The beginnings of trigonometry are obscure. So far as the pre-Hellenic period is concerned, there are some problems in the Rhind papyrus (ca. 1650 B.C.) that involve the cotangent of the dihedral angle at the base of a regular square pyramid, and there is the Babylonian cuneiform tablet known as Plimpton 322* (1900 to 1600 B.C.), which essentially contains a remarkable table of secants of fifteen angles ranging between 45° and 30°. It may well be that further studies into the mathematics of ancient Mesopotamia will disclose a substantial development of practical trigonometry. Babylonian astronomers had amassed a considerable collection of observational data, and we know that much of this information passed on to the Greeks. It was this early astronomy that gave birth to spherical trigonometry.

One of the earliest of the Greek astronomers was Aristarchus of Samos (ca. 310-230 B.C.), who is said to have applied mathematics to astronomy and to be the first to put forward the heliocentric theory of the solar system.† Nothing of his writings has come down to us, but it has been reported that, in his tract *On Sizes and Distances of the Sun and Moon*, he used the equivalent of the fact that

$$
\frac{\sin a}{\sin b} < \frac{a}{b} < \frac{\tan a}{\tan b},
$$

where $0 < b < a < \pi/2$.

*That is, the item with catalogue number 322 in the G. A. Plimpton archaeological collection at Columbia University.

†Aristarchus is sometimes referred to as the “Copernicus of antiquity.”
The next eminent Greek mathematician-astronomer of antiquity known to us was Hipparchus, who was born in Nicaea of Asia Minor and who flourished around 140 B.C. Though Hipparchus reported an observation of the vernal equinox at Alexandria in 146 B.C., his most important astronomical observations were made at the famous observatory at Rhodes. Renowned as a careful and precise observer, he is credited with such accomplishments as the determination of the mean lunar month to within one second of the present accepted value, an accurate calculation of the inclination of the ecliptic, and the discovery and estimation of the annual precession of the equinoxes. He is said also to have computed the lunar parallax, to have determined the moon's perigee, and to have catalogued 850 fixed stars. He advocated the use of latitude and longitude to locate positions on the earth's surface, and he may have been the first to introduce into Greece the division of a circle into 360°. Though our knowledge of these achievements is only second-hand—inasmuch as almost nothing of Hipparchus' writings has come down to us—the implication is that Hipparchus was aware of the basic trigonometry of the celestial sphere.

A more direct, and very important, connection of Hipparchus with trigonometry is the crediting to Hipparchus, by the fourth-century commentator Theon of Alexandria, of a 12-book treatise dealing with the construction of a table of chords. This table is lost to us, but a subsequent table, given by Claudius Ptolemy (ca. 85-ca. 165)* and believed to have been adapted from Hipparchus' treatise, has survived. Ptolemy's table gives the lengths of the chords of all central angles of a given circle by half-degree intervals from $\frac{1}{2}^\circ$ to $180^\circ$. The radius of the circle is divided into 60 equal parts and the chord lengths then expressed sexagesimally in terms of one of these parts as a unit. Thus, using the symbolism $\text{crd} \alpha$ to represent the length of the chord of a central angle $\alpha$, one finds recordings like $\text{crd} 36^\circ = 37p 4' 55''$, meaning that the chord of a central angle of $36^\circ$ is equal to $37/60$ (or 37 small parts) of the radius, plus $4/60$ of one of these small parts,

*Not to be confused with any of the erstwhile kings of Egypt bearing the name Ptolemy.
plus $55/3600$ more of one of these small parts. It is evident from Figure 30 that a table of chords is equivalent to a table of trigonometric sines, for

$$\sin \alpha = \frac{AM}{OA} = \frac{AB}{\text{diameter of circle}} = \frac{\text{crd } 2\alpha}{120}.$$ 

Thus Ptolemy's table of chords gives, in reality, the sines of angles by quarter-degree intervals from $0^\circ$ to $90^\circ$. There are reports that Hipparchus made systematic use of his table of chords and apparently was aware of the equivalent of several formulas now used in the solution of right spherical triangles.

Theon has also mentioned a six-book treatise on chords in a circle written by Menelaus of Alexandria (ca. 100), but this work also, along with a variety of others by Menelaus, is lost to us. There is, however, a three-book treatise by Menelaus, called *Sphaerica*, that has been preserved in the Arabic. This work throws considerable light on the Greek development of trigonometry.

The disappearance of so much of the early Greek work in astronomy is due to the fact that Ptolemy wrote a treatise that so
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eclipsed these earlier works that they were rendered superfluous. It was about A.D. 150 that Ptolemy wrote his great definitive Greek work on astronomy. This highly influential treatise, called the *Syntaxis mathematica*, or "Mathematical Collection," is remarkable for its completeness, compactness, and elegance. To distinguish it from other, lesser works on astronomy, later commentators assigned to it the superlative *magiste*, or "greatest." Still later, the Arabian translators prefixed the Arabian article *al*, and the work has ever since been known as the *Almagest*. The treatise is in thirteen books, and it is in Book I that we find, among some preliminary astronomical material, the table of chords referred to above, along with a succinct explanation of its derivation from a fertile geometrical proposition now known as Ptolemy's theorem: *In a cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the two pairs of opposite sides.*

Certainly practical trigonometry cannot progress very far without the use of so-called trigonometric tables. The earliest systematic construction of an applicable trigonometric table therefore marks a great moment in mathematics. It is the purpose of the rest of this lecture to give the gist of Ptolemy's method of constructing his highly useful table of chords, or table of sines.* For convenience of presentation and ease of understanding, we shall use modern algebraic notation, employ modern decimal fractions in place of the ancient sexagesimal fractions, and carry out the details of the development in small steps.

1. We start by establishing Ptolemy’s theorem quoted above. To this end let $ABCD$ (see Figure 31) be any simple quadrilateral inscribed in a circle and let $E$ be the point on the diagonal $AC$ such that $\angle ABE = \angle DBC$. From the similar triangles $ABE$ and $DBC$ we have $AB/AE = DB/DC$, whence $(AB)(DC) = (DB)(AE)$. Again, from the similar triangles $ABD$ and $EBC$ we have $AD/DB = EC/CB$, whence $(AD)(CB) = (DB)(EC)$. It follows that

*It is likely that the method had been earlier employed by Hipparchus. The much briefer secant table found in Plimpton 322 appears to have been cleverly constructed from a collection of primitive Pythagorean triangles and is not nearly as applicable as Ptolemy's table.*
\[(AB)(DC) + (AD)(CB) = DB(AE + EC) = (DB)(AC),\]
and the theorem is established.

We now establish three corollaries to Ptolemy’s theorem.

2. **Corollary 1.** If \(a\) and \(b\) are the chords of two arcs of a circle of unit radius, then

\[s = \frac{a}{2}(4 - b^2)^{1/2} + \frac{b}{2}(4 - a^2)^{1/2}\]

is the chord of the sum of the two arcs.

Apply Ptolemy’s theorem to the quadrilateral of Figure 32 where \(AC\) is a diameter, \(BC = a\), and \(CD = b\).

3. **Corollary 2.** If \(a\) and \(b\), \(a \geq b\), are the chords of two arcs of a circle of unit radius, then

\[d = \frac{a}{2}(4 - b^2)^{1/2} - \frac{b}{2}(4 - a^2)^{1/2}\]

is the chord of the difference of the two arcs.
Apply Ptolemy's theorem to the quadrilateral of Figure 33, where $AB$ is a diameter, $BD = a$, and $BC = b$.

4. **COROLLARY 3.** If $t$ is the chord of a minor arc of a circle of unit radius, then

$$h = [2 - (4 - t^2)^{1/2}]^{1/2}$$

is the chord of half the arc.

Apply Ptolemy's theorem to the quadrilateral of Figure 34 where $AC$ is a diameter, $BD = t$ and $BD$ is perpendicular to $AC$. We obtain

$$2t = 2h(4 - h^2)^{1/2},$$

whence, by squaring and rearranging the terms, we find

$$h^4 - 4h^2 + t^2 = 0.$$ Solving this as a quadratic in $h^2$ we obtain

$$h^2 = 2 \pm (4 - t^2)^{1/2}.$$
Since $h$ represents the chord of half the minor arc of the chord $BD$, we require the minus sign in the above result. Taking square roots we finally obtain

$$h = \sqrt{2 - (4 - t^2)^{1/2}}.$$

5. Consider an isosceles triangle $AOB$ (see Figure 35) with vertex angle $AOB = 36^\circ$. Draw $AC$ bisecting $\triangle BAO$. Then from similar triangles $AOB$ and $BAC$ we have $AB/CB = OB/AB$. Setting $AB = x$ and taking $OB = 1$, we find

$$x/(1 - x) = 1/x \text{ or } x^2 + x - 1 = 0,$$

whence (to four-decimal-place accuracy)

$$x = (\sqrt{5} - 1)/2 = 0.6180.$$

It follows that in a circle of unit radius, $\text{crd } 36^\circ = 0.6180$.*

---

*This is, of course, the golden ratio considered in Lecture 5.
6. Since in a circle of unit radius, \( \text{crd} \, 60° = 1 \), we now find, by Corollary 2 above, that in the unit circle

\[
\text{crd} \, 24° = \text{crd}(60° - 36°) = 0.4158.
\]

7. By Corollary 3 we may now successively calculate the chords, in the unit circle, of 12°, 6°, 3°, 90′, and 45′, obtaining

\[
\text{crd} \, 90′ = 0.0262 \quad \text{and} \quad \text{crd} \, 45′ = 0.0131.
\]

8. By the relation

\[
\frac{\sin a}{\sin b} < \frac{a}{b}, \quad b < a < 90°,
\]

the equivalent of which we pointed out early in our lecture was known to Aristarchus, we have

\[
\text{crd} \, 60′/\text{crd} \, 45′ < 60/45 = 4/3,
\]

or

\[
\text{crd} \, 1° < (4/3)(0.0131) = 0.01747.
\]

Also

\[
\text{crd} \, 90′/\text{crd} \, 60′ < 90/60 = 3/2,
\]

or

\[
\text{crd} \, 1° > (2/3)(0.0262) = 0.01747.
\]

It follows that, to four decimal place accuracy, \( \text{crd} \, 1° = 0.0175 \).

9. By Corollary 3 we may now find \( \text{crd} \, \frac{1}{2}° \).

10. Now one can construct a table of chords in the unit circle for \( \frac{1}{2}° \) intervals.

Much of the subsequent work in practical trigonometry was the construction of ever better trigonometric tables. Thus the tenth-century Moslem mathematician Abū'l-Wefā (940–998) computed a table of sines and tangents for 15′ intervals. Later, a table of sines was computed by the Viennese mathematician Georg von Peurbach (1423–1461) and a table of tangents by the German mathematician
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Johann Müller* (1436–1476). George Joachim Rhaeticus (1514–1576), the leading Teutonic mathematical astronomer of the sixteenth century, spent 12 years with hired computers forming two remarkable and still useful trigonometric tables. One was a 10-place table of all six of the trigonometric functions, for every 10" of arc; the other was a 15-place table for sines of every 10" of arc, along with the first, second, and third differences. It is interesting that the Mathematics Advisory Board of the well-known CRC Handbook of Tables for Mathematics almost voted to delete the trigonometric tables from the Handbook's fifth edition; the wide proliferation of pocket calculators has rendered these tables rather superfluous.

We conclude with a brief word about the meanings of the present names of the trigonometric functions. With the exception of sine, the meanings are all clear from the geometrical interpretations of the functions when the angle is taken as a central angle of a circle of unit radius. Thus, in Figure 36, if the radius of the circle is one unit, the measures of \( \tan \alpha \) and \( \sec \alpha \) are given by the lengths of the tangent segment \( CD \) and the secant segment \( OD \). And, of course, cotangent merely means "complement's tangent," and so on. The tangent, cotangent, secant, and cosecant functions have been known by various other names, these present ones not appearing until the end of the sixteenth century.

The origin of our word sine is deeper. The Hindu mathematician, Āryabhata the Elder (ca. 475–ca. 550), called it \( \text{ardhā-jyā} \) ("half-chord") and also \( \text{jyā-ardhā} \) ("chord-half"), and then abbreviated the term by simply using \( \text{jiyā} \) ("chord"). From \( \text{jyā} \) the Arabs phonetically derived \( \text{jiba} \), which, following the Arabian practice of omitting vowels, was written as \( \text{jb} \). Now \( \text{jiba} \), aside from its technical significance, is a meaningless word in Arabic. Later writers, coming across \( \text{jb} \), as an abbreviation for the meaningless \( \text{jiwa} \) and knowing Arabic but no Sanskrit, substituted \( \text{jaib} \) instead, which contains the same letters and is a good Arabic word meaning "cove" or "bay." Still later, Gherardo of Cremona (ca. 1150), when he made his translations from the Arabic into Latin, replaced the Arabic \( \text{jaib} \) by its Latin equivalent \( \text{sinus} \), whence came our present word \( \text{sine} \).

*Better known as Regiomontanus, a Latinized form of his birthplace of Königsberg ("king's mountain").
Exercises

10.1. From a knowledge of the graphs of the functions $\sin x$ and $\tan x$ show that $(\sin x)/x$ decreases and $(\tan x)/x$ increases as $x$ increases from 0 to $\pi/2$, and thus establish the inequalities

$$\frac{\sin a}{\sin b} < \frac{a}{b} < \frac{\tan a}{\tan b},$$

where $0 < b < a < \pi/2$.

10.2. Show that Corollaries 1, 2, and 3 of the lecture text are equivalent to the trigonometric identities

$$\sin(a + \beta) = \sin a \cos \beta + \cos a \sin \beta,$$
$$\sin(a - \beta) = \sin a \cos \beta - \cos a \sin \beta,$$
$$\sin(\theta/2) = [(1 - \cos \theta)/2]^{1/2},$$

where $0 < \alpha, \beta, \theta/2 < \pi/2$.

10.3. Establish the following consequences of Ptolemy's theorem:
If $P$ lies on the arc $AB$ of the circumcircle of...
(a) an equilateral triangle \(ABC\), then \(PC = PA + PB\).
(b) a square \(ABCD\), then \((PA + PC)PC = (PB + PD)PD\).
(c) a regular pentagon \(ABCDE\), then \(PC + PE = PA + PB + PD\).
(d) a regular hexagon \(ABCDEF\), then \(PD + PE = PA + PB + PC + PF\).

10.4. A point lying on a side line of a triangle, but not coinciding with a vertex of the triangle, is called a menelaus point of the triangle for this side. Prove the following chain of theorems, wherein all segments and angles are directed (or sensed) segments and angles:

(a) **Menelaus' theorem.** A necessary and sufficient condition for three menelaus points \(D, E, F\) for the sides \(BC, CA, AB\) of a triangle \(ABC\) to be collinear is that

\[
\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1.
\]

(b) If vertex \(O\) of a triangle \(BOC\) is joined to a point \(D\) (other than \(B\) or \(C\)) on line \(BC\), then

\[
\frac{BD}{DC} = \frac{OB \sin BOD}{OC \sin DOC}.
\]

(c) Let \(D, E, F\) be menelaus points on the sides \(BC, CA, AB\) of a triangle \(ABC\), and let \(O\) be a point in space not in the plane of triangle \(ABC\). Then the points \(D, E, F\) are collinear if and only if

\[
\frac{\sin BOD}{\sin DOC} \cdot \frac{\sin COE}{\sin EOA} \cdot \frac{\sin AOF}{\sin FOB} = -1.
\]

(d) Let \(D', E', F'\) be three menelaus points on the sides \(B'C', C'A', A'B'\) of a spherical triangle \(A'B'C'\). Then \(D', E', F'\), lie on a great circle of the sphere if and only if

\[
\frac{\sin B'D'}{\sin D'C'} \cdot \frac{\sin C'E'}{\sin E'A'} \cdot \frac{\sin A'F'}{\sin F'B'} = -1.
\]
(This is the spherical case of the Menelaus theorem used by Menelaus in his *Sphaerica*.)

10.5. Establish the following chain of theorems:

(a) The product of two sides of a triangle is equal to the product of the altitude on the third side and the diameter of the circumscribed circle.

(b) Let $ABCD$ be a cyclic quadrilateral of diameter $t$. Denote the lengths of sides $AB$, $BC$, $CD$, $DA$ by $a$, $b$, $c$, $d$, the diagonals $BD$ and $AC$ by $m$ and $n$, and the angle between either diagonal and the perpendicular upon the other by $\theta$. Then

$$mt \cos \theta = ab + cd, \quad nt \cos \theta = ad + bc.$$ 

(c) In the above quadrilateral

$$m^2 = \frac{(ac + bd)(ab + cd)}{ad + bc},$$

$$n^2 = \frac{(ac + bd)(ad + bc)}{ab + cd}.$$ 

(d) If, in the above quadrilateral, the diagonals are perpendicular to each other, then

$$t^2 = \frac{(ad + bc)(ab + cd)}{ac + bd}.$$ 

(e) *Ptolemy's second theorem*. In the above quadrilateral

$$\frac{n}{m} = \frac{ad + bc}{ab + cd}.$$ 

For those interested we here state an extension of Ptolemy's theorem and a singularly beautiful generalization of the theorem.

*Extension of Ptolemy's theorem*. In a convex quadrilateral $ABCD$,

$$(BC)(AD) + (CD)(AB) \geq (BD)(AC),$$

with equality if and only if the quadrilateral is cyclic.
Generalization of Ptolemy's theorem. Let $T_1T_2T_3T_4$ be a convex quadrilateral inscribed in a circle $C$. Let $C_1, C_2, C_3, C_4$ be four circles touching circle $C$ externally at $T_1, T_2, T_3, T_4$, respectively. Then

$$t_{12}t_{34} + t_{23}t_{41} = t_{13}t_{24},$$

where $t_{ij}$ is the length of a common external tangent to circles $C_i$ and $C_j$. [This is a special case of a more general theorem due to John Casey (1820-1891).]

Further Reading
