4 June 2001

PHYSICS LETTERS A

Physics Letters A 284 (2001) 112–117

www.elsevier.nl/locate/pla

Collisional diffusion in a 2-dimensional point vortex gas

Daniel H.E. Dubin *, Dezhe Z. Jin [1]

Department of Physics, University of California at San Diego, La Jolla, CA 92093, USA

Received 25 April 2001; accepted 26 April 2001
Communicated by M. Porkolab

Abstract

Simulations showing the effect of shear on the collisional diffusion of a 2D point vortex gas are compared to theory. For finite shear the diffusion is considerably smaller than previous zero-shear theories predict. Surprisingly, changing the sign of the applied shear changes the diffusion by an order of magnitude. © 2001 Elsevier Science B.V. All rights reserved.

The self-diffusion of a 2-dimensional (2D) gas of interacting point vortices is a classic problem in non-equilibrium statistical physics, with relevance to the behavior of type-II superconductors, dislocations in solids, rotating superfluid helium, and turbulence and transport in Euler fluids and plasmas. This Letter considers the effect of an applied shear on the diffusion.

Early work on this problem focused on the case of a quiescent, homogeneous shear-free gas [1,2]. When the vortices are distributed randomly, representing high-temperature thermal fluctuations, Taylor and McNamara showed that the diffusion coefficient (for diffusion in 1 direction) has the following simple form:

$$D^{\mathrm{TM}} = \frac{1}{8\pi}\sqrt{\sum_\alpha \frac{N_\alpha \gamma_\alpha^2}{\pi}}, \tag{1}$$

where N_α is the number of point vortices of type α, each with circulation γ_α. The diffusion coefficient is not an intensive quantity because the diffusion process is dominated by large "Dawson–Okuda vortices" whose size is of order the system size [2].

We find that in the presence of applied shear, the Dawson–Okuda vortices are disrupted and the diffusive transport is greatly reduced compared to Eq. (1). This result may be relevant to current experiments and theories in fusion plasmas, which also observe reduced transport in the presence of shear [3]. In such plasmas, the fluctuations are unstable and turbulent; and so the transport is difficult to determine theoretically. However, in a stable gas of point vortices, statistical theory determines the transport explicitly. Our theory may therefore be considered as a simple paradigm for the shear-reduction of transport seen in more complex turbulent systems.

Our theory also applies directly to experiments measuring collisional diffusion in a cylindrical pure electron plasma column [4,5], confined by a uniform magnetic field $B\hat{z}$. Such plasmas can be held in conditions such that individual electrons act as rods of charge that $\mathbf{E} \times \mathbf{B}$ drift in the fields of the other rods. Under such conditions, these $\mathbf{E} \times \mathbf{B}$ drifts are the main cause of collisional diffusion, dominating over the classical diffusion [6] caused by velocity-scattering collisions. The $\mathbf{E} \times \mathbf{B}$ dynamics of a collection of charged rods

* Corresponding author.
E-mail address: dhdubin@ucsd.edu (D.H.E. Dubin).
[1] Present address: MIT, E25-210, Cambridge, MA 02139, USA.

0375-9601/01/$ – see front matter
PII: S0375-9601(01)00296-1

is isomorphic to a gas of point vortices [7]: each rod, with charge $q < 0$ per unit length, is equivalent to a point vortex with circulation $\gamma = -4\pi cq/B > 0$. Furthermore, such plasmas rotate with a rotation frequency $\omega(r) = v_\theta(r)/r > 0$ that may have substantial radial shear, characterized by a local shear rate $S(r) \equiv r\, d\omega/dr$.

In order to evaluate the diffusion in such a sheared plasma/point vortex gas, we consider N identical vortices confined to a cylindrical patch of radius R, with uniform density $n = N/\pi R^2$, giving an average interparticle spacing $a = (\pi n)^{-1/2}$. To this vortex patch an external sheared rotation $\omega(r)$ is applied, with uniform shear rate S. For this system a dimensionless measure of the shear rate can be defined:

$$s = 2S/n\gamma, \tag{2}$$

which is the shear rate compared to the rotation frequency of the uniform patch in the absence of an applied shear.

In this Letter, we focus on the case of moderate to strong dimensionless shear, i.e., $|s| > 1$. We derive the results of our statistical theory for self-diffusion, and compare these results to molecular dynamics and particle in cell simulations. We find that the simulations agree with our theory provided that s is negative (i.e., negative shear, the usual circumstance in a stable pure electron plasma). Surprisingly, however, when s is positive the transport observed in the simulations is roughly an order of magnitude smaller than our theory predicts. We will discuss a qualitative explanation of this effect, but at present no precise theory exists.

We first describe theory for the diffusion that applies to the negative shear regime $s \lesssim -1$. (The case $s \gtrsim 1$ will be discussed in relation to the simulations.) In the shear range $s \lesssim -1$, we find that two separate collisional processes are responsible for radial diffusion in the presence of the applied shear: small impact parameter collisions between vortices, described by a Boltzmann formalism, and large impact parameter collisions, described by a quasilinear formalism. Here small and large impact parameters mean radial displacements between vortices that are smaller or larger than a distance $2l$, where the "trapping distance" l is defined as

$$l \equiv \sqrt{-\gamma/4\pi S} = a/\sqrt{-2s}. \tag{3}$$

Note that for $s > 0$, the trapping distance is undefined. We will presently see that the lack of a trapping distance for $s > 0$ has a profound effect on the diffusion. The trapping distance arises when one considers the trajectories of 2 identical vortices in the shear flow, at positions r_1 and r_2. For simplicity, we take $|r_1 - r_2| \ll r_1$, and introduce local Cartesian coordinates in a moving frame with an origin initially at the initial average position $\mathbf{R}(0) = [\mathbf{r}_1(0) + \mathbf{r}_2(0)]/2$, and moving with the local fluid rotation velocity $\omega(R(0))$. The x-axis of this frame corresponds to the radial direction, and the y-axis corresponds to the direction of local flow (the θ-direction). In this coordinate frame, the vortices have positions $\Delta\mathbf{r}_1 = (x_1, y_1)$ and $\Delta\mathbf{r}_2 = (x_2, y_2)$, where $\Delta\mathbf{r} \equiv \mathbf{r} - \mathbf{R}$. Their interaction is described by the stream function

$$\psi(\Delta\mathbf{r}_1, \Delta\mathbf{r}_2) = \frac{S}{2}\left(x_1^2 + x_2^2\right) + \frac{\gamma}{4\pi}\ln\left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]. \tag{4}$$

Here the term proportional to the shear rate S is the stream function due to the shear flow, and the logarithmic term describes the vortex interaction. The motion of vortex 1 then follows from the Hamiltonian equations

$$\frac{dx_1}{dt} = -\frac{\partial\psi}{\partial y_1}, \qquad \frac{dy_1}{dt} = \frac{\partial\psi}{\partial x_1},$$

and similarly for vortex 2. Under this dynamics, ψ is a conserved quantity, and it is straightforward to show that $\Delta\mathbf{r}_1 + \Delta\mathbf{r}_2$ is also conserved, taking the value

$$\Delta\mathbf{r}_1 + \Delta\mathbf{r}_2 = 0. \tag{5}$$

Applying Eq. (5) to Eq. (4), ψ can be written as a function of the position of vortex 1 alone:

$$\psi(x_1, y_1) = S\left[x_1^2 - l^2\ln\left[4\left(x_1^2 + y_1^2\right)\right]\right], \tag{6}$$

where the trapping distance l is given by Eq. (3). Contours of constant ψ are displayed in Fig. 1, with $(x_1, y_1) = 0$ at the center of the figure.

For $s < 0$, vortices are retrograde (rotating against the shear), and they follow the trajectories shown in Fig. 1(a). According to Eq. (5), vortex pairs have reflection symmetry through the center of the figure, moving in opposite directions in this frame of reference. The separatrix in Fig. 1(a) has stagnation points

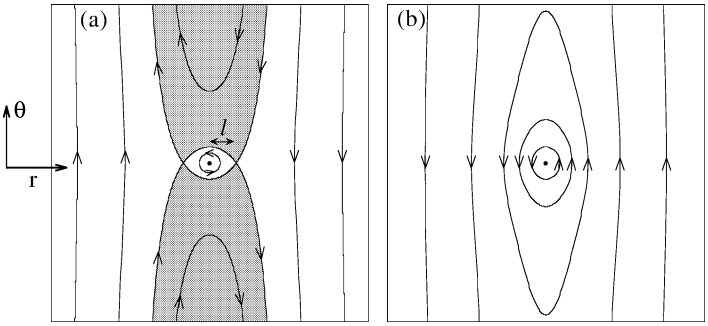

Fig. 1. Streamlines for the interaction of 2 vortices in a shear flow. (a) $s < 0$ (retrograde vortices). (b) $s > 0$ (prograde vortices). Arrows show flow direction assuming $\gamma > 0$.

at $x = \pm l$, $y = 0$, and using Eq. (6) this implies that the separatrix $x_s(y)$ is determined by

$$x_s^2 = l^2\Big(1 + \ln\big[\big(x_s^2 + y^2\big)/l^2\big]\Big) \approx l^2\big(1 + \ln\big[y^2/l^2\big]\big), \tag{7}$$

where the second line is correct for $y \gg l$.

Vortices that begin in the shaded region inside the separatrix take a radial step due to their interaction. Provided that $a > l$ (i.e., $|s| > 1$), an uncorrelated sequence of these small impact parameter collisions will occur, causing radial diffusion of the vortices. This diffusion can be estimated as $\nu \Delta r^2$, where the radial step Δr is of order l, and the collision rate $\nu \sim n|S|l^2$ is the number of vortices per unit time carried by the shear flow into the shaded trapping region of a given vortex (Fig. 1(a)). This results in a diffusion coefficient $\nu \Delta r^2 \sim n|S|l^4 \sim \gamma/|s|$.

A rigorous Boltzmann calculation of the radial diffusion due to these small impact parameter collisions agrees with this estimate. To determine the diffusion coefficient D^B, we integrate over random steps due to a flux Γ_y of vortices carried by the shear flow: $\Gamma_y = n|S\rho|$, where n is the areal density (in cm^{-2}) and $\rho = x_2 - x_1$ is the x-displacement (impact parameter) between the vortices when the vortices are well-separated in y (i.e., before the collision begins, but after the previous collision with some other vortex has ended). Let us call this y-displacement y_0. Then Fig. 1(a) shows that if $|\rho| \leqslant 2x_s(y_0)$, the two vortices will take a step in the x-direction of magnitude $|\rho|$ as they exchange x-positions in the collision. The Boltzmann diffusion coefficient is therefore

$$D^B = \frac{1}{2}\int_{-2x_s(y_0)}^{2x_s(y_0)} d\rho\, \rho^2 \Gamma_y. \tag{8}$$

Using our previous expression for Γ_y, the integral can be performed yielding $D^B = 4n|S|x_s^4(y_0)$. Finally, Eq. (7) implies that $x_s(y_0)$ depends only logarithmically on y_0, so an estimate for y_0 is sufficient. Therefore, for y_0 we use the mean y displacement between collision events (the mean free path), $y_0 \simeq 1/(4ln)$. Then the Boltzmann diffusion coefficient becomes

$$D^B = \frac{\gamma}{2\pi^2|s|}\ln^2\left(\frac{e\pi^2 s^2}{4}\right), \tag{9}$$

where we have employed Eqs. (2), (3) and (7). The logarithm increases D^B over our previous estimate of $\gamma/|s|$ because the shaded region in Fig. 1(a) diverges logarithmically with increasing $|y|$, increasing both the size of the radial step Δr and the collision rate ν.

However, the Boltzmann result for self-diffusion given by Eq. (9) neglects diffusion from large impact parameters. In the Boltzmann description, two vortices with a large impact parameter (outside the shaded region) stream by one another and suffer no net change in radial position. Actually, large impact parameter collisions are not isolated events; many are occurring simultaneously, leading to random motion in the fluctuations.

An estimate of the diffusion from these distant collisions is easily obtained. For these collisions the step size is now smaller than before, because interacting vortices are farther apart. If the impact parameter between vortices is ρ, the radial step Δr is of order $\Delta t\gamma/\rho$, where Δt is the time over which the interaction takes place. For particles streaming along unperturbed circular orbits, $\Delta t \sim 1/|S|$, which implies a small step $\Delta r \sim l^2/\rho$. There are many of these interactions per unit time; the collision rate is $\nu \sim n|S|\rho^2$, leading to a diffusion coefficient $\nu\Delta r^2 \sim n|S|l^4 \sim \gamma/|s|$, which is the same order as Eq. (9).

This diffusion from multiple distant collisions can be obtained more quantitatively from a quasilinear calculation [9] based on the Kubo formula $D^K = \int_0^\infty dt\, \langle \delta v_r(t)\delta v_r(0)\rangle$. Here $\delta v_r(t)$ is the radial velocity fluctuation of a test vortex at position (r,θ) due to $p = 1,\ldots,N$ other vortices at positions (r_p,θ_p), and $\langle\ \rangle$ denotes an ensemble average over a random distribution of the vortices within the vortex patch. The velocity fluctuation is determined by a superposition of N flow fields [8],

$$\delta v_r(t) = -\frac{\gamma}{4\pi r}\sum_{p=1}^{N}\frac{\partial}{\partial\theta}\ln|\mathbf{r}-\mathbf{r}_p|^2, \tag{10}$$

easily written in terms of Fourier modes:

$$\delta v_r(t) = \frac{\gamma}{4\pi r}\sum_{p=1}^{N}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{im}{|m|}\,e^{im(\theta-\theta_p)}(r_</r_>)^{|m|}, \tag{11}$$

where $r_{<(>)}$ is the lesser (greater) of r and r_p. The time integral in the Kubo formula can then be done using integration along unperturbed orbits, assuming that each vortex merely rotates about the center of the vortex patch, i.e., $r_p = \text{const}$, $\theta_p(t) = \omega(r_p)t + \theta_{p0}$. The ensemble average can also be easily calculated using standard techniques for random distributions, converting $\langle\sum_p\sum_{\bar p}\rangle$ to $\sum_{\bar p}\delta_{p\bar p}\int r_p\,dr_p\,d\theta_{p0}\,n(r_p)$.

The result, after performing the θ_{p0} and t integrals, is

$$D^K = \frac{\gamma^2}{(4\pi r)^2}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}2\pi^2\int_0^\infty r_p\,dr_p\,n(r_p) \times \delta\big(m\big[\omega(r)-\omega(r_p)\big]\big)\left(\frac{r_<}{r_>}\right)^{2m}. \tag{12}$$

The δ-function, arising from the time integral over unperturbed orbits, implies that resonant interactions are the most important to the transport process. If we then assume that $\omega(r)$ is monotonic in r so that only $r = r_p$ contributes, the radial integral yields

$$D^K = \frac{n\gamma^2}{8r|\partial\omega/\partial r|}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{1}{|m|}. \tag{13}$$

The divergent sum occurs because nearby vortices following unperturbed orbits take a long time to separate and therefore take a large radial step. However, the sum can be cut off by noting that there is a minimum separation d for which unperturbed orbits are a good approximation. Adding the cutoff to Eq. (13) implies

$$D^K = \frac{\gamma}{2|s|}\ln[r/d], \tag{14}$$

where we have used Eq. (2).

One possible estimate for d is the trapping distance l, since vortices separated by l do not follow unperturbed orbits. Another possibility is that vortices diffuse apart before they are carried away by the shear, and so cannot be treated with unperturbed orbit theory. For vortices separated in r by a distance δ, the time to shear apart a distance of order δ is given by $1/|S|$, and the time to diffusively separate by a distance δ is $\delta^2/4D^K$. Setting the two times equal gives the diffusion-limited minimum separation $\delta = [4D^K/|S|]^{1/2}$. Accordingly, in Eq. (14) we take the maximum of our two estimates:

$$d = \max(\delta, l). \tag{15}$$

Note that Eqs. (2), (3) and (14) imply that $\delta/l = \sqrt{(8\pi/|s|)\ln[r/d]}$, so the shear required to make $\delta < l$ is rather large. In our simulations it will turn out that $\delta > l$, so we use $d = \delta$ in Eq. (14).

Finally, the total diffusion coefficient is the sum of the Boltzmann and Kubo diffusion from small and large impact parameter collisions:

$$D = D^B + D^K. \tag{16}$$

Eq. (16) is correct only when the shear is large enough so that $D < D^{\text{TM}}$, where D^{TM} is the zero-shear result given by Eq. (1). However, comparing Eqs. (1) and (16), we see that only a small shear, $s \sim O(1/N^{1/2})$, is required to meet this inequality. When $|s| \lesssim 1$, Boltzmann collisions no longer occur since $a < l$, so $D^B = 0$; but Eq. (14) for D^K still holds. This implies that Eq. (16) is valid when $D^K \lesssim D^{\mathrm{TM}}$, or $|s| \gtrsim \ln[r/d]\sqrt{16\pi^3/N}$. In other words, small shears wipe out the large-scale Dawson–Okuda vortices responsible for the diffusion predicted by Eq. (1).

We have tested this theory using numerical simulations of N identical point vortices, initially placed randomly inside a circular patch, with an applied uniform external shear rate S. We find that Eq. (16) works well for $s \lesssim -1$, but overestimates the diffusion for $s \gtrsim 1$.

As a check of the numerics, we employ two separate simulation techniques, a 2D molecular dynamics (MD) method for point vortices, and a 2D particle in cell (PIC) simulation. The MD simulation is a standard N^2 code using the fourth-order Runge–Kutta method. The PIC simulation has been described previously [10]. In the PIC simulation the diffusion coefficient is an increasing function of the number of grid points, but for sufficiently fine grid the diffusion is independent of the number of grid points to within our measurement error of about 30%. We use up to a 2048×2048 square grid in the largest PIC simulations. In both the PIC and MD codes, time steps are chosen to conserve energy at the 0.1% level or better over the course of the simulation, and angular momentum (mean square radius of the cylindrical patch) is typically conserved even more accurately. Also, the time step was varied by factors of two in both codes, with no observable change in the diffusion.

In order to measure the diffusion coefficient, we chose as test particles all vortices in the band of radii from $0.43R$ to $0.57R$. For these vortices we followed the mean square change in radial position, $\langle \delta r^2(t) \rangle$, where for vortex i, $\delta r_i(t) = r_i(t) - r_i(0)$. (We also verified that $\langle \delta r(t) \rangle = 0$.)

Two examples with $N = 50{,}000$ are shown in Fig. 2, for the cases $s = \pm 12$. Here, the unexpectedly low diffusion for $s > 0$ is apparent. The diffusion coefficient is found from the equation $\langle \delta r^2(t) \rangle = 2Dt$. More precisely, we fit a straight line to the segment of the curve that has nearly constant slope, and we take D as half the slope of that line.

Figs. 3 and 4 summarize our results for the self-diffusion coefficient. Fig. 3 displays the diffusion coefficient as a function of the particle number N for four different values of the shear parameter, $s = 0$, -1.2, -12, and $+12$. The first case corresponds to a shear free plasma, and can be seen to match the expected Taylor–McNamara scaling with particle number, Eq. (1), shown by the dashed line. The cases $s = -1.2$ and -12 correspond to moderate and strong negative shear. The measured diffusion matches Eq. (16) quite well. However, for $s = +12$ the diffusion is an order of magnitude smaller.

In Fig. 4, the diffusion coefficient is shown as a function of shear for fixed particle number $N = 10^4$. The scaling with s matches Eq. (16) when $s < 0$. At large s, we obtain $D \propto 1/|s|$, although both Boltzmann and Kubo logarithms also introduce dependence on shear, which at low shear can be quite strong. Generally at low to moderate shear, the Kubo result dominates, but at low N or large shear the Boltzmann result dominates.

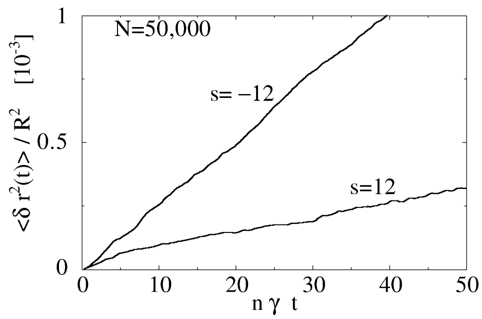

Fig. 2. Mean square change in radial position of vortices vs. time, $\langle \delta r^2(t) \rangle$, for $N = 50{,}000$. Two shear rates are shown, $s = 12$ and $s = -12$.

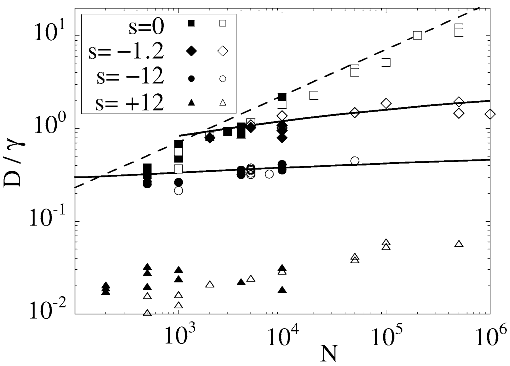

Fig. 3. Self-diffusion coefficient vs. particle number N, for shear rates $s = 0$, -1.2, -12 and 12. Solid points are from MD simulations, open points are from PIC simulations. The dashed line is Eq. (1). The solid lines are Eq. (16), evaluated at $s = -1.2$ and $s = -12$.

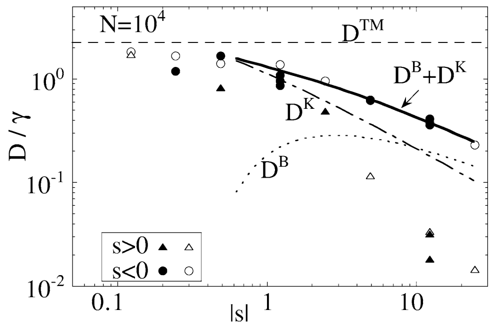

Fig. 4. Self-diffusion coefficient vs. the shear rate s, for $N = 10{,}000$. Solid points are from MD simulations, open points are from PIC simulations. The dashed line is Eq. (1), and the solid line is Eq. (16). The dot-dashed line and the dotted line are the separate Kubo and Boltzmann contributions to the diffusion, Eqs. (14) and (9).

However, for $s > 0$ the simulations do not match Eq. (16): the measured diffusion is up to an order of magnitude less than the theory, depending on the shear rate. It is not surprising that the Boltzmann diffusion theory fails to work for this case; the Boltzmann picture of nearby particles reflecting off one another is no longer correct. For $s > 0$ vortices are prograde (rotating with the shear). Two isolated vortices orbit around one another rather than suffer reflections, as shown in Fig. 1(b). As a result, the measured $\langle \delta r^2(t) \rangle$ has a large slope at early times as vortices begin to rotate around one another, but then relaxes to a smaller slope as vortices return to their initial radii (see Fig. 2).

Fig. 4 shows that even if one neglects Boltzmann diffusion, Kubo diffusion by itself also overestimates the $s > 0$ simulation results. We believe that fluctuations now consist of several self-trapped vortices following elliptical orbits similar to the streamlines of Fig. 1(b). Vortices return to their initial radii several times, and net transport occurs only through the break-up of these fluctuations through interaction with other similar fluctuations. The Kubo theory fails because the unperturbed orbit approximation fails for such fluctuations. A proper transport theory must go beyond the unperturbed orbit approximation in this case; such a theory will be the subject of future work.

Acknowledgement

The authors thank Prof. C.F. Driscoll for suggestions that improved the manuscript. This research was supported by National Science Foundation grant PHY-9876999 and Office of Naval Research grant N00014-96-1-0239.

References

[1] J.B. Taylor, B. McNamara, Phys. Fluids 14 (1971) 1492.
[2] J.M. Dawson, H. Okuda, R.N. Carlile, Phys. Rev. Lett. 27 (1971) 491;
H. Okuda, J.M. Dawson, Phys. Fluids 16 (1973) 408.
[3] K.H. Burrell, Phys. Plasmas 4 (1997) 1499;
P.W. Terry, Rev. Mod. Phys. 72 (2000) 109;
H. Biglari et al., Phys. Fluids B 2 (1990) 1.
[4] F. Anderegg et al., Phys. Rev. Lett. 78 (1997) 2128.
[5] D.H.E. Dubin, Phys. Rev. Lett. 79 (1997) 2678.
[6] C.L. Longmire, M.N. Rosenbluth, Phys. Rev. 103 (1956) 507.
[7] R.H. Levy, Phys. Fluids 8 (1965) 1288;
R.H. Levy, Phys. Fluids 11 (1968) 920.
[8] P.G. Saffman, Vortex Dynamics, Cambridge University Press, 1992, p. 116.
[9] D.H.E. Dubin, D.Z. Jin, in: Non-Neutral Plasma Physics III, AIP Conf. Proc., Vol. 498, American Institute of Physics, New York, 1999, p. 233.
[10] D.A. Schecter, D.H.E. Dubin, K.S. Fine, C.F. Driscoll, Phys. Fluids 11 (1999) 905.