta, \alpha)} = \frac{\beta\alpha}{\Phi(\sqrt{\beta/\alpha}(\alpha - (x_0/\beta)))} + e^{2\alpha} \Phi(\sqrt{\beta/\alpha}(\alpha + (x_0/\beta)))
\]
\[
- e^{2\alpha} \Phi(\sqrt{\beta/\alpha}(\alpha - (x_0/\beta))) - e^{2\alpha} \Phi(\sqrt{\beta/\alpha}(\alpha + (x_0/\beta)))
\].

Equation (14) gives \(E[(Z - \gamma)|Z > y_j; \theta_{(h,h+1)}]\), and other conditional expectations in Equations (11) and (12) can be derived by using Equation (13) and substituting \(r = 2, 1, 0\), successively to obtain
\[
E[(Z - \gamma)^2|Z > y_j; \theta_{(h,h+1)}] = \mu_2(x_0) = \beta(\mu_1 + \beta\alpha^2 + 2p_0x_0^2),
\]
\[
E\left[ \frac{1}{(Z - \gamma)} | Z > y_j; \theta_{(h,h+1)} \right] = \mu_{-1}(x_0) = \frac{\beta + \mu_1 - 2p_0\beta x_0}{\beta^2\alpha^2}.
\]
The final MLEs of the parameters are local MLEs corresponding to the EM-algorithm, and we will denote them here by LMLE2.

4. Asymptotic variances and covariances of the estimates from the EM-algorithm

Asymptotic variances and covariances of the MLEs when the EM-algorithm is used can be obtained by using the missing information principle of Louis [30] and Tanner [31]. Basically, this principle states that

\[
\text{Observed information} = \text{Complete information} - \text{Missing information}.
\]

Based on this principle, Louis [30] developed a procedure for finding the observed information matrix when the EM-algorithm is used to find the MLEs in an incomplete data situation. We adopt this principle in our case of progressive Type-II right censoring. We will denote the complete, observed and missing information by \(I(\theta)\), \(I_Y(\theta)\) and \(I_Z|Y(\theta)\), respectively. The complete information \(I(\theta)\) is given by

\[
I(\theta) = -E \left[ \frac{\partial^2 \text{E}[\ell(Y, Z, \theta)|Y = y]}{\partial \theta^2} \right],
\]

(14)

where \(E[\ell(Y, Z, \theta)|Y = y]\) is as given in Equation (8). The Fisher information matrix for a single observation which is censored at the time of the \(j\)th failure is given by

\[
I_{Z|Y}^{(j)}(\theta) = -E \left[ \frac{\partial^2 \log f_{Zjk}(z_{jk}|y_j; \theta)}{\partial \theta^2} \right],
\]

(15)

where \(f_{Zjk}(z_{jk}|y_j; \theta)\) is as given in Equation (9). Then, the missing (expected) information is simply

\[
I_{Z|Y}(\theta) = \sum_{j=1}^{m} R_j I_{Z|Y}^{(j)}(\theta),
\]

(16)

where \(I_{Z|Y}^{(j)}(\theta)\) is as given in Equation (15). The observed information \(I_Y(\theta)\) is then obtained as

\[
I_Y(\theta) = I(\theta) - I_{Z|Y}(\theta),
\]

(17)

where \(I(\theta)\) and \(I_{Z|Y}(\theta)\) are given by Equations (14) and (16), respectively, and expressions are derived for these quantities below. Finally, by inverting the information matrix \(I_Y(\theta)\) in Equation (17), we obtain the asymptotic variances and covariances of the MLEs when the EM-algorithm is used.

4.1. Complete information matrix \(I(\theta)\)

The log-likelihood function \(l^*(\theta)\) based on \(n\) complete observations \(y_i, i = 1, 2, \ldots, n\), is given by

\[
l^*(\theta) = \sum_{i=1}^{n} \log f(y_i; \theta),
\]

(18)

where \(f(y_i; \theta)\) is as given in Equation (3). By differentiating the log-likelihood function in Equation (18) and equating to zero, one obtains the following estimating equations (which are
indeed the same as in Equation (6) with \( m = n \), and \( R_j = 0; \ j = 1, 2, \ldots, m \):

\[
\frac{\partial l^*}{\partial \gamma} = \sum_{j=1}^{n} \frac{3}{2(y_j - \gamma)} + \frac{n}{2\beta} - \frac{\beta \alpha^2}{2(y_j - \gamma)^2} = 0
\]

\[
\frac{\partial l^*}{\partial \beta} = \frac{n}{2\beta} + \sum_{j=1}^{n} \frac{y_j - \gamma}{2\beta^2} - \frac{\alpha^2}{2(y_j - \gamma)} = 0
\]

\[
\frac{\partial l^*}{\partial \alpha} = \frac{m}{\alpha} + n - \sum_{j=1}^{n} \frac{\beta \alpha}{y_j - \gamma} = 0
\]

The second derivatives of the log-likelihood function \( l^*(\theta) \) are obtained by appropriately differentiating the first derivatives in Equation (19), and are as follows:

\[
\frac{\partial^2 l^*}{\partial \gamma^2} = \sum_{j=1}^{n} \frac{3}{2(y_j - \gamma)^2} - \frac{\beta \alpha^2}{2(y_j - \gamma)^3}
\]

\[
\frac{\partial^2 l^*}{\partial \beta^2} = -\frac{n}{2\beta^2} - \frac{\alpha^2}{2(y_j - \gamma)^2}
\]

\[
\frac{\partial^2 l^*}{\partial \alpha^2} = -\sum_{j=1}^{n} \frac{\beta}{y_j - \gamma}
\]

\[
\frac{\partial^2 l^*}{\partial \gamma \partial \beta} = -\frac{n}{2\beta^2} - \frac{\alpha^2}{2(y_j - \gamma)^2}
\]

\[
\frac{\partial^2 l^*}{\partial \gamma \partial \alpha} = -\sum_{j=1}^{n} \frac{\beta \alpha}{(y_j - \gamma)^2}
\]

\[
\frac{\partial^2 l^*}{\partial \beta \partial \alpha} = -\sum_{j=1}^{n} \frac{\alpha}{y_j - \gamma}
\]

For the IG3 distribution, it is easy to see that

\[
E[Y_i - \gamma] = \beta \alpha
\]

\[
E\left[ \frac{1}{Y_i - \gamma} \right] = \frac{\alpha + 1}{\beta \alpha^2}
\]

\[
E\left[ \frac{1}{(Y_i - \gamma)^2} \right] = \frac{\alpha^2 + 3\alpha + 3}{\beta^2 \alpha^4}
\]

\[
E\left[ \frac{1}{(Y_i - \gamma)^3} \right] = \frac{\alpha^3 + 6\alpha^2 + 15\alpha + 15}{\beta^3 \alpha^6}
\]

One gets the complete information matrix \( I(\theta) \) in Equation (14), by using Equations (20) and (21) and noting that \( E[\ell(Y, Z, \theta)] \) for \( \ell(Y, Z, \theta) \) in Equation (8) is the same as \( E[l^*(\theta)] \) for \( l^*(\theta) \)
in Equation (18), as follows:

\[ I(\theta) = n \begin{pmatrix}
\frac{2\alpha^3 + 9\alpha^2 + 21\alpha + 21}{2\beta^2\alpha^4} & \frac{2\alpha^2 + 3\alpha + 3}{2\beta^2\alpha^2} & \frac{\alpha^2 + 3\alpha + 3}{\beta\alpha^3} \\
\frac{2\alpha^2 + 3\alpha + 3}{2\beta^2\alpha^2} & \frac{2\alpha + 1}{\beta^2} & \frac{\alpha + 1}{\beta\alpha^2} \\
\frac{\alpha^2 + 3\alpha + 3}{\beta\alpha^3} & \frac{\alpha + 1}{\beta\alpha^2} & \frac{\alpha + 2}{\alpha^2}
\end{pmatrix}. \quad (22) \]

The asymptotic variance–covariance matrix of the MLE of \( \theta \) in the complete sample case is obtained by inverting the matrix \( I(\theta) \) in Equation (22). Denoting \( H = [2\alpha^4 + 19\alpha^3 + 66\alpha^2 + 87\alpha + 24]^{-1} \), the asymptotic variances and covariances are as follows:

\[
\begin{align*}
\text{Var}(\hat{\gamma}) &= 4H(\alpha^2 + 3\alpha + 1)\beta^2\alpha^4, \\
\text{Var}(\hat{\beta}) &= 2H(\alpha^3 + 9\alpha^2 + 27\alpha + 24)\beta^2, \\
\text{Var}(\hat{\alpha}) &= H(4\alpha^4 + 28\alpha^3 + 81\alpha^2 + 108\alpha + 33)\alpha^2, \\
\text{Cov}(\hat{\gamma}, \hat{\beta}) &= 2H(\alpha + 3)\beta^2\alpha^3, \\
\text{Cov}(\hat{\gamma}, \hat{\alpha}) &= -2H(2\alpha^3 + 9\alpha^2 + 12\alpha + 3)\beta\alpha^3, \\
\text{Cov}(\hat{\beta}, \hat{\alpha}) &= -4H(\alpha^3 + 6\alpha^2 + 12\alpha + 6)\beta\alpha.
\end{align*}
\]

For the complete sample case, Jones and Cheng [32] obtained the asymptotic variances and covariances of the MLEs of the parameters \((\gamma, \mu, \lambda)\), while those corresponding to the parameters \((\gamma, \beta, \alpha)\) were obtained by Koutrouvelis et al. [33].

### 4.2. Missing information matrix \( I_{Z|Y}(\theta) \)

The logarithm of the density function of an observation \( z_{jk} \) censored at \( y_j \), the time of the \( j \)th failure, is given by (see Equation (9))

\[
\log f_{Z|y}(z > y_j; \theta) = \text{const.} + \frac{1}{2} \log \beta + \log \alpha - \frac{3}{2} \log(z - \gamma) - \frac{(z - \gamma - \beta\alpha)}{2\beta(z - \gamma)} \\
- \log[1 - \Phi(\psi_{1j}) - e^{2\alpha} \Phi(\psi_{2j})]. \quad (23)
\]

Differentiating Equation (23) with respect to \( \gamma \), \( \beta \) and \( \alpha \), we get

\[
\begin{align*}
\frac{\partial \log f_{Z|y}}{\partial \gamma} &= \frac{3(z - \gamma) + \alpha}{2(z - \gamma)^2} - L_{1j}, \\
\frac{\partial \log f_{Z|y}}{\partial \beta} &= \frac{\beta + 1}{2\beta^2} - \frac{y_j - \gamma}{\beta} L_{1j}, \\
\frac{\partial \log f_{Z|y}}{\partial \alpha} &= \frac{2(z - \gamma) + \alpha}{2\alpha(z - \gamma)} - 2 \left( \frac{y_j - \gamma}{\alpha} L_{1j} - e^{2\alpha} M_{2j} \right) \Bigg). \quad (24)
\end{align*}
\]
Once again, upon using Equations (13) and (14), we get successively the following expressions:

\[
\begin{align*}
E\left[\frac{1}{(Z - \gamma)^{3}} \mid Z > y_j; \theta\right] &= \mu_{-3}(x_0) = \frac{5\mu_{-2} + a\mu_{-1} - 2p_0x_0^{-1}}{\lambda} \\
&= \frac{5\beta\mu_{-2} + \mu_{-1} - 2p_0\beta x_0^{-1}}{\beta^{2}a^{2}} \\
I_{ZY}(\theta) &= \frac{7\beta\mu_{-3} + \mu_{-2} - 2p_0\beta x_0^{-2}}{\beta^{2}a^{2}}
\end{align*}
\]

(25)

Then the Fisher information matrix based on one observation \( z_{jk} \) censored at \( y_j \), \( I_{ZY}^{(j)}(\theta) \), in Equation (15), can be obtained by using Equations (24) and (25); the expressions are long, but straightforward. Using \( I_{ZY}(\theta) \) from above and \( I(\theta) \) from Equation (22), the observed information \( I_Y(\theta) \) can be readily obtained by using the missing information principle in Equation (17). Finally, one can get the asymptotic variances and covariances of the MLEs, while using the EM-algorithm, by inverting this observed information matrix \( I_Y(\theta) \).

5. Alternative estimators

As an alternative estimator, one can use modified MLE (MMLE). MMLEs use first-order statistic for estimating \( \gamma \) and are therefore easier to compute than the MLEs. For small samples when LMLEs fail to converge, these MMLEs (which always exist) produce reasonable estimates. For progressively censored data, one can have two versions of MMLEs corresponding to the Newton–Raphson algorithm and the EM-algorithm. We will denote them by MMLE1 and MMLE2, respectively. In both versions, the likelihood equation \( \frac{\partial l(\theta)}{\partial \gamma} = 0 \) is replaced by

\[
\gamma + \beta\alpha + \beta \sqrt{\alpha} \cdot IG^{-1}\left(\frac{1}{n+1}\right) = y_1,
\]

(26)

where \( IG^{-1} \) is the inverse of the cdf \( IG3(0, 1, 1) \) and \( n = m + \sum_{j=1}^{m} R_j \) is the total sample size. Equation (26) is used as a test equation and it replaces the first equation in (6) for the Newton–Raphson algorithm version and Equation (12) for the EM-algorithm version. One starts with a first approximation \( \gamma_1 \) of \( \gamma \), then calculates conditional estimates \( \beta_i = \beta(\gamma_i) \) and \( \alpha_j = \alpha(\gamma_i) \) by using the second and third equations of (6) for the Newton–Raphson algorithm and Equation (11) for the EM-algorithm. The values \( (\gamma_1, \beta_1, \alpha_1) \) are substituted in Equation (26). If the test equation is satisfied, then no further iteration is required. Otherwise, a second approximation \( \gamma_2 \) is selected and the cycle of calculations described above is repeated. The iterations are continued until two sufficiently close values \( \gamma_i \) and \( \gamma_{i+1} \) are found such that the following is satisfied:

\[
\gamma_i + \beta_i\alpha_i + \beta_i \sqrt{\alpha} \cdot IG^{-1}\left(\frac{1}{n+1}\right) < (>) y_1 < (>) \gamma_{i+1} + \beta_{i+1}\alpha_{i+1} + \beta_{i+1} \sqrt{\alpha_{i+1}} \cdot IG^{-1}\left(\frac{1}{n+1}\right).
\]
Thus, the final MMLE $\hat{\gamma}$ of $\gamma$ is obtained using which the final MMLEs $\hat{\beta}$ and $\hat{\alpha}$ of $\beta$ and $\alpha$ are obtained.

6. Illustrative examples

We use three examples – two given in Cohen and Whitten [25] and one given in von Alven [34] – to illustrate the methods of estimation detailed in the preceding sections. We modified these original examples to consider progressively censored data. We then assume that each data come from a three-parameter inverse Gaussian distribution.

In order to obtain LMLE1, we use a variation of the method described in Section 2. We start with an initial $\gamma(0)$ which is less than $y_1$ and use the second and third equations of (6) to get $\beta(0) = \beta(\gamma(0))$ and $\alpha(0) = \alpha(\gamma(0))$. The first equation is used as the test equation. If the left-hand side of this equation equals zero when $(\gamma(0), \beta(0), \alpha(0))$ substituted there, then no further iteration is required. Otherwise, we choose another $\gamma(1)$ which is less than $y_1$ yielding $\beta(1) = \beta(\gamma(1))$ and $\alpha(1) = \alpha(\gamma(1))$ and look for the sign change in the left-hand side of the first equation of (6). Finally, interpolating on $\gamma$ values, we get the final estimate $(\hat{\gamma}, \hat{\beta}, \hat{\alpha})$ which when substituted in the first equation of (6) produces zero.

Next to obtain LMLE2, we again use a variation of the method described in Section 3. We start with an initial $\gamma(0)$ which is less than $y_1$ and get $\beta(0) = \beta(\gamma(0))$ and $\alpha(0) = \alpha(\gamma(0))$ by using Equation (10). Then we get $\beta(1)$ and $\alpha(1)$ by using Equation (11). Equation (12) is treated as the test equation. If the left-hand side of that equation equals zero when $(\gamma(0), \beta(1), \alpha(1))$ substituted there, then no further iteration is needed. Otherwise, by choosing another $\gamma(1)$ which is less than $y_1$, we repeat the procedure and look for the sign change in Equation (12). Finally, interpolating on $\gamma$ values, we get the final estimate $(\hat{\gamma}, \hat{\beta}, \hat{\alpha})$, which when substituted in the first equation of (12) produce zero.

For the determination of MMLE, as discussed in Section 5, $\text{IG}^{-1}(1/(n + 1))$ for Examples 1–3 are given by 0.1812126954 (for $n = 20$), 0.2285680510 (for $n = 10$) and 0.1435281129 (for $n = 46$), respectively. Proceeding as explained in Section 5, we get two versions of MMLE, viz., MMLE1 and MMLE2 (for the Newton–Raphson algorithm and EM-algorithm, respectively).

Example 1 The maximum flood levels in millions of cubic feet per second for 20 four-year periods from 1890 to 1969 in the Susquehanna river at Harrisburg, Pennsylvania, are given in Cohen and Whitten [25]. We modified the data to make it progressively censored with $m = 12$ stages, and these progressively censored data are presented in Table 1.

<table>
<thead>
<tr>
<th>Table 1. Progressively censored data for Example 1.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>$Y_i$</td>
</tr>
<tr>
<td>$R_i$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2. Estimates and standard errors of estimates for $\gamma$, $\beta$ and $\alpha$ based on data in Table 1.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
</tr>
<tr>
<td>Estimator</td>
</tr>
<tr>
<td>LMLE2</td>
</tr>
</tbody>
</table>
Table 3. Progressively censored data for Example 2.

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_i$</td>
<td>152.7</td>
<td>172.0</td>
<td>172.5</td>
<td>173.3</td>
<td>193.0</td>
<td>204.7</td>
<td>234.9</td>
</tr>
<tr>
<td>$R_i$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 4. Estimates and standard errors of estimates for $\gamma$, $\beta$ and $\alpha$ based on data in Table 3.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$\hat{\gamma}$</th>
<th>$\sigma_{\hat{\gamma}}$</th>
<th>$\hat{\beta}$</th>
<th>$\sigma_{\hat{\beta}}$</th>
<th>$\hat{\alpha}$</th>
<th>$\sigma_{\hat{\alpha}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LMLE2</td>
<td>99.312</td>
<td>7.884</td>
<td>6.231</td>
<td>1.217</td>
<td>10.138</td>
<td>1.482</td>
</tr>
</tbody>
</table>

Table 5. Progressively censored data for Example 3.

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_i$</td>
<td>0.2</td>
<td>0.3</td>
<td>0.5</td>
<td>0.5</td>
<td>0.6</td>
<td>0.6</td>
<td>0.7</td>
<td>0.7</td>
<td>0.8</td>
<td>0.8</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>$R_i$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>30</td>
</tr>
</tbody>
</table>

Table 6. Estimates, standard errors and approximate 95% confidence intervals (CIs) for $\gamma$, $\beta$ and $\alpha$ based on data in Table 5.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$\hat{\gamma}$</th>
<th>$\sigma_{\hat{\gamma}}$</th>
<th>95% CI</th>
<th>$\hat{\beta}$</th>
<th>$\sigma_{\hat{\beta}}$</th>
<th>95% CI</th>
<th>$\hat{\alpha}$</th>
<th>$\sigma_{\hat{\alpha}}$</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>LMLE2</td>
<td>-0.018</td>
<td>0.005</td>
<td>(-0.028, -0.008)</td>
<td>6.997</td>
<td>0.593</td>
<td>(5.835, 7.159)</td>
<td>0.498</td>
<td>0.053</td>
<td>(0.394, 0.602)</td>
</tr>
</tbody>
</table>

For the data in Table 1, the LMLE1 are given by $\hat{\gamma} = 0.173$, $\hat{\beta} = 0.077$ and $\hat{\alpha} = 3.460$. The LMLE2 and their standard errors are presented in Table 2. The MMLE1 are given by $\tilde{\gamma} = 0.089$, $\tilde{\beta} = 0.064$ and $\tilde{\alpha} = 3.506$, and the MMLE2 are given by $\tilde{\gamma} = 0.212$, $\tilde{\beta} = 0.068$ and $\tilde{\alpha} = 3.466$.

Example 2 In this example, the data refer to the fatigue life in hours of 10 bearings of a certain type. The data were used by McCool [35] and is also given in Cohen and Whitten [25]. We modified these data to make it progressively censored with $m = 7$ stages, and these progressively censored data are presented in Table 3.

For the data in Table 3, the LMLE1 are given by $\hat{\gamma} = 91.5$, $\hat{\beta} = 6.807$ and $\hat{\alpha} = 9.123$. The LMLE2 and their standard errors are presented in Table 4. The MMLE1 are given by $\tilde{\gamma} = 102.781$, $\tilde{\beta} = 5.923$ and $\tilde{\alpha} = 8.932$, and the MMLE2 are given by $\tilde{\gamma} = 105.623$, $\tilde{\beta} = 6.156$ and $\tilde{\alpha} = 9.793$.

Example 3 In this example, the complete data are given in von Alven [34] and were also used by Koutrouvelis et al. [33]. The complete data consists of 46 observations, which represent active repair times in hours for an airborne communicator transceiver. We modified these data to make it progressively censored with $m = 12$ stages, and these progressively censored data are presented in Table 5.

For the data in Table 5, the LMLE1 are given by $\hat{\gamma} = -0.027$, $\hat{\beta} = 7.230$ and $\hat{\alpha} = 0.517$. The LMLE2 and their standard errors are presented in Table 6. The MMLE1 are given by $\tilde{\gamma} = -0.032$, $\tilde{\beta} = 6.322$ and $\tilde{\alpha} = 0.612$, and the MMLE2 are given by $\tilde{\gamma} = -0.024$, $\tilde{\beta} = 6.956$ and $\tilde{\alpha} = 0.596$. 
In order to examine the effect of delayed censoring, we considered two schemes of progressive censoring. In one scheme, the censoring pattern was delayed than the other. That is, the censoring was done at the early stages in one while it was done at the late stages in the other. Keeping the stages \( m = 12 \) same as in Table 5, for the delayed censoring scheme, we considered PCS-1:

\[
R_1 = 0, \quad R_2 = 0, \quad R_3 = 0, \quad R_4 = 0, \quad R_5 = 0, \quad R_6 = 0, \quad R_7 = 0, \quad R_8 = 0, \quad R_9 = 0, \quad R_{10} = 2, \quad R_{11} = 2, \quad R_{12} = 30.
\]

For the other scheme, we took PCS-2:

\[
R_1 = 30, \quad R_2 = 2, \quad R_3 = 2, \quad R_4 = 0, \quad R_5 = 0, \quad R_6 = 0, \quad R_7 = 0, \quad R_8 = 0, \quad R_9 = 0, \quad R_{10} = 0, \quad R_{11} = 0, \quad R_{12} = 0.
\]

The progressively censored data obtained under these two schemes are presented in Tables 7 and 8, respectively.

For the data in Table 7, the LMLE1 are given by \( \hat{\gamma} = -0.024, \hat{\beta} = 7.352 \) and \( \hat{\alpha} = 0.454 \). The LMLE2 estimates and their standard errors are presented in Table 9. The MMLE1 are given by \( \tilde{\gamma} = -0.028, \tilde{\beta} = 6.775 \) and \( \tilde{\alpha} = 0.529 \), and the MMLE2 are given by \( \tilde{\gamma} = -0.020, \tilde{\beta} = 7.002 \) and \( \tilde{\alpha} = 0.568 \).

For the data in Table 8, the LMLE1 are given by \( \hat{\gamma} = -0.018, \hat{\beta} = 7.162 \) and \( \hat{\alpha} = 0.529 \). The LMLE2 estimates and their standard errors are presented in Table 10. The MMLE1 are given by \( \tilde{\gamma} = -0.025, \tilde{\beta} = 7.353 \) and \( \tilde{\alpha} = 0.526 \), and the MMLE2 are given by \( \tilde{\gamma} = -0.027, \tilde{\beta} = 7.223 \) and \( \tilde{\alpha} = 0.522 \).

We can see from Tables 6, 9 and 10 that the estimates of all the parameters, viz., \( \gamma, \beta \) and \( \alpha \), remain more or less the same in all the situations of censoring.

### 7. Simulation study

We performed a simulation study for different sample sizes and different degrees of censoring in order to compare the performance of LMLE1 and LMLE2 in terms of coverage probabilities of 95% CIs for the parameters \( \gamma, \beta \) and \( \alpha \). Ten thousand samples were simulated from the inverse distribution.
Gaussian distribution with $\gamma = 100$, $\beta = 5.0$ and $\alpha = 1.0$ with sample sizes $n = 20$ (with $m = 12, 14, 16, 18$ stages of censoring), and $n = 40$ (with $m = 24, 28, 32, 36$ stages of censoring). In each of these cases, we considered two censoring schemes with one reflecting comparatively delayed censoring than the other. The coverage probabilities of 95% CIs from this simulation study for the parameters $\gamma$, $\beta$ and $\alpha$ are reported in Tables 11–13, respectively. It should be mentioned here that for both these methods, after determining the estimates and the standard errors by using the formulas and expressions presented earlier in Sections 2–4, CIs for the parameters were obtained simply by using the asymptotic normality of the estimates. After that, the coverage probabilities of these CIs were found as the proportion of times the CIs contained the true values of the parameters.

Another 10,000 samples were simulated from the inverse Gaussian distribution with $\gamma = 100$, $\beta = 5.0$ and $\alpha = 2.0$ with samples sizes $n = 20$ (with $m = 12, 14, 16, 18$ stages of censoring), and $n = 40$ (with $m = 24, 28, 32, 36$ stages of censoring) as in the previous case. We kept the

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>$R$</th>
<th>LMLE1</th>
<th>LMLE2</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>12</td>
<td>(6, 2, 0, ..., 0, 0, 0)</td>
<td>91.38</td>
<td>92.32</td>
</tr>
<tr>
<td>20</td>
<td>12</td>
<td>(0, 0, 0, ..., 0, 2, 6)</td>
<td>91.41</td>
<td>92.86</td>
</tr>
<tr>
<td>20</td>
<td>14</td>
<td>(5, 1, 0, ..., 0, 0, 0)</td>
<td>91.93</td>
<td>93.79</td>
</tr>
<tr>
<td>20</td>
<td>14</td>
<td>(0, 0, 0, ..., 0, 1, 5)</td>
<td>92.22</td>
<td>93.97</td>
</tr>
<tr>
<td>20</td>
<td>16</td>
<td>(3, 1, 0, ..., 0, 0, 0)</td>
<td>93.44</td>
<td>94.84</td>
</tr>
<tr>
<td>20</td>
<td>16</td>
<td>(0, 0, 0, ..., 0, 1, 3)</td>
<td>93.92</td>
<td>94.76</td>
</tr>
<tr>
<td>20</td>
<td>18</td>
<td>(1, 1, 0, ..., 0, 0, 0)</td>
<td>94.05</td>
<td>94.53</td>
</tr>
<tr>
<td>20</td>
<td>18</td>
<td>(0, 0, 0, ..., 0, 1, 1)</td>
<td>94.35</td>
<td>95.76</td>
</tr>
<tr>
<td>40</td>
<td>24</td>
<td>(14, 2, 0, ..., 0, 0, 0)</td>
<td>92.45</td>
<td>93.05</td>
</tr>
<tr>
<td>40</td>
<td>24</td>
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<td>93.04</td>
<td>93.88</td>
</tr>
<tr>
<td>40</td>
<td>28</td>
<td>(10, 2, 0, ..., 0, 0, 0)</td>
<td>93.51</td>
<td>93.79</td>
</tr>
<tr>
<td>40</td>
<td>28</td>
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<td>94.76</td>
<td>94.28</td>
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<tr>
<td>40</td>
<td>32</td>
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<td>94.73</td>
<td>94.62</td>
</tr>
<tr>
<td>40</td>
<td>32</td>
<td>(0, 0, 0, ..., 0, 2, 6)</td>
<td>93.62</td>
<td>94.97</td>
</tr>
<tr>
<td>40</td>
<td>36</td>
<td>(2, 2, 0, ..., 0, 0, 0)</td>
<td>94.03</td>
<td>94.82</td>
</tr>
<tr>
<td>40</td>
<td>36</td>
<td>(0, 0, 0, ..., 0, 2, 2)</td>
<td>94.87</td>
<td>94.58</td>
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</tbody>
</table>

Table 11. Coverage probabilities of 95% CIs for $\gamma$ based on Monte Carlo simulations with $\gamma = 100$, $\beta = 5.0$, $\alpha = 1.0$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>$R$</th>
<th>LMLE1</th>
<th>LMLE2</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>12</td>
<td>(6, 2, 0, ..., 0, 0, 0)</td>
<td>93.43</td>
<td>94.01</td>
</tr>
<tr>
<td>20</td>
<td>12</td>
<td>(0, 0, 0, ..., 0, 2, 6)</td>
<td>93.88</td>
<td>94.34</td>
</tr>
<tr>
<td>20</td>
<td>14</td>
<td>(5, 1, 0, ..., 0, 0, 0)</td>
<td>93.57</td>
<td>93.97</td>
</tr>
<tr>
<td>20</td>
<td>14</td>
<td>(0, 0, 0, ..., 0, 1, 5)</td>
<td>95.06</td>
<td>94.89</td>
</tr>
<tr>
<td>20</td>
<td>16</td>
<td>(3, 1, 0, ..., 0, 0, 0)</td>
<td>95.42</td>
<td>95.12</td>
</tr>
<tr>
<td>20</td>
<td>16</td>
<td>(0, 0, 0, ..., 0, 1, 3)</td>
<td>95.42</td>
<td>95.12</td>
</tr>
<tr>
<td>20</td>
<td>18</td>
<td>(1, 1, 0, ..., 0, 0, 0)</td>
<td>94.89</td>
<td>95.43</td>
</tr>
<tr>
<td>20</td>
<td>18</td>
<td>(0, 0, 0, ..., 0, 1, 1)</td>
<td>95.83</td>
<td>95.66</td>
</tr>
<tr>
<td>40</td>
<td>24</td>
<td>(14, 2, 0, ..., 0, 0, 0)</td>
<td>93.74</td>
<td>94.53</td>
</tr>
<tr>
<td>40</td>
<td>24</td>
<td>(0, 0, 0, ..., 0, 2, 14)</td>
<td>94.01</td>
<td>94.87</td>
</tr>
<tr>
<td>40</td>
<td>28</td>
<td>(10, 2, 0, ..., 0, 0, 0)</td>
<td>94.81</td>
<td>94.07</td>
</tr>
<tr>
<td>40</td>
<td>28</td>
<td>(0, 0, 0, ..., 0, 2, 10)</td>
<td>95.32</td>
<td>94.63</td>
</tr>
<tr>
<td>40</td>
<td>32</td>
<td>(6, 2, 0, ..., 0, 0, 0)</td>
<td>94.87</td>
<td>94.88</td>
</tr>
<tr>
<td>40</td>
<td>32</td>
<td>(0, 0, 0, ..., 0, 2, 6)</td>
<td>95.23</td>
<td>95.64</td>
</tr>
<tr>
<td>40</td>
<td>36</td>
<td>(2, 2, 0, ..., 0, 0, 0)</td>
<td>95.47</td>
<td>95.55</td>
</tr>
<tr>
<td>40</td>
<td>36</td>
<td>(0, 0, 0, ..., 0, 2, 2)</td>
<td>95.86</td>
<td>95.76</td>
</tr>
</tbody>
</table>

Table 12. Coverage probabilities of 95% CIs for $\beta$ based on Monte Carlo simulations with $\gamma = 100$, $\beta = 5.0$, $\alpha = 1.0$. 
design of $R$ unchanged from Tables 11–13. The corresponding coverage probabilities of 95% CIs from this simulation study for the parameters $\gamma$, $\beta$ and $\alpha$ are reported in Tables 14–16, respectively.

From the tables, we observe that the LMLE1 and LMLE2 have nearly the same coverage probabilities. Also the coverage probabilities do not seem to change with the censoring schemes. The coverage probabilities usually increase with the larger proportion of uncensored data. The estimation of the parameter $\gamma$ seems to be most sensitive to the censoring while the CIs for the parameters $\beta$ and $\alpha$ seem to be quite satisfactory and close to the nominal level of 95%. In general, we have found that for the Monte Carlo simulation with $\gamma = 100$, $\beta = 5.0$, $\alpha = 2.0$, the number of steps taken for convergence was little bit more than the one with $\gamma = 100$, $\beta = 5.0$, $\alpha = 1.0$ for $n = 20$. 

Table 13. Coverage probabilities of 95% CIs for $\alpha$ based on Monte Carlo simulations with $\gamma = 100$, $\beta = 5.0$, $\alpha = 1.0$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>$R$</th>
<th>LMLE1</th>
<th>LMLE2</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>12</td>
<td>(6.2.0,0,0)</td>
<td>93.04</td>
<td>93.63</td>
</tr>
<tr>
<td>20</td>
<td>12</td>
<td>(0.0,0,0)</td>
<td>93.28</td>
<td>93.52</td>
</tr>
<tr>
<td>20</td>
<td>14</td>
<td>(5.1.0,0,0)</td>
<td>93.35</td>
<td>93.21</td>
</tr>
<tr>
<td>20</td>
<td>14</td>
<td>(0.0,0,0)</td>
<td>93.94</td>
<td>93.69</td>
</tr>
<tr>
<td>20</td>
<td>16</td>
<td>(3.1.0,0,0)</td>
<td>94.91</td>
<td>95.25</td>
</tr>
<tr>
<td>20</td>
<td>16</td>
<td>(0.0,0,0)</td>
<td>94.78</td>
<td>94.72</td>
</tr>
<tr>
<td>20</td>
<td>18</td>
<td>(1.1,0,0)</td>
<td>95.22</td>
<td>95.23</td>
</tr>
<tr>
<td>20</td>
<td>18</td>
<td>(0.0,0,0)</td>
<td>95.23</td>
<td>95.74</td>
</tr>
<tr>
<td>40</td>
<td>24</td>
<td>(14.2.0,0,0)</td>
<td>93.01</td>
<td>93.33</td>
</tr>
<tr>
<td>40</td>
<td>24</td>
<td>(0.0,0,0,0)</td>
<td>93.56</td>
<td>93.87</td>
</tr>
<tr>
<td>40</td>
<td>28</td>
<td>(10.2.0,0,0)</td>
<td>93.43</td>
<td>93.12</td>
</tr>
<tr>
<td>40</td>
<td>28</td>
<td>(0.0,0,0,0)</td>
<td>94.01</td>
<td>93.94</td>
</tr>
<tr>
<td>40</td>
<td>32</td>
<td>(6.2.0,0,0)</td>
<td>95.21</td>
<td>95.23</td>
</tr>
<tr>
<td>40</td>
<td>32</td>
<td>(0.0,0,0,0)</td>
<td>95.24</td>
<td>95.33</td>
</tr>
<tr>
<td>40</td>
<td>36</td>
<td>(2.2.0,0,0)</td>
<td>95.22</td>
<td>95.41</td>
</tr>
<tr>
<td>40</td>
<td>36</td>
<td>(0.0,0,0,0)</td>
<td>95.39</td>
<td>95.67</td>
</tr>
</tbody>
</table>

Table 14. Coverage probabilities of 95% CIs for $\gamma$ based on Monte Carlo simulations with $\gamma = 100$, $\beta = 5.0$, $\alpha = 2.0$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>$R$</th>
<th>LMLE1</th>
<th>LMLE2</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>12</td>
<td>(6.2.0,0,0)</td>
<td>92.52</td>
<td>92.93</td>
</tr>
<tr>
<td>20</td>
<td>12</td>
<td>(0.0,0,0,0,0)</td>
<td>92.26</td>
<td>92.89</td>
</tr>
<tr>
<td>20</td>
<td>14</td>
<td>(5.1.0,0,0)</td>
<td>92.68</td>
<td>93.87</td>
</tr>
<tr>
<td>20</td>
<td>14</td>
<td>(0.0,0,0,0,0)</td>
<td>93.05</td>
<td>93.95</td>
</tr>
<tr>
<td>20</td>
<td>16</td>
<td>(3.1.0,0,0,0)</td>
<td>93.72</td>
<td>93.91</td>
</tr>
<tr>
<td>20</td>
<td>16</td>
<td>(0.0,0,0,0,0)</td>
<td>93.97</td>
<td>94.23</td>
</tr>
<tr>
<td>20</td>
<td>18</td>
<td>(1.1,0,0,0,0)</td>
<td>94.56</td>
<td>94.88</td>
</tr>
<tr>
<td>20</td>
<td>18</td>
<td>(0.0,0,0,0,0)</td>
<td>94.67</td>
<td>95.82</td>
</tr>
<tr>
<td>40</td>
<td>24</td>
<td>(14.2.0,0,0,0)</td>
<td>92.93</td>
<td>93.53</td>
</tr>
<tr>
<td>40</td>
<td>24</td>
<td>(0.0,0,0,0,0,0)</td>
<td>93.12</td>
<td>93.78</td>
</tr>
<tr>
<td>40</td>
<td>28</td>
<td>(10.2.0,0,0,0,0)</td>
<td>94.78</td>
<td>94.68</td>
</tr>
<tr>
<td>40</td>
<td>28</td>
<td>(0.0,0,0,0,0,0)</td>
<td>94.76</td>
<td>94.65</td>
</tr>
<tr>
<td>40</td>
<td>32</td>
<td>(6.2.0,0,0,0,0)</td>
<td>93.89</td>
<td>94.91</td>
</tr>
<tr>
<td>40</td>
<td>32</td>
<td>(0.0,0,0,0,0,0)</td>
<td>94.86</td>
<td>95.02</td>
</tr>
<tr>
<td>40</td>
<td>36</td>
<td>(2.2.0,0,0,0,0)</td>
<td>94.35</td>
<td>94.82</td>
</tr>
<tr>
<td>40</td>
<td>36</td>
<td>(0.0,0,0,0,0,0)</td>
<td>94.35</td>
<td>94.82</td>
</tr>
</tbody>
</table>
The coverage probabilities based on a Monte Carlo simulation study also produced similar results. It has also been observed that the coverage probabilities for both methods are better for two versions of MMLEs. We have considered three examples and we have observed in all these cases that the EM-algorithm and the Newton–Raphson method produce very similar results. The coverage probabilities based on a Monte Carlo simulation study also produced similar results. It has also been observed that the coverage probabilities for both methods are better and closer to the nominal level of 95% when we had a larger proportion of uncensored data in general.

8. Concluding remarks

In this article, we have proposed an EM-algorithm for the MLE based on progressively Type-II censored samples from a three-parameter Inverse Gaussian distribution to estimate the parameters. The Newton–Raphson method has also been used for the same purpose. We have discussed two versions of MMLEs. We have considered three examples and we have observed in all these cases that the EM-algorithm and the Newton–Raphson method produce very similar results. The coverage probabilities based on a Monte Carlo simulation study also produced similar results. It has also been observed that the coverage probabilities for both methods are better and closer to the nominal level of 95% when we had a larger proportion of uncensored data in general.
References