LOWER RECORD VALUES AND CHARACTERIZATIONS OF EXPONENTIAL DISTRIBUTION

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ABSTRACT

Let \( X_1, X_2, \cdots \), be a sequence of independent and identically distributed random variables with continuous distribution function \( F(x) \). For \( n = 1, 2, \cdots \), we denote the order statistics of \( X_1, X_2, \cdots, X_n \) by \( X_{1:n}, X_{2:n}, \cdots, X_{n:n} \). Consider the following.

\[
L^{(k)}(1) = k, \quad \text{and} \quad L^{(k)}(n + 1) = \min \{ j : j > L^{(k)}(n), X_j < X_{j-k,j-1} \},
\]

\[
X^{(k)}(n) = X_{L^{(k)}(n)-k+1;L^{(k)}(n)}, n \geq 1.
\]

The sequence \( L^{(k)}(1), L^{(k)}(2), \cdots \) and \( X^{(k)}(1), X^{(k)}(2), \cdots \) are called lower \( k \)-order record times and lower \( k \)-order record values respectively. In this article, some characterizations of the exponential distribution are proved through properties of \( (L^{(k)}(n) - k + 1)X^{(k)}(n) \) and \( X^{(k)}(1) \) generalizing the results of Ahsanullah and Kirmani (1991).

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1 INTRODUCTION

Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed (iid) random variables with continuous distribution function (df) $F(x)$, and probability density function (pdf) $f(x)$. For $n = 1, 2, \cdots$, we denote the order statistics of $X_1, X_2, \cdots, X_n$ by

$$X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}.$$  \hspace{1cm} (1.1)

Since $F(\cdot)$ is continuous, removing a null set will give us $X_i \neq X_j; i \neq j$. That is we get the strict inequality in (1.1). Consider the following,

$$L^{(k)}(1) = k, \text{ and } L^{(k)}(n+1) = \min \left\{ j : j > L^{(k)}(n), X_j < X_{j-k-1} \right\},$$

$$X^{(k)}(n) = X_{L^{(k)}(n)-k+1,L^{(k)}(n)}, n \geq 1.$$  

The sequence of random variables $\{L^{(k)}(n)\}$ and $\{X^{(k)}(n)\}$ are called lower $k$–order record times and lower $k$–order record values respectively. One can go from lower records to upper records by replacing the original sequence of random variables by $\{-X_i, i \geq 1\}$ or (if $P(X_i > 0) = 1$) by $\{1/X_i, i > 1\}$. We will write $\tilde{F}(x) = 1 - F(x)$ for the survival function, $R(x) = -\ln \tilde{F}(x)$, and $r(x) = f(x)/\tilde{F}(x)$, the hazard rate function.

Record values of iid random variables and their properties have been extensively studied in the literature. See Ahsanullah (1995), Nagaraja (1988), and Nevzorov (1987) for recent reviews. In this article, we prove some characterizations of the exponential distribution through properties of $(L^{(k)}(n) - k + 1)X^{(k)}(n)$ when $F(0) = 0$. Theorem 1 shows that, under certain assumptions, $(L^{(k)}(n) - k + 1)X^{(k)}(n)$ and $X^{(k)}(1)$ are identically distributed if and only if the underlying distribution $F(x)$ is exponential. In Theorem 2 we characterize the exponential distribution through the conditional distribution of $(L^{(k)}(n) - k + 1)X^{(k)}(n)$ given $L^{(k)}(n) - k + 1 = m$ for some $m \geq n$. In Theorem 3, it is proved that, when the underlying distribution $F(\cdot)$ is harmonic new better (worse) than used in expectation, the condition $(L^{(k)}(n) - k + 1)X^{(k)}(n) \overset{d}{=} X^{(k)}(1)$ of Theorem 1 could be replaced by $E \left[ (L^{(k)}(n) - k + 1)X^{(k)}(n) \right] = E \left[ X^{(k)}(1) \right]$, where $E[\cdot]$ represents expected value.

2 MAIN RESULTS

Let $X_1, X_2, \cdots$, be iid random variables having a common continuous df $F(\cdot)$ with $F(0) = 0$. Rényi (1962) shows that the marginal distribution of $L^{(1)}(n)$

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is independent of the underlying $F(\cdot)$ and is given by
\[
P \left( L^{(1)}(n) = m \right) = \frac{|S^{n-1}_{m-1}|}{m!}; \quad m \geq n,
\]
where $S^m_n$ are the Stirling numbers of the first kind, defined by
\[
x(x - 1) \cdots (x - m + 1) = \sum_{n \geq 0} S^m_n x^n.
\]

Similarly, it can be shown that $L^{(k)}(n)$ is independent of the underlying $F(\cdot)$ and is given by
\[
P \left( L^{(k)}(n) = m \right) = \frac{|S^{n-1}_{m-k}|}{(m - k + 1)!}; \quad m \geq n + k - 1. \tag{2.1}
\]

In order to prove our main results, we will need the following lemma.

**Lemma 1** $P \left( (L^{(k)}(n) - k + 1)X^{(k)}(n) > x \right) = \sum_{m=n}^{\infty} \frac{|S^{n-1}_{m-k}|}{m!} \bar{G}^m(x/m)$, for all $x \geq 0$, where $\bar{G}(x) = \bar{F}^k(x)$.

**Proof** It is easy to show that the joint probability of $X^{(k)}(1), L^{(k)}(2), X^{(k)}(2), \ldots, L^{(k)}(n), X^{(k)}(n)$ is given by
\[
h(x_1, m_2, x_2, \ldots, m_n, x_n) = \prod_{i=1}^{n-1} g(x_i) \bar{G}^{m_{i+1} - m_i - 1}(x_i) g(x_n), \tag{2.2}
\]
where $m_1 = k$, and $g(x_i) = -\frac{d}{dx_i} \bar{G}(x_i) = k \bar{F}^{k-1}(x_i) f(x_i), i = 1, 2, \ldots, n$.

Now integrating and summing up equation (2.2) we get the joint probability function of $X^{(k)}(n)$ and $L^{(k)}(n)$ to be
\[
h(x_n, m_n) = \frac{|S^{n-1}_{m_n-k}|}{(m_n - k + 1)!} \bar{G}^{m_n-k}(x_n) g(x_n); m_n \geq n + k - 1. \tag{2.3}
\]

Let $Y = (L^{(k)}(n) - k + 1)X^{(k)}(n)$ and $M = L^{(k)}(n) - k + 1$. Then from equation (2.3), the joint probability function of $Y$ and $M$ is given by
\[
h(y, m) = \frac{|S^{m-1}_{m-k}|}{m!} \bar{G}^{m-1}(y/m) g(y/m); \quad m \geq n,
\]
which gives

\[ P(Y > x, M = m) = \int_x^\infty \frac{|S_{m-1}^n|}{m!} G_{m-1}^{-1}(y/m)g(y/m)dy \]

\[ = \frac{|S_{m-1}^n|}{m!} G_m(x/m). \]

So,

\[ P \left( (L^{(k)}(n) - k + 1)X^{(k)}(n) > x \right) = P(Y > x) \]

\[ = \sum_{m=n}^{\infty} P(Y > x, M = m) \]

\[ = \sum_{m=n}^{\infty} \frac{|S_{m-1}^n|}{m!} G_m(x/m). \] \(\square\)

**Remark 1** Lemma 1 also gives us the equation (2.1).

**Theorem 1** Suppose that \( \lim_{x \to 0^+} \frac{F(x)}{x} = 0, 0 < \theta < \infty \). The random variables \( Y = (L^{(k)}(n) - k + 1)X^{(k)}(n) \) and \( X = X^{(k)}(1) \) are identically distributed if and only if \( F(x) = \exp(-\theta x), x > 0 \), for some \( \theta > 0 \).

**Proof (If Part)** Easy to verify.

(Only if Part) Let \( Y \overset{d}{=} X \) and define \( u(x) = \frac{F(x)}{x} \), \( x > 0 \). Since \( \lim_{x \to 0^+} \frac{F(x)}{x} = 0 \), we have \( u(0) = u(0^+) \). Then

\[ \sum_{m=n}^{\infty} \frac{|S_{m-1}^n|}{m!} e^{-k\lambda u(x/m)} = e^{-k\lambda u(x)}, \quad x > 0. \] \( (2.4) \)

The theorem is proved if it can be shown that \( u(x) \) in equation (2.4) is a constant. Following Ahsanullah and Kirmani (1991) we show that \( u(x) \) is indeed a constant. Define for any given \( T > 0 \),

\[ a_0 = \min_{x \in [0, T]} u(x), \quad x_0 = \inf \{ x \in [0, T] | u(x) = a_0 \}, \]

\[ a_1 = \max_{x \in [0, T]} u(x), \quad x_1 = \inf \{ x \in [0, T] | u(x) = a_1 \}. \]

By continuity of \( u(x) \), we have \( x \in [0, T] \) and \( u(x_0) = a_0 \). Therefore,

\[ u(x_0) \leq u(x_0/m) \] for all \( m \geq 1. \] \( (2.5) \)
If in (2.5) equality holds for all \( m \geq n \) then \( u(x_0) = u(0) \), which gives \( x_0 = 0 \) by the definition of \( x_0 \). Now suppose that \( x_0 > 0 \). Then in (2.5) we must have strict inequality for at least one value of \( m \geq n \) and hence

\[
e^{-kx_0 u(x_0)} - \sum_{m=n}^{\infty} \frac{|S_{m-1}^{n-1}|}{m!} e^{-kx_0 u(x/m)} = \sum_{m=n}^{\infty} \frac{|S_{m-1}^{n-1}|}{m!} \left\{ e^{-kx_0 u(x_0)} - e^{-kx_0 u(x_0/m)} \right\} > 0,
\]

which contradicts (2.4). Therefore, \( x_0 = 0 \). Similarly, it can be shown that \( x_1 = 0 \). This proves that \( \min_{x \in [0, \tau]} u(x) = \max_{x \in [0, \tau]} u(x) \) and therefore \( u(x) \) is constant. \( \blacksquare \)

**Theorem 2** Suppose that \( \lim_{x \to 0^+} \frac{F(x)}{x} = 0 < \theta < \infty \). Then \( X_1 \) has the exponential distribution (with mean \( 1/\theta \)) if and only if, for some \( m \geq 2 \), the conditional distribution of \( (L^{(k)}(n) - k + 1)X^{(k)}(n) \) given \( L^{(k)}(n) - k + 1 = m \) is identical with the unconditional distribution of \( (L^{(k)}(n) - k + 1)X^{(k)}(n) \).

**Proof** From Lemma 1 and equation (2.1), the conditional survival function of \( Y = (L^{(k)}(n) - k + 1)X^{(k)}(n) \) given \( M = L^{(k)}(n) - k + 1 = m \) is given by

\[
P(Y > x | M = m) = G^m(x/m).
\]

So \( K(x) = P(Y \leq x) \) is independent of \( m \) if and only if, for all \( x > 0 \),

\[
K(x) = \frac{G^m(x/m)}{G(x/m)} = \lim_{r \to \infty} \frac{G^m(x/m)}{G^r(x/m)} = \exp(-k\theta x).
\]

It follows that \( F(x) = \exp(-\theta x), x > 0 \). \( \blacksquare \)

A distribution function \( F(\cdot) \) with \( F(0) = 0 \) and \( \mu = E(X_1) < \infty \) is said to be harmonic new better (worse) than used in expectation abbreviated HNBUE (HNWUE), if \( \int_0^\infty F(t)\,dt \leq (\geq) \mu e^{-t/\mu}, t \geq 0 \). See Basu and Ebrahimi (1986), Kefes (1982) and references there in. These distributions contain all new better (worse) than used in expectation (NBUE, NWUE) distributions and also IFR (DFR), IFRA (DFRA), and NBU (NWU) distributions.

**Theorem 3** Suppose the underlying distribution \( F(\cdot) \) is HNBUE (HNWUE). Then \( E[(L^{(k)}(n) - k + 1)X^{(k)}(n)] = E[X^{(k)}(1)] \) if and only if \( F(\cdot) \) is exponential.
Proof Let $\mu = E \left[ X^{(k)}(1) \right]$ and suppose $F(\cdot)$ is HNBUE (HNWUE). Then we have

$$E \left[ (L^{(k)}(n) - k + 1)X^{(k)}(n) \right] = \sum_{m=n}^{\infty} \frac{|S_{m-1}|}{m!} \int_{0}^{\infty} G^{m}(x/m)dx$$

$$= \sum_{m=n}^{\infty} \frac{|S_{m-1}|}{(m-1)!} \int_{0}^{\infty} G^{m}(y)dy$$

$$\geq (\leq) \sum_{m=n}^{\infty} \frac{|S_{m-1}|}{m!} \mu,$$

with equality if and only if

$$\int_{0}^{\infty} \tilde{G}^{m}(y)dy = \frac{\mu}{m} \text{ for all } m \geq n. \quad (2.6)$$

But note that (2.6) is a necessary and sufficient condition for $F(x)$ to be exponential when $F(x)$ is HNBUE (HNWUE) (see Basu and Kirmani (1986)).

References


