Morse and Morse-Bott Homology

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Betti numbers, Morse theory, and homology

- Betti numbers
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- Transversality
- Morse homology

Perturbations

- Generic perturbations
- Applications of the perturbation approach
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Cascades

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- Applications of the cascade approach
- Cascades and perturbations

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- Multicomplexes and spectral sequences
- The Morse-Bott-Smale multicomplex
Reference


Special volume in honor of Professor Augustin Banyaga on the occasion of his 65th birthday.
Examples for Betti numbers

\[ S^1 \]
\[ S^2 \]

\[ b_0(S^1) = 1 \]
\[ b_1(S^1) = 1 \]
\[ b_2(S^1) = 0 \]

\[ b_0(S^2) = 1 \]
\[ b_1(S^2) = 0 \]
\[ b_2(S^2) = 1 \]
Examples for Betti numbers

\[ b_0(T^2) = 1 \]
\[ b_1(T^2) = 2 \]
\[ b_2(T^2) = 1 \]
Fundamental Idea: There should be a connection between the critical points of a Morse function $f: M \to \mathbb{R}$ and the Betti numbers of $M$. 

Morse functions
Fundamental Idea: There should be a connection between the critical points of a Morse function \( f : M \to \mathbb{R} \) and the Betti numbers of \( M \).
The Morse inequalities

**Weak Morse inequalities:** \( \nu_k(f) \geq b_k(M) \) for all \( k = 0, \ldots, m \), where \( \nu_k(f) = \#Cr_k(f) \) is the number of critical points of index \( k \).

**Strong Morse inequalities:**

\[
\begin{align*}
\nu_0 & \geq b_0 \\
\nu_1 - \nu_0 & \geq b_1 - b_0 \\
\nu_2 - \nu_1 + \nu_0 & \geq b_2 - b_1 + b_0 \\
\vdots & \vdots \\
v_{m-1} - v_{m-2} + \cdots \pm v_0 & \geq b_{m-1} - b_{m-2} + \cdots \pm b_0 \\
v_m - v_{m-1} + v_{m-2} - \cdots \pm v_0 & = b_m - b_{m-1} + b_{m-2} - \cdots \pm b_0
\end{align*}
\]

**Corollary:** \( \chi(M) = b_0 - b_1 + \cdots \pm b_m = (-1)^m \chi(M) \)
The polynomial Morse inequalities

The **Poincaré polynomial** of $M$ is defined to be

$$P_t(M) = \sum_{k=0}^{m} b_k(M)t^k$$

and the **Morse polynomial** of $f : M \to \mathbb{R}$ is defined to be

$$M_t(f) = \sum_{k=0}^{m} \nu_k(f)t^k.$$ 

**Theorem (Polynomial Morse Inequalities):** For any Morse function $f : M \to \mathbb{R}$ on a smooth manifold $M$ we have

$$M_t(f) = P_t(M) + (1 + t)R(t)$$

where $R(t)$ is a polynomial with non-negative integer coefficients.
Stable and unstable manifolds

Let \( p \in M \) be a critical point of a smooth function \( f : M \rightarrow \mathbb{R} \) on a smooth Riemannian manifold \( M \) of dimension \( m < \infty \), and let \( \varphi : \mathbb{R} \times M \rightarrow M \) be the 1-parameter family of diffeomorphisms determined by \(-\nabla f\). The **stable manifold** of \( p \) is

\[
W^s(p) = \{ x \in M | \lim_{t \to \infty} \varphi_t(x) = p \}
\]

and the **unstable manifold** of \( p \) is

\[
W^u(p) = \{ x \in M | \lim_{t \to -\infty} \varphi_t(x) = p \}.
\]

**The Stable/Unstable Manifold Theorem:** If \( p \) is a nondegenerate critical point, then the stable manifold \( W^s(p) \) is a smoothly embedded open disk of dimension \( m - \lambda_p \) and the unstable manifold \( W^u(p) \) is a smoothly embedded open disk of dimension \( \lambda_p \).
Examples for stable and unstable manifolds

\[ f(x, y, z) = z \]
Examples for stable and unstable manifolds
Morse-Smale transversality

A Morse function $f : M \to \mathbb{R}$ is called Morse-Smale if and only if all its stable and unstable manifolds intersect transversally, i.e. $W^u(q) \cap W^s(p)$ for all $p, q \in Cr(f)$.

If $W^u(q) \cap W^s(p) \neq \emptyset$, then this condition implies that $W^u(q) \cap W^s(p)$ is a manifold of dimension $\lambda_q - \lambda_p$ and the moduli space

$$\mathcal{M}(q, p) = (W^u(q) \cap W^s(p)) / \mathbb{R}$$

is a manifold of dimension $\lambda_q - \lambda_p - 1$.

Note: The dimension of $M$ does not affect the dimension of the moduli space $\mathcal{M}(q, p)$.
A Morse-Smale function on a 2-torus
The Morse-Smale-Witten chain complex

Let $f : M \to \mathbb{R}$ be a Morse-Smale function on a compact smooth Riemannian manifold $M$ of dimension $m < \infty$, and assume that orientations for the unstable manifolds of $f$ have been chosen. Let $C_k(f)$ be the free abelian group generated by the critical points of index $k$, and let

$$C_*(f) = \bigoplus_{k=0}^{m} C_k(f).$$

Define a homomorphism $\partial_k : C_k(f) \to C_{k-1}(f)$ by

$$\partial_k(q) = \sum_{p \in C_{r_{k-1}}(f)} n(q, p)p$$

where $n(q, p)$ is the number of gradient flow lines from $q$ to $p$ counted with sign. The pair $(C_*(f), \partial_*)$ is called the Morse-Smale-Witten chain complex of $f$. 

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The height function on the 2-sphere

\[ f(x, y, z) = z \]

\[ C_2(f) \xrightarrow{\partial_2} C_1(f) \xrightarrow{\partial_1} C_0(f) \xrightarrow{} 0 \]

\[ \langle n \rangle \xrightarrow{\partial_2} \langle 0 \rangle \xrightarrow{\partial_1} \langle s \rangle \xrightarrow{} 0 \]
The height function on a deformed 2-sphere

\[
\begin{align*}
C_2(f) &\xrightarrow{\partial_2} C_1(f) & \xrightarrow{\partial_1} C_0(f) &\rightarrow 0 \\
\approx & \quad \approx & \quad \approx & \\
\langle r, s \rangle &\xrightarrow{\partial_2} \langle q \rangle & \xrightarrow{\partial_1} \langle p \rangle &\rightarrow 0
\end{align*}
\]
The height function on a tilted 2-torus

\[
\begin{align*}
\mathbb{T}_s M & \quad \mathbb{T}_r M & \quad \mathbb{T}_q M & \quad \mathbb{T}_p M \\
\uparrow & \quad \uparrow & \quad \uparrow & \quad \uparrow \\
\approx & \quad \approx & \quad \approx & \quad \approx \\
< s > & \quad \mathbb{D}_2 & \quad < q, r > & \quad \mathbb{D}_1 & \quad < p > & \quad \rightarrow 0
\end{align*}
\]
References for Morse homology

A Morse-Bott function on the 2-sphere

\[ f(x, y, z) = z^2 \]

\[ B_0 \quad B_2 \]

\[ S^2 \]

\[ n \quad s \]
A Morse-Bott function on the 2-sphere

Can we construct a chain complex for this function? a spectral sequence? a multicomplex?

\[ f(x, y, z) = z^2 \]
Generics perturbations

**Theorem (Morse 1932)**

Let $M$ be a finite dimensional smooth manifold. Given any smooth function $f : M \rightarrow \mathbb{R}$ and any $\varepsilon > 0$, there is a Morse function $g : M \rightarrow \mathbb{R}$ such that

$$\sup \{|f(x) - g(x)| \mid x \in M\} < \varepsilon.$$

**Theorem**

Let $M$ be a finite dimensional compact smooth manifold. The space of all $C^r$ Morse functions on $M$ is an open dense subspace of $C^r(M, \mathbb{R})$ for any $2 \leq r \leq \infty$ where $C^r(M, \mathbb{R})$ denotes the space of all $C^r$ functions on $M$ with the $C^r$ topology.
**Theorem (Morse 1932)**

Let $M$ be a finite dimensional smooth manifold. Given any smooth function $f : M \rightarrow \mathbb{R}$ and any $\varepsilon > 0$, there is a Morse function $g : M \rightarrow \mathbb{R}$ such that $\sup \{ |f(x) - g(x)| \mid x \in M \} < \varepsilon$.

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Why not just perturb the Morse-Bott function $f : M \rightarrow \mathbb{R}$ to a Morse function?
Fiber bundles and group actions

If $\pi : E \to B$ is a smooth fiber bundle with fiber $F$, and $f$ is a Morse function on $B$, then $f \circ \pi$ is a Morse-Bott function with critical submanifolds diffeomorphic to $F$.

\[
\begin{array}{ccc}
F & \to & E \\
\downarrow & \pi & \\
B & \to & \mathbb{R}
\end{array}
\]
Fiber bundles and group actions

If \( \pi : E \rightarrow B \) is a smooth fiber bundle with fiber \( F \), and \( f \) is a Morse function on \( B \), then \( f \circ \pi \) is a Morse-Bott function with critical submanifolds diffeomorphic to \( F \).

\[
\begin{array}{ccc}
F & \longrightarrow & E \\
\downarrow \pi & & \downarrow \\
B & \longrightarrow & \mathbb{R}
\end{array}
\]

In particular, if \( G \) is a Lie group acting on \( M \), then this might be useful for studying equivariant homology.

\[
\begin{array}{ccc}
M & \longrightarrow & EG \times_G M \\
\downarrow \pi & & \downarrow \\
BG & \longrightarrow & \mathbb{R}
\end{array}
\]
Symplectic and Instanton Floer homology


Generalizations: Donaldson polynomials for 4-manifolds with boundary, knot homology groups, comparing the quantum cup product to the pair of pants product.
An explicit perturbation of $f : M \to \mathbb{R}$

Let $T_j$ be a small tubular neighborhood around each connected component $C_j \subseteq Cr(f)$ for all $j = 1, \ldots, l$. Pick a positive Morse function $f_j : C_j \to \mathbb{R}$ and extend $f_j$ to a function on $T_j$ by making $f_j$ constant in the direction normal to $C_j$ for all $j = 1, \ldots, l$. 

Let $\tilde{T}_j \subset T_j$ be a smaller tubular neighborhood of $C_j$ with the same coordinates as $T_j$, and let $\rho_j$ be a smooth bump function which is constant in the coordinates parallel to $C_j$, equal to 1 on $\tilde{T}_j$, equal to 0 outside of $T_j$, and decreases on $T_j - \tilde{T}_j$ as the coordinates move away from $C_j$. For small $\varepsilon > 0$ (and a careful choice of the metric) this determines a Morse-Smale function $h_\varepsilon = f + \varepsilon \left( \sum_{j=1}^{l} \rho_j f_j \right)$.
An explicit perturbation of \( f : M \to \mathbb{R} \)

Let \( T_j \) be a small tubular neighborhood around each connected component \( C_j \subseteq C^r(f) \) for all \( j = 1, \ldots, l \). Pick a positive Morse function \( f_j : C_j \to \mathbb{R} \) and extend \( f_j \) to a function on \( T_j \) by making \( f_j \) constant in the direction normal to \( C_j \) for all \( j = 1, \ldots, l \).

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An explicit perturbation of $f : M \to \mathbb{R}$

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Let $\tilde{T}_j \subset T_j$ be a smaller tubular neighborhood of $C_j$ with the same coordinates as $T_j$, and let $\rho_j$ be a smooth bump function which is constant in the coordinates parallel to $C_j$, equal to 1 on $\tilde{T}_j$, equal to 0 outside of $T_j$, and decreases on $T_j - \tilde{T}_j$ as the coordinates move away from $C_j$. For small $\varepsilon > 0$ (and a careful choice of the metric) this determines a Morse-Smale function

$$h_\varepsilon = f + \varepsilon \left( \sum_{j=1}^{l} \rho_j f_j \right).$$
Critical points of the perturbed function

If \( p \in C_j \) is a critical point of \( f_j : C_j \to \mathbb{R} \) of index \( \lambda_j \), then \( p \) is a critical point of \( h_\varepsilon \) of index

\[
\lambda_p^{h_\varepsilon} = \lambda_j + \lambda_j^j
\]

where \( \lambda_j \) is the Morse-Bott index of \( C_j \).
Critical points of the perturbed function

If $p \in C_j$ is a critical point of $f_j : C_j \to \mathbb{R}$ of index $\lambda_j^p$, then $p$ is a critical point of $h_\varepsilon$ of index

$$\lambda_p^{h_\varepsilon} = \lambda_j + \lambda_j^p$$

where $\lambda_j$ is the Morse-Bott index of $C_j$.

**Theorem (Morse-Bott Inequalities)**

Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a finite dimensional oriented compact smooth manifold, and assume that all the critical submanifolds of $f$ are orientable. Then there exists a polynomial $R(t)$ with non-negative integer coefficients such that

$$MB_t(f) = P_t(M) + (1 + t)R(t).$$

(Different orientation assumptions in [Banyaga-H 2009] than the proof using the Thom Isomorphism Theorem.)
The idea behind the Banyaga-H proof

\begin{align*}
MB_t(f) &= \sum_{j=1}^{l} P_t(C_j) t^{\lambda_j} \\
&= \sum_{j=1}^{l} \left( M_t(f_j) - (1 + t) R_j(t) \right) t^{\lambda_j} \\
&= \sum_{j=1}^{l} M_t(f_j) t^{\lambda_j} - (1 + t) \sum_{j=1}^{l} R_j(t) t^{\lambda_j} \\
&= M_t(h) - (1 + t) \sum_{j=1}^{l} R_j(t) t^{\lambda_j} \\
&= P_t(M) + (1 + t) R_h(t) - (1 + t) \sum_{j=1}^{l} R_j(t) t^{\lambda_j}
\end{align*}
Cascades (Frauenfelder 2003 and Bourgeois 2002)

Let \( f : M \to \mathbb{R} \) be a Morse-Bott function and suppose

\[
\text{Cr}(f) = \bigsqcup_{j=1}^{l} C_j,
\]

where \( C_1, \ldots, C_l \) are disjoint connected critical submanifolds of Morse-Bott index \( \lambda_1, \ldots, \lambda_l \) respectively. Let \( f_j : C_j \to \mathbb{R} \) be a Morse function on the critical submanifold \( C_j \) for all \( j = 1, \ldots, l \).

**Definition**

If \( q \in C_j \) is a critical point of the Morse function \( f_j : C_j \to \mathbb{R} \) for some \( j = 1, \ldots, l \), then the **total index** of \( q \), denoted \( \lambda_q \), is defined to be the sum of the Morse-Bott index of \( C_j \) and the Morse index of \( q \) relative to \( f_j \), i.e.

\[
\lambda_q = \lambda_j + \lambda^j_q.
\]
A 3-cascade

\[ C_j \quad q \quad x_1(t) \quad y_1(t_1) \quad y_1(0) \quad y_2(t_2) \quad x_2(t) \quad y_2(t) \quad x_3(t) \quad p \quad C_i \quad \]

\[ n=3 \]

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Definition
Denote the space of flow lines from \( q \) to \( p \) with \( n \) cascades by \( W_n^c(q, p) \), and denote the quotient of \( W_n^c(q, p) \) by the action of \( \mathbb{R}^n \) by \( M_n^c(q, p) = W_n^c(q, p)/\mathbb{R}^n \). The set of unparameterized flow lines with cascades from \( q \) to \( p \) is defined to be

\[
M^c(q, p) = \bigcup_{n \in \mathbb{Z}_+} M_n^c(q, p)
\]

where \( M_0^c(q, p) = W_0^c(q, p)/\mathbb{R} \).
The $\mathbb{Z}_2$-cascade chain complex

Define the $k^{\text{th}}$ chain group $C^c_k(f)$ to be the free abelian group generated by the critical points of total index $k$ of the Morse-Smale functions $f_j$ for all $j = 1, \ldots, l$, and define $n^c(q, p; \mathbb{Z}_2)$ to be the number of flow lines with cascades between a critical point $q$ of total index $k$ and a critical point $p$ of total index $k - 1$ counted mod 2. Let

$$C^c_*(f) \otimes \mathbb{Z}_2 = \bigoplus_{k=0}^{m} C^c_k(f) \otimes \mathbb{Z}_2$$

and define a homomorphism $\partial^c_k : C^c_k(f) \otimes \mathbb{Z}_2 \to C^c_{k-1}(f) \otimes \mathbb{Z}_2$ by

$$\partial^c_k(q) = \sum_{p \in Cr(f_{k-1})} n^c(q, p; \mathbb{Z}_2)p.$$

The pair $(C^c_*(f) \otimes \mathbb{Z}_2, \partial^c_*)$ is called the cascade chain complex with $\mathbb{Z}_2$ coefficients.
A cascade chain complex for the 2-sphere

\[ f(x, y, z) = z^2 \]

\[ C_2^c(f) \xrightarrow{\partial_2^c} C_1^c(f) \xrightarrow{\partial_1^c} C_0^c(f) \rightarrow 0 \]

\[ \langle n, s \rangle \xrightarrow{\partial_2^c} \langle q \rangle \xrightarrow{\partial_1^c} \langle p \rangle \rightarrow 0 \]
A cascade chain complex for the 2-torus

\[ C^c_2(f) \xrightarrow{\partial^c_2} C^c_1(f) \xrightarrow{\partial^c_1} C^c_0(f) \rightarrow 0 \]

\[ \langle s \rangle \xrightarrow{\partial^c_2} \langle q, r \rangle \xrightarrow{\partial^c_1} \langle p \rangle \rightarrow 0 \]
The Arnold-Givental conjecture

Let \((M, \omega)\) be a \(2n\)-dimensional compact symplectic manifold, \(L \subset M\) a compact Lagrangian submanifold, and \(R \in \text{Diff}(M)\) an antisymplectic involution, i.e. \(R^*\omega = -\omega\) and \(R^2 = \text{id}\), whose fixed point set is \(L\).

**Conjecture.** Let \(H_t\) be a smooth family of Hamiltonian functions on \(M\) for \(0 \leq t \leq 1\) and denote by \(\Phi_H\) the time-1 map of the flow of the Hamiltonian vector field of \(H_t\). If \(L\) intersects \(\Phi_H(L)\) transversally, then

\[
\#(L \cap \Phi_H(L)) \geq \sum_{k=0}^{n} b_k(L; \mathbb{Z}_2).
\]

Proved by Frauenfelder for a class of Lagrangians in Marsden-Weinstein quotients by letting \(H \to 0\) (2004).
The explicit perturbation of $f : M \to \mathbb{R}$

Let $T_j$ be a small tubular neighborhood around each connected component $C_j \subseteq Cr(f)$ for all $j = 1, \ldots, l$. Pick a positive Morse function $f_j : C_j \to \mathbb{R}$ and extend $f_j$ to a function on $T_j$ by making $f_j$ constant in the direction normal to $C_j$ for all $j = 1, \ldots, l$. Let $\tilde{T}_j \subset T_j$ be a smaller tubular neighborhood of $C_j$ with the same coordinates as $T_j$, and let $\rho_j$ be a smooth bump function which is constant in the coordinates parallel to $C_j$, equal to 1 on $\tilde{T}_j$, equal to 0 outside of $T_j$, and decreases on $T_j - \tilde{T}_j$ as the coordinates move away from $C_j$. For small $\varepsilon > 0$ (and a careful choice of the metric) this determines a Morse-Smale function

$$h_\varepsilon = f + \varepsilon \left( \sum_{j=1}^{l} \rho_j f_j \right).$$
A perturbed Morse-Bott function on the 2-sphere

\[ f(x, y, z) = z^2 \]

\[ C_2(h_\varepsilon) \xrightarrow{\partial_2} C_1(h_\varepsilon) \xrightarrow{\partial_1} C_0(h_\varepsilon) \rightarrow 0 \]

\[ \langle n, s \rangle \xrightarrow{\partial_2} \langle q \rangle \xrightarrow{\partial_1} \langle p \rangle \rightarrow 0 \]
A cascade chain complex for the 2-sphere

\[ C^c_2(f) \xrightarrow{\partial^c_2} C^c_1(f) \xrightarrow{\partial^c_1} C^c_0(f) \rightarrow 0 \]

\[ < n, s > \xrightarrow{\partial^c_2} < q > \xrightarrow{\partial^c_1} < p > \rightarrow 0 \]

\[ f(x, y, z) = z^2 \]
Comparing the cascade and Morse chain complexes

For every sufficiently small \( \varepsilon > 0 \) and \( k = 0, \ldots, m \) we have

\[
C^c_k(f) \approx C_k(h_\varepsilon) = \bigoplus_{\lambda_j+n=k} C_n(f_j).
\]
For every sufficiently small $\varepsilon > 0$ and $k = 0, \ldots, m$ we have

$$C^c_k(f) \approx C_k(h_\varepsilon) = \bigoplus_{\lambda_j + n = k} C_n(f_j).$$

Is $\mathcal{M}^c(q, p) \approx \mathcal{M}_{h_\varepsilon}(q, p)$ when $\lambda_q - \lambda_p = 1$?
Comparing the cascade and Morse chain complexes

For every sufficiently small $\varepsilon > 0$ and $k = 0, \ldots, m$ we have

$$C^c_k(f) \approx C_k(h_\varepsilon) = \bigoplus_{\lambda_j + n = k} C_n(f_j).$$

Is $M^c(q, p) \approx M_{h_\varepsilon}(q, p)$ when $\lambda_q - \lambda_p = 1$?

If so, then we can use the orientations on $M_{h_\varepsilon}(q, p)$ to define the cascade chain complex over $\mathbb{Z}$ so that $\partial^c_k = -\partial_k$ for all $k = 0, \ldots, m$, where $\partial_k$ is the Morse-Smale-Witten boundary operator of $h_\varepsilon$. In particular,

$$H_*((C^c_*(f), \partial^c_*)) \approx H_*(M; \mathbb{Z}).$$
Theorem (Banyaga-H 2013)

Assume that \( f \) satisfies the Morse-Bott-Smale transversality condition with respect to the Riemannian metric \( g \) on \( M \), \( f_k : C_k \to \mathbb{R} \) satisfies the Morse-Smale transversality condition with respect to the restriction of \( g \) to \( C_k \) for all \( k = 1, \ldots, l \), and the unstable and stable manifolds \( W^u_{f_j}(q) \) and \( W^s_{f_i}(p) \) are transverse to the beginning and endpoint maps.

1. When \( n = 0, 1 \) the set \( \mathcal{M}^c_n(q, p) \) is either empty or a smooth manifold without boundary.

2. For \( n > 1 \) the set \( \mathcal{M}^c_n(q, p) \) is either empty or a smooth manifold with corners.

3. The set \( \mathcal{M}^c(q, p) \) is either empty or a smooth manifold without boundary.

In each case the dimension of the manifold is \( \lambda_q - \lambda_p - 1 \). The above manifolds are orientable when \( M \) and \( C_k \) are orientable.
Correspondence of moduli spaces

**Theorem (Banyaga-H 2013)**

Let \( p, q \in \text{Cr}(h_\varepsilon) \) with \( \lambda_q - \lambda_p = 1 \). For any sufficiently small \( \varepsilon > 0 \) there is a bijection between unparameterized cascades and unparameterized gradient flow lines of the Morse-Smale function \( h_\varepsilon : M \to \mathbb{R} \) between \( q \) and \( p \),

\[
\mathcal{M}^c(q, p) \leftrightarrow \mathcal{M}_{h_\varepsilon}(q, p).
\]

**Definition**

Let \( p, q \in \text{Cr}(h_\varepsilon) \) with \( \lambda_q - \lambda_p = 1 \), define an orientation on the zero dimensional manifold \( \mathcal{M}^c(q, p) \) by identifying it with the left hand boundary of \( \mathcal{M}_{h_\varepsilon}(q, p) \times [0, \varepsilon] \).
Main idea: The Exchange Lemma

\[ x_k(t) \quad y_k(0) \quad r_k \quad \tilde{r}_k \quad x_{k+1}(t) \quad y_k(t_k) \]

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Correspondence of chain complexes

Theorem (Banyaga-H 2011)

For $\varepsilon > 0$ sufficiently small we have $C^c_k(f) = C_k(h_{\varepsilon})$ and $\partial^c_k = -\partial_k$ for all $k = 0, \ldots, m$, where $\partial_k$ denotes the Morse-Smale-Witten boundary operator determined by the Morse-Smale function $h_{\varepsilon}$. In particular, $(C^c_*(f), \partial^c_*)$ is a chain complex whose homology is isomorphic to the singular homology $H_*(M; \mathbb{Z})$.

Moral: The cascade chain complex of a Morse-Bott function $f : M \to \mathbb{R}$ is the same as the Morse-Smale-Witten complex of a small perturbation of $f$. 
Let $R$ be a principal ideal domain. A first quadrant multicomplex $X$ is a bigraded $R$-module $\{X_{p,q}\}_{p,q \in \mathbb{Z}^+}$ with differentials

$$d_i : X_{p,q} \to X_{p-i,q+i-1} \quad \text{for all } i = 0, 1, \ldots$$

that satisfy

$$\sum_{i+j=n} d_i d_j = 0 \quad \text{for all } n.$$

A first quadrant multicomplex can be assembled to form a filtered chain complex $((CX)_*, \partial)$ by summing along the diagonals, i.e.

$$(CX)_n \equiv \bigoplus_{p+q=n} X_{p,q} \quad \text{and} \quad F_s(CX)_n \equiv \bigoplus_{p+q=n, p \leq s} X_{p,q}$$

and $\partial_n = d_0 \oplus \cdots \oplus d_n$ for all $n \in \mathbb{Z}^+$. The above relations then imply that $\partial_n \circ \partial_{n+1} = 0$ and $\partial_n(F_s(CX)_*) \subseteq F_s(CX)_*$. 

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A bicomplex has two filtrations, but a general multicomplex only has one filtration.
\[ \cdots \xrightarrow{d_0} X_{3,0} \xrightarrow{d_0} 0 \]
\[ \cdots \xrightarrow{d_0} X_{2,1} \xrightarrow{d_0} X_{2,0} \xrightarrow{d_0} 0 \]
\[ \cdots \xrightarrow{d_0} X_{1,2} \xrightarrow{d_0} X_{1,1} \xrightarrow{d_0} X_{1,0} \xrightarrow{d_0} 0 \]
\[ \cdots \xrightarrow{d_0} X_{0,3} \xrightarrow{d_0} X_{0,2} \xrightarrow{d_0} X_{0,1} \xrightarrow{d_0} X_{0,0} \xrightarrow{d_0} 0 \]
\[ \cdots \xrightarrow{\partial_3} (CX)_3 \xrightarrow{\partial_2} (CX)_2 \xrightarrow{\partial_1} (CX)_1 \xrightarrow{\partial_0} (CX)_0 \xrightarrow{\partial_0} 0 \]
The bigraded module associated to the filtration

\[ F_s(CX)_n \equiv \bigoplus_{p+q=n \atop p \leq s} X_{p,q} \]

is

\[ G((CX)_*)_s,t = F_s(CX)_{s+t}/F_{s-1}(CX)_{s+t} \approx X_{s,t} \]

for all \( s, t \in \mathbb{Z}_+ \), and the \( E^1 \) term of the associated spectral sequence is given by

\[ E^1_{s,t} = Z^1_{s,t} / (Z^0_{s-1,t+1} + \partial Z^0_{s,t+1}) \]

where

\[ Z^1_{s,t} = \{ c \in F_s(CX)_{s+t} | \partial c \in F_{s-1}(CX)_{s+t-1} \} \]

\[ Z^0_{s,t} = \{ c \in F_s(CX)_{s+t} | \partial c \in F_s(CX)_{s+t-1} \} = F_s(CX)_{s+t}. \]
Theorem

Let $(\{X_{p,q}\}_{p,q \in \mathbb{Z}^+}, \{d_i\}_{i \in \mathbb{Z}^+})$ be a first quadrant multicomplex and $((CX)_*, \partial)$ the associated assembled chain complex. Then the $E^1$ term of the spectral sequence associated to the filtration of $(CX)_*$ determined by the restriction $p \leq s$ is given by

$$E^1_{s,t} \approx H_{s+t}(X_{s,*}, d_0)$$

where $(X_{s,*}, d_0)$ denotes the following chain complex.

$$\cdots \xrightarrow{d_0} X_{s,3} \xrightarrow{d_0} X_{s,2} \xrightarrow{d_0} X_{s,1} \xrightarrow{d_0} X_{s,0} \xrightarrow{d_0} 0$$

Moreover, the $d^1$ differential on the $E^1$ term of the spectral sequence is induced from the homomorphism $d_1$ in the multicomplex.
However, $d^r$ is not induced from $d_r$ for $r \geq 2$

Consider the following first quadrant double complex

\[
\begin{align*}
0 & \leftarrow d_1 & 0 & \leftarrow d_1 & 0 \\
\downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 \\
<x_{0,1}> & \leftarrow d_1 & <x_{1,1}> & \leftarrow d_1 & 0 \\
\downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 \\
0 & \leftarrow d_1 & <x_{1,0}> & \leftarrow d_1 & <x_{2,0}>
\end{align*}
\]

where $<x_{p,q}>$ denotes the free abelian group generated by $x_{p,q}$, the groups $X_{p,q} = 0$ for $p + q > 2$, and the homomorphisms $d_0$ and $d_1$ satisfy the following: $d_0(x_{1,1}) = x_{1,0}$, $d_1(x_{1,1}) = x_{0,1}$, and $d_1(x_{2,0}) = x_{1,0}$. In this case, $d_2 = 0$ but $d^2 \neq 0$. 
The Morse-Bott-Smale multicomplex

Let $C_p(B_i)$ be the group of “$p$-dimensional chains” in the critical submanifolds of index $i$. Assume that $f : M \to \mathbb{R}$ is a Morse-Bott-Smale function and the manifold $M$, the critical submanifolds, and their negative normal bundles are all orientable.

If $\sigma : P \to B_i$ is a singular $C_p$-space in $S^\infty_p(B_i)$, then for any $j = 1, \ldots, i$ composing the projection map $\pi_2$ onto the second component of $P \times_{B_i} \overline{M}(B_i, B_{i-j})$ with the endpoint map $\partial_+ : \overline{M}(B_i, B_{i-j}) \to B_{i-j}$ gives a map

$$P \times_{B_i} \overline{M}(B_i, B_{i-j}) \xrightarrow{\pi_2} \overline{M}(B_i, B_{i-j}) \xrightarrow{\partial_+} B_{i-j}.$$
\[
\begin{array}{cccc}
C_1(B_2) & \xrightarrow{\partial_0} & C_0(B_2) & \xrightarrow{\partial_0} 0 \\
\oplus & \downarrow & \downarrow & \uparrow \\
C_2(B_1) & \xrightarrow{\partial_0} & C_1(B_1) & \xrightarrow{\partial_0} C_0(B_1) & \xrightarrow{\partial_0} 0 \\
\oplus & \downarrow & \downarrow & \uparrow & \downarrow & \uparrow \\
C_3(B_0) & \xrightarrow{\partial_0} & C_2(B_0) & \xrightarrow{\partial_0} C_1(B_0) & \xrightarrow{\partial_0} C_0(B_0) & \xrightarrow{\partial_0} 0 \\
\| & \| & \| & \| \\
C_3(f) & \xrightarrow{\partial} & C_2(f) & \xrightarrow{\partial} C_1(f) & \xrightarrow{\partial} C_0(f) & \xrightarrow{\partial} 0 \\
\end{array}
\]
The Morse-Bott Homology Theorem

Theorem (Banyaga-H 2010)

The homology of the Morse-Bott-Smale multicompex \((C_\ast(f), \partial)\) is independent of the Morse-Bott-Smale function \(f : M \to \mathbb{R}\).
Therefore,

\[ H_\ast(C_\ast(f), \partial) \cong H_\ast(M; \mathbb{Z}). \]

Note: If \(f\) is constant, then \((C_\ast(f), \partial)\) is the chain complex of singular \(N\)-cube chains. If \(f\) is Morse-Smale, then \((C_\ast(f), \partial)\) is the Morse-Smale-Witten chain complex. This gives a new proof of the Morse Homology Theorem.
References

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Let $P \to N$ be a (trivial) principal $SU(2)$-bundle over an oriented closed 3-manifold $N$, and let $\mathcal{A}$ be the space of connections on $P$. Define $CS : \mathcal{A} \to \mathbb{R}$ by

$$CS(A) = \frac{1}{4\pi^2} \int_M tr(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A).$$

The above functional descends to a function $cs : \mathcal{A}/\mathcal{G} \to \mathbb{R}/\mathbb{Z}$ whose critical points are gauge equivalence classes of flat connections. Extending everything to $P \times \mathbb{R} \to N \times \mathbb{R}$, the gradient flow equation becomes the instanton equation

$$F + *F = 0,$$

where $F$ denotes the curvature and $*$ is the Hodge star operator.

**Theorem.** When $\mathcal{N}$ is a homology 3-sphere the Chern-Simons functional can be perturbed so that it has discrete critical points and defines $\mathbb{Z}_8$-graded homology groups $I_*(\mathcal{N})$ analogous to the Morse homology groups.

Generalizations: Donaldson polynomials for 4-manifolds with boundary, knot homology groups
The symplectic action functional

Let \((M, \omega)\) be a closed symplectic manifold and \(S^1 = \mathbb{R}/\mathbb{Z}\). A time-dependent Hamiltonian \(H : M \times S^1 \rightarrow \mathbb{R}\) determines a time-dependent vector field \(X_H\) by

\[
\omega(X_H(x, t), v) = v(H)(x, t) \quad \text{for } v \in T_x M.
\]

Let \(\mathcal{L}(M)\) be the space of free contractible loops on \(M\) and

\[
\tilde{\mathcal{L}}(M) = \{(x, u) | x \in \mathcal{L}(M), u : D^2 \rightarrow M \text{ such that } u(e^{2\pi i t}) = x(t)\} / \sim
\]

its universal cover with covering group \(\pi_2(M)\). The symplectic action functional \(a_H : \tilde{\mathcal{L}}(M) \rightarrow \mathbb{R}\) is defined by

\[
a_H((x, u)) = \int_{D^2} u^* \omega + \int_0^1 H(x(t), t) \, dt.
\]

**Theorem.** Let \((P, \omega)\) be a compact symplectic manifold. If \(I_\omega\) and \(I_c\) are proportional, then the fixed point set of every exact diffeomorphism of \((P, \omega)\) satisfies the Morse inequalities with respect to any coefficient ring whenever it is nondegenerate.

Generalizations: Allowing \(H\) to be degenerate (e.g. \(H = 0\)) leads to critical submanifolds and Morse-Bott homology.
The Yang-Mills gradient flow

Let $(\Sigma, g)$ be a closed oriented Riemann surface, $G$ a compact Lie group, $\mathfrak{g}$ its Lie algebra, and $P$ a principal $G$-bundle over $\Sigma$. Pick an ad-invariant inner product on $\mathfrak{g}$, let $\mathcal{A}(P)$ denote the affine space of $\mathfrak{g}$-valued connection 1-forms on $P$, and define

$$\mathcal{YM} : \mathcal{A}(P) \to \mathbb{R}$$

by

$$\mathcal{YM}(A) = \int_{\Sigma} F_A \wedge *F_A$$

where $F_A = dA + \frac{1}{2}[A \wedge A]$ is the curvature of $A$.

The Yang-Mills function is a Morse-Bott function studied by Atiyah-Bott and by Swoboda (2011) using cascades.
Closed Reeb orbits

Let $M$ be a compact, orientable manifold of dimension $2n - 1$ with contact form $\alpha$. The **Reeb vector field** $R_\alpha$ associated to the contact form $\alpha$ is characterized by

$$
\begin{align*}
  d\alpha(R_\alpha, -) &= 0 \\
  \alpha(R_\alpha) &= 1.
\end{align*}
$$

Closed trajectories of the Reeb vector field are critical points of the action functional $A : C^\infty(S^1, M) \to \mathbb{R}$

$$
A(\gamma) = \int_\gamma \alpha.
$$

**Lemma.** For any contact structure $\xi$ on $M$, there exists a contact form $\alpha$ for $\xi$ such that all closed orbits of $R_\alpha$ are nondegenerate.
Contact homology

Let $A$ be the graded supercommutative algebra freely generated by the “good” closed Reeb orbits over the graded ring $\mathbb{Q}[H_2(M;\mathbb{Z})/\mathcal{R}]$, i.e. $\gamma_1 \gamma_2 = (-1)^{|\gamma_1||\gamma_2|} \gamma_2 \gamma_1$.

**Theorem.** (Eliashberg-Hofer 2000) There is a differential $d : A \to A$ defined by counting $J$-holomorphic curves in the symplectization $(\mathbb{R} \times M, d(e^t \alpha))$ such that $(A, d)$ is a differential graded algebra. Moreover, $HC_*(M, \xi) \overset{\text{def}}{=} H_*(A, d)$ is an invariant of the contact structure $\xi$.

**Theorem.** (Bourgeois 2002) Assume that $\alpha$ is a contact form of Morse-Bott type for $(M, \xi)$ and that $J$ is an almost complex structure on the symplectization that is $S^1$-invariant along the critical submanifolds $N_T$. Then there is a chain complex with a boundary operator defined by counting cascades whose homology is isomorphic to the contact homology $HC_*(M, \xi)$.
Viterbo’s symplectic homology

Definition
A compact symplectic manifold \((W, \omega)\) has **contact type** boundary if and only if there exists a vector field \(X\) defined in a neighborhood of \(M = \partial W\) transverse and pointing outward along \(M\) such that \(\mathcal{L}_X \omega = \omega\).

In this case, \(\lambda = \omega(X, \cdot)|_M\) is a contact form on \(M\), and the symplectic homology of \(W\) combines the 1-periodic orbits of a Hamiltonian on \(W\) with the Reeb orbits on \(M = \partial W\).

Bourgeois and Oancea have defined the cascade chain complex for a time-independent Hamiltonian on \(W\) whose 1-periodic orbits are transversally nondegenerate (2009). They have also proved that there is an exact sequence relating the symplectic homology groups of \(W\) with the linearized contact homology groups of \(M\) (2009).
Compactness

Denote the space of nonempty closed subsets of $M \times \mathbb{R}^l$ in the topology determined by the Hausdorff metric by $\mathcal{P}^c(M \times \mathbb{R}^l)$, and map a broken flow line with cascades $(v_1, \ldots, v_n)$ to its image $\text{Im}(v_1, \ldots, v_n) \subset M$ and the time $t_j$ spent flowing along or resting on each critical submanifold $C_j$ for all $j = 1, \ldots, l$.

**Theorem (Banyaga-H 2013)**

The space $\overline{M}^c(q, p)$ of broken flow lines with cascades from $q$ to $p$ is compact, and there is a continuous embedding

$$\mathcal{M}^c(q, p) \hookrightarrow \overline{M}^c(q, p) \subset \mathcal{P}^c(M \times \mathbb{R}^l).$$

Hence, every sequence of unparameterized flow lines with cascades from $q$ to $p$ has a subsequence that converges to a broken flow line with cascades from $q$ to $p$. 