# The Picard group of the moduli space of vector bundles on the quadric surface

arXiv: 2007.11666

Dmitrii Pedchenko

FRG Workshop on Moduli Spaces and Stability, December 2020

The Pennsylvania State University

$X \;=\;$ smooth projective surface $/\mathbb{C}$,

$H \;=\;$ ample divisor on $X$.

For a vector bundle $\mathcal{V}$, set $\mu(\mathcal{V}) = \frac{c_1(\mathcal{V})\cdot H}{r(\mathcal{V})}$.

## Definition

Vector bundle $\mathcal{V}$ is slope (semi)stable if for any subbundle $\mathcal{E} \subset \mathcal{V}$, we have

$$\mu(\mathcal{E}) \underset{(-)}{<} \mu(\mathcal{V}).$$

Key fact: for $\mathcal{V}, \mathcal{W}$ semistable with $\mu(\mathcal{V}) > \mu(\mathcal{W})$, we have $\mathrm{Hom}(\mathcal{V}, \mathcal{W}) = 0$.

---

Fix numerical invariants $\mathbf{v} = (r, ch_1, ch_2) \in K(X)$.

## Theorem *(Mumford, Gieseker, Maruyama, Simpson, Álvarez-Cónsul, King)*

There is a projective moduli space $M(\mathbf{v})$ for semistable bundles on $X$.

We will be interested in the Picard group of the moduli space:

$$\mathrm{Pic}\,(M(\mathbf{v})),$$

when $X = \mathbb{P}^1 \times \mathbb{P}^1$. Previous work: Yoshioka, Nakashima, Qin.

## Constructing line bundles on $M(\mathbf{v})$

$\mathcal{U}/S =$ flat family of bundles of Chern character $\mathbf{v}$ on $X$:

$$\begin{array}{ccc} & \mathcal{U} & \\ & | & \\ & X \times S & \\ {}^{q}\swarrow & & \searrow^{p} \\ X & & S \end{array}$$

The Donaldson homomorphism is a map $\lambda_{\mathcal{U}} : K(X) \to \mathrm{Pic}(S)$ defined by

$$\begin{array}{ccccccccc} K(X) & \xrightarrow{q^*} & K^0(X \times S) & \xrightarrow{\mathcal{U}\cdot} & K^0(X \times S) & \xrightarrow{p_!} & K^0(S) & \xrightarrow{\det} & \mathrm{Pic}(S) \\ \mathcal{E} & \longmapsto & q^*\mathcal{E} & \longmapsto & \mathcal{U} \otimes q^*\mathcal{E} & \longmapsto & p_!(\mathcal{U} \otimes q^*\mathcal{E}) & \longmapsto & \det(p_!(\mathcal{U} \otimes q^*\mathcal{E})) \end{array}$$

---

$\mathcal{U}/M(\mathbf{v}) =$ universal family of semistable bundles of character $\mathbf{v}$ on $X$.

Set $\mathbf{v}^{\perp} = \{\mathbf{e} \in K(X) \mid \chi(\mathbf{e} \cdot \mathbf{v}) = 0\}$.

The Donaldson homomorphism $\mathbf{v}^{\perp} \xrightarrow{\lambda} \mathrm{Pic}(M(\mathbf{v}))$ gives a natural way to construct line bundles on $M(\mathbf{v})$.

# The $X = \mathbb{P}^2$ case

## Definition

Vector bundle $E$ is exceptional if

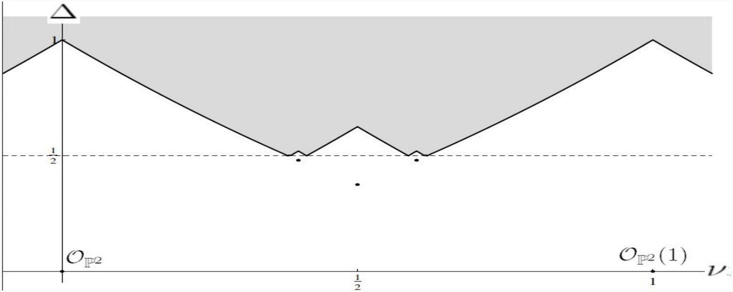
$$\mathrm{Hom}(E, E) = \mathbb{C},$$
$$\mathrm{Ext}^i(E, E) = 0 \text{ for } i > 0.$$

Exceptional bundles are semistable.

Introduce $\mathbf{v} = (r, \nu, \Delta)$ with $\nu = \frac{c_1}{r}$, $\Delta = \frac{1}{2}\nu^2 - \frac{ch_2}{r}$.

## Theorem (Drézet, Le Potier '85)

$$\dim M(\mathbf{v}) > 0 \iff \Delta \geq \mathrm{DLP}(\nu)$$



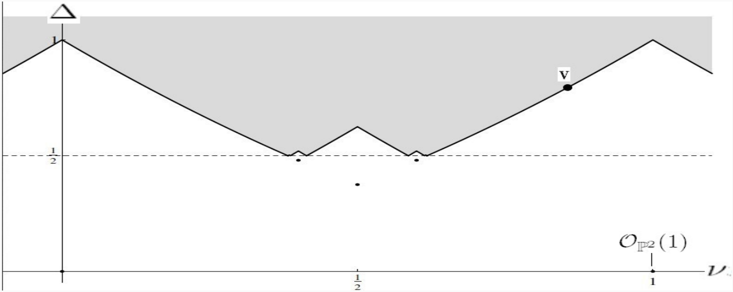

The DLP curve: graph of $\Delta = \mathrm{DLP}(\nu)$ for $X = \mathbb{P}^2$.

The branches of the DLP curve are constructed using exceptional bundles: if $E$ satisfies $0 \leq \mu(E) - \mu(\mathbf{v}) < 3$, then

$$\mathrm{Hom}(E, \mathcal{V}) = 0 \text{ by semistability, } \quad \mathrm{Ext}^2(E, \mathcal{V}) = 0 \text{ by Serre duality and semistability } \implies \chi(E, \mathcal{V}) \leq 0.$$

Using Riemann-Roch, this is a numerical condition $\Delta \geq \mathrm{DLP}(\nu)$ on $\mathbf{v} = (r, \nu, \Delta)$.

# The $X = \mathbb{P}^2$ case




The DLP curve: graph of $\Delta = \mathrm{DLP}(\nu)$ for $X = \mathbb{P}^2$.

Recall the Donaldson homomorphism $\mathbf{v}^{\perp} \xrightarrow{\lambda} \mathrm{Pic}(M(\mathbf{v}))$.
We have $K(\mathbb{P}^2) \simeq \mathbb{Z}^3$ and $\mathbf{v}^{\perp} = \{\mathbf{e} \in K(\mathbb{P}^2) \mid \chi(\mathbf{e} \cdot \mathbf{v}) = 0\} \simeq \mathbb{Z}^2$.

## Definition

If $\mathbf{v}$ lies on the branch of the DLP curve given by the exceptional bundle $E$, then we say $E$ is associated to $\mathbf{v}$.

$$\chi(E, \mathcal{V}) = 0 \implies \mathrm{Ext}^i(E, \mathcal{V}) = 0$$

## Theorem *(Drézet '88)*

Let $\mathbf{v} = (r, \nu, \Delta)$ be a character with $\dim M(\mathbf{v}) > 0$.

1. Above DLP: if $\Delta > \mathrm{DLP}(\nu)$, then $\lambda$ is an isomorphism and
$$\mathrm{Pic}(M(\mathbf{v})) \simeq \mathbb{Z}^2.$$
2. On DLP: if $\Delta = \mathrm{DLP}(\nu)$, then $\lambda$ is surjective,
$$\mathrm{Pic}(M(\mathbf{v})) \simeq \mathbb{Z},$$
and $\ker(\lambda) = \mathbb{Z}[\overline{E}]$, where $E$ is associated to $\mathbf{v}$ and $\overline{E}$ is either $E$ or $E^{\vee}$.

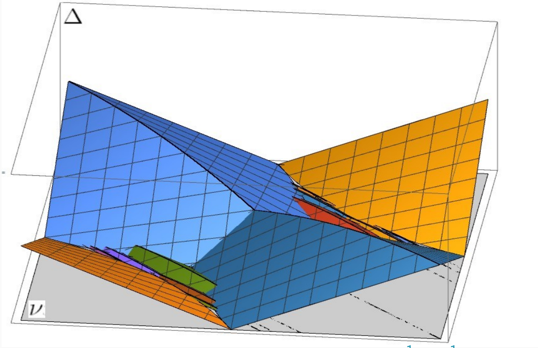
**Theorem** *(Rudakov '94)*

$$\dim M(\mathbf{v}) > 0, \quad H = F_1 + (1 \pm \epsilon) F_2$$
$$\Updownarrow$$
$$\Delta \geq \text{DLP}(\nu)$$

$\nu \in \text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)_{\mathbb{Q}} = \{aF_1 + bF_2 \,|\, a, b \in \mathbb{Q}\}$.

$\mathbf{v}^{\perp} = \{\mathbf{e} \in K(\mathbb{P}^1 \times \mathbb{P}^1) \,|\, \chi(\mathbf{e} \cdot \mathbf{v}) = 0\} \simeq \mathbb{Z}^3$.

The Picard number is no longer controlled only by the exceptional bundles.




The DLP surface: graph of $\Delta = \text{DLP}(\nu)$ for $X = \mathbb{P}^1 \times \mathbb{P}^1$

**Theorem** *(P. '20)*

Let $\mathbf{v} = (r, \nu, \Delta)$ be a character with $\dim M(\mathbf{v}) > 0$. Let $\mathbf{v}^{\perp} \xrightarrow{\lambda} \text{Pic}(M(\mathbf{v}))$ be the Donaldson homomorphism.

1. Above DLP: $\lambda$ is an isomorphism and $\text{Pic}(M(\mathbf{v})) \simeq \mathbb{Z}^3$.
2. On one branch of DLP: $\lambda$ is surjective, and if $E$ is associated to $\mathbf{v}$, then either

$$a)\ \text{Pic}(M(\mathbf{v})) \simeq \mathbb{Z}^2,\ \ker(\lambda) = \mathbb{Z}[\overline{E}], \text{ or}$$

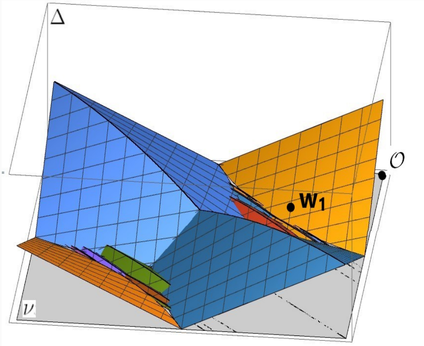
$$(!)\ b)\ \text{Pic}(M(\mathbf{v})) \simeq \mathbb{Z},\ \ker(\lambda) \supsetneq \mathbb{Z}[\overline{E}].\ (!)$$

3. On two branches of DLP: $\lambda$ is surjective, and if $E_1, E_2$ are associated to $\mathbf{v}$, then

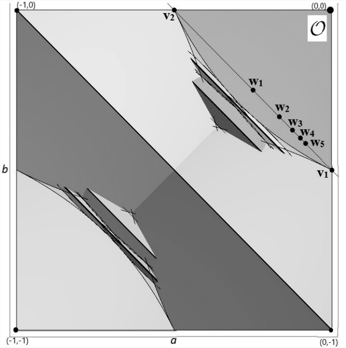
$$\text{Pic}(M(\mathbf{v})) \simeq \mathbb{Z}, \quad \ker(\lambda) \simeq \mathbb{Z}[\overline{E_1}] + \mathbb{Z}[\overline{E_2}].$$

Take $\mathbf{w}_1 = (r, \nu, \Delta) = (4, -\frac{1}{4}F_1 - \frac{1}{4}F_2, \frac{9}{16})$; line bundle $\mathcal{O}$ is associated to $\mathbf{w}_1$. Then

$$M(\mathbf{w}_1) \simeq \mathbb{P}^3.$$




The DLP surface: graph of $\Delta = \mathrm{DLP}(\nu)$ for $X = \mathbb{P}^1 \times \mathbb{P}^1$




Top-down view of the DLP surface

One can iteratively construct an infinite sequence $\{\mathbf{w}_k\}_{k\in\mathbb{N}}$ with $\mathrm{Pic}(M(\mathbf{w}_k)) \simeq \mathbb{Z}$.

# Idea of the proof

1. Build a family $\mathcal{V}_t/T$ of bundles of character $\mathbf{v}$ admitting convenient resolutions
2. Show $\mathrm{Pic}(M(\mathbf{v})) \simeq \mathrm{Pic}^G(T^{ss})$
3. Compute $\mathrm{Pic}^G(T)$ (easy)
4. Analyze the unstable locus $T^{un} \subset T$ (codimension, irreducibility) to find $\mathrm{Pic}^G(T^{ss})$ (hard)

## Good characters

In the good case, $\mathrm{codim}_T(T^{un}) \geq 2$ and $\mathrm{Pic}(M(\mathbf{v})) \simeq \mathrm{Pic}^G(T^{ss}) \simeq \mathrm{Pic}^G(T) \simeq \mathbb{Z}^2$.

## Bad characters

In the bad case, the locus $T^{un} \subset T$ has an irreducible divisorial component, which gives

$$\mathbb{Z} \to \mathrm{Pic}^G(T) \xrightarrow{res} \mathrm{Pic}^G(T^{ss}) \to 0,$$

and causes the Picard rank to drop:

$$\mathrm{Pic}(M(\mathbf{v})) \simeq \mathbb{Z}$$

instead of $\mathbb{Z}^2$.

# Thank you!

**Theorem** *(P. '20)*

Let $\mathbf{v} = (r, \nu, \Delta)$ be a character with $\dim M(\mathbf{v}) > 0$. Let $\mathbf{v}^\perp \xrightarrow{\lambda} \mathrm{Pic}(M(\mathbf{v}))$ be the Donaldson homomorphism. If $\mathbf{v}$ is

1. Above DLP: $\lambda$ is an isomorphism and
$$\mathrm{Pic}(M(\mathbf{v})) \simeq \mathbb{Z}^3.$$

2. On one branch of DLP: $\lambda$ is surjective, and if $E$ is associated to $\mathbf{v}$, then either
$$a)\ \mathrm{Pic}(M(\mathbf{v})) \simeq \mathbb{Z}^2,\ \ker(\lambda) = \mathbb{Z}[\overline{E}], \text{ or}$$
$$(!)\ b)\ \mathrm{Pic}(M(\mathbf{v})) \simeq \mathbb{Z},\quad \ker(\lambda) \supsetneq \mathbb{Z}[\overline{E}].\ (!)$$

3. On two branches of DLP: $\lambda$ is surjective, and if $E_1, E_2$ are associated to $\mathbf{v}$, then
$$\mathrm{Pic}(M(\mathbf{v})) \simeq \mathbb{Z},\quad \ker(\lambda) \simeq \mathbb{Z}[\overline{E_1}] + \mathbb{Z}[\overline{E_2}].$$