Chapter 2

Eulerian cosmological Perturbation Theory

Cosmological Perturbation Theory (Bernardeau et al., 2002, and references therein) provides the unique theoretical framework of studying the evolution of the density and velocity fields of matter fluctuation in the Universe. While the non-linear gravitational instability breaks down the validity of linear Perturbation Theory on smaller scales ($k \gtrsim 0.1 \, [h/\text{Mpc}]$ at present), we expect to model the non-linear evolution of cosmic matter field by using higher order Perturbation Theory. Yet, there is a fundamental limitation of Perturbation Theory: it improves upon the linear theory only in the very small region when non-linearity is too strong (this happens around $z \sim 0$), and breaks down on the scales where non-perturbative effects such as shell-crossing and violent relaxation take place. Therefore, we define quasi-nonlinear regime where higher-order Perturbation Theory correctly models the non-linear evolution of cosmic matter field.

Quasi-nonlinear regime in standard Perturbation Theory satisfies following three conditions.

- [1] Quasi-nonlinear regime is small compare to the Hubble length so that evolution of cosmic matter field is governed by Newtonian fluid equations.
- [2] Quasi-nonlinear regime is large enough to neglect baryonic pressure so that we can treat dark matter and baryon as a single component of pressureless matter.
- [3] In quasi-nonlinear regime, vorticity developed by non-linear gravitational interaction is negligibly small.

With these three conditions, we can approximate cosmic matter field as a pressureless, single component Newtonian fluid which is completely described by its density contrast and velocity gradient. In Section 2.1, we shall present the Perturbation Theory calculation of the non-linear evolution of cosmic matter field based on these conditions.

Extended studies of standard Perturbation Theory by relaxing one of these conditions are also available in literature. Noh & Hwang (2008) have studied the single fluid
equation in the full General Relativistic context and show that, if one use the proper gauge (temporal comoving gauge, to be specific), the General Relativistic perturbation equations exactly coincide with their Newtonian counterparts up to 2nd order; thus, the General Relativistic correction appears from third order in density perturbation. In Noh et al. (2009), we showed that the 3rd order General Relativistic correction term is sub-dominant on sub-horizon scales, so that the Newtonian PT approach is valid on quasi-nonlinear regime. It is because the purely General Relativistic effect comes through the gravitational potential, and the gravitational potential is much smaller than the density field on sub-horizon scales. At the same time, this correction term increases on large scales comparable to the Hubble radius, because gravitational potential sharply increases as $P_{\phi} \propto k^{n_s-4}$; thus, it eventually exceeds the linear power spectrum near horizon scale. Shoji & Komatsu (2009) have included a pressureful component to the analysis and have found a perturbative solution of double-fluid equations up to 3rd order. Finally, Pueblas & Scoccimarro (2009) measures the vorticity power spectrum from N-body simulations, and show that vorticity effect on density power spectrum is indeed negligible in the quasi-nonlinear regime.

These studies have indicated that non-linear effects coming from violating three conditions are not significant on scales which are most relevant for upcoming high redshift galaxy surveys. One notable exception is when including massive neutrinos. Massive neutrinos suppress the linear power spectrum below the neutrino free streaming scale (Takada et al., 2006), and change nonlinear matter power spectrum, correspondingly. Although the non-linear effect to the matter power spectrum is marginal due to the small energy fraction of neutrino, $f_\nu \equiv \Omega_\nu/\Omega_m$, this effect has to be included in order to measure neutrino mass from galaxy surveys (Shoji & Komatsu, 2009; Saito et al., 2009).

Once we model the non-linear evolution of density field and velocity field of cosmic matter fluctuation, we can calculate the galaxy power spectrum we would observe from galaxy surveys. Here, we have to model two more non-linearities: nonlinear redshift space distortion and nonlinear bias. In order to understand those non-linear effects separately, we first present the non-linear galaxy power spectrum in real space in Section 2.5, then present the non-linear redshift space matter power spectrum in Section 2.6. We combine all the non-linearities and present the non-linear galaxy power spectrum in redshift space in Section 2.7. For each section, we also analyze the effect of primordial non-Gaussianity on the power spectrum of large scale structure in the Perturbation Theory framework.

While we focus on the Eulerian Perturbation Theory in this chapter, Lagrangian Perturbation Theory provide yet another intuition on the non-linear growth of the structure.
In particular, Lagrangian perturbation theory (or its linear version which is also known as Zel’dovich approximation) is widely used to generate the initial condition for cosmological N-body simulations. We review the Lagrangian Perturbation Theory in Appendix E.

Although the material in this chapter is self-contained, we by no means aim for the complete review. For more detailed review on Perturbation Theory, we refer readers to Bernardeau et al. (2002).

2.1 Eulerian Perturbation Theory solution

We review calculation of non-linear Eulerian Perturbation Theory following the pioneering work in the literature (Vishniac, 1983; Fry, 1984; Goroff et al., 1986; Suto & Sasaki, 1991; Makino et al., 1992; Jain & Bertschinger, 1994; Scoccimarro & Frieman, 1996). We treat dark matter and baryons as pressureless dust particles, as we are interested in the scales much larger than the Jeans length. We also assume that peculiar velocity is much smaller than the speed of light, which is always an excellent approximation, and that the fluctuations we are interested in are deep inside the horizon; thus, we treat the system as Newtonian. Then, the evolution of the matter fluctuation, \( \delta(x, \tau) \equiv \rho(x, \tau)/\bar{\rho}(\tau) - 1 \), follows Newtonian fluid equations in expanding universe:

\[
\dot{\delta} + \nabla \cdot [(1 + \delta)v] = 0, \tag{2.1}
\]
\[
\dot{v} + (v \cdot \nabla)v = -Hv - \nabla \phi, \tag{2.2}
\]
\[
\nabla^2 \phi = 4\pi G a^2 \bar{\rho} \delta, \tag{2.3}
\]

where the dots denote \( \partial/\partial \tau \) (\( \tau \) is the conformal time), \( \nabla \) denotes \( \partial/\partial x \) (\( x \) is the comoving coordinate), \( v = dx/d\tau \) is the peculiar velocity field, and \( \phi \) is the peculiar gravitational potential field from density fluctuations, and \( H \equiv d\ln a/d\tau = aH \). As we ignore the vorticity, \( v \) is curl-free, which motivates our using \( \theta \equiv \nabla \cdot v \), the velocity divergence field.

In Fourier space, the Newtonian fluid equations become two coupled integro-differential equations for \( \delta_k(\tau) \) and \( \theta_k(\tau) \). Using equation (2.3) and the Friedmann equation, we write
the continuity equation [Eq. (2.1)] and the Euler equation [Eq. (2.2)] in Fourier space

\[
\frac{\partial \delta_k (\tau)}{\partial \tau} + \theta_k (\tau) = - \int \frac{d^3k_1}{(2\pi)^3} \int d^3k_2 \delta^0 (k_1 + k_2 - k) \frac{k \cdot k_1}{k_1^2} \theta_{k_1} (\tau), \delta_{k_2} (\tau),
\]

(2.4)

\[
\frac{\partial \theta_k (\tau)}{\partial \tau} + \mathcal{H} (\tau) \theta_k (\tau) + \frac{3}{2} \dot{\mathcal{H}}^2 (\tau) \Omega_m (\tau) \delta_k (\tau) = - \int \frac{d^3k_1}{(2\pi)^3} \int d^3k_2 \delta^0 (k_1 + k_2 - k) \frac{k_2 \cdot k_1}{2k_1^2 k_2^2} \theta_{k_1} (\tau) \theta_{k_2} (\tau),
\]

(2.5)

respectively. Note that left hand side of equations above are linear in perturbation variables, and non-linear evolution is described by the right hand side as coupling between different Fourier modes.

2.1.1 Linear solution for density field and velocity field

When density and velocity fluctuations are small, we can neglect the mode coupling terms in the right hand side of equation (2.4) and equation (2.5). Then, the continuity and the Euler equation are linearized as

\[
\frac{\partial \delta_1 (k, \tau)}{\partial \tau} + \theta_1 (k, \tau) = 0,
\]

(2.6)

\[
\frac{\partial \theta_1 (k, \tau)}{\partial \tau} + \mathcal{H} (\tau) \theta_1 (k, \tau) + \frac{3}{2} \dot{\mathcal{H}}^2 (\tau) \Omega_m (\tau) \delta_1 (k, \tau) = 0.
\]

(2.7)

Combining these two equations, we have a second order differential equation for \( \delta_1 (k, \tau) \) as

\[
\frac{\partial^2 \delta_1 (k, \tau)}{\partial \tau^2} + \mathcal{H} (\tau) \frac{\partial \delta_1 (k, \tau)}{\partial \tau} + \frac{3}{2} \dot{\mathcal{H}}^2 (\tau) \Omega_m (\tau) \delta_1 (k, \tau) = 0,
\]

(2.8)

whose solution is given by

\[
\delta_1 (k, a) = C_+ (k) H (a) \int_0^a \frac{da'}{a'^3 H (a')^3} + C_- (k) H (a).
\]

(2.9)

Here, the first term is a growing mode and the second term is a decaying mode.

Let us only consider a growing mode. There are two conventions in the literature about normalizing a growing mode. One normalization convention is requiring that a growing mode is equal to the scale factor in the matter dominated epoch: \( D_+ (a) \big|_{\text{EdS}} = a \). Here, EdS stands for the ‘Einstein-de-Sitter’ Universe which is a flat, matter dominated universe. Therefore, a growing solution becomes

\[
D_+ (a) = \frac{5}{2} \Omega_m H (a) \int_0^a \frac{da'}{[a' H (a')/H_0]^3}.
\]

(2.10)
Figure 2.1: The linear growth factor, $D(a)$, for three different cosmologies: sCDM ($\Omega_m = 1$, $\Omega_\Lambda = 0$) $\Lambda$CDM ($\Omega_m = 0.277$, $\Omega_\Lambda = 0.723$) oCDM ($\Omega_m = 0.277$, $\Omega_\Lambda = 0$)

where $\Omega_m$ takes its present value. Another convention is normalizing its value to be unity at present:

$$D(a) = \frac{D_+(a)}{D_+(a = 1)}.$$ \hspace{1cm} (2.11)

Throughout this dissertation, we use the later convention, and call $D(a)$ the ‘linear growth factor’. Note that the two different conventions differ by a factor of 0.765 for the cosmological parameters in Table 1 (“WMAP+BAO+SN”) of Komatsu et al. (2009).

Figure 2.1 shows the linear growth factor for three different cosmologies: standard Cold Dark Matter (sCDM) model ($\Omega_m = 1$), Cold Dark Matter with cosmological constant ($\Lambda$CDM) model ($\Omega_m = 0.277$, $\Omega_\Lambda = 0.723$), and open Cold Dark Matter (oCDM) model ($\Omega_m = 0.277$, $\Omega_\Lambda = 0$). For given density fluctuations today, at high redshifts, the density fluctuations have to be larger for the oCDM universe, and smaller for sCDM universe compared to the standard $\Lambda$CDM universe. It is because in $\Lambda$CDM and oCDM universe, energy density is dominated by dark energy and curvature, respectively; both of them retard the growth of structure by speeding up the expansion of universe faster than the sCMD universe.

We calculate the velocity gradient field $\theta_1(k, \tau)$ as

$$\theta_1(k, \tau) = -\frac{\partial \delta_1(k, \tau)}{\partial \tau} = -\frac{\delta_1(k, \tau)}{D(\tau)} \frac{dD(\tau)}{d\tau} = -f(\tau)\mathcal{H}(\tau)\delta_1(k, \tau),$$ \hspace{1cm} (2.12)
The logarithmic derivative of the linear growth factor, \( f(a) \equiv \frac{d \ln D}{d \ln a} \), for three different cosmologies: sCDM (\( \Omega_m = 1, \Omega_\Lambda = 0 \)) \( \Lambda \)CDM (\( \Omega_m = 0.277, \Omega_\Lambda = 0.723 \)) oCDM (\( \Omega_m = 0.277, \Omega_\Lambda = 0 \))

where

\[
f(\tau) \equiv \frac{d \ln D}{d \ln a} = \frac{1}{2} \left( \frac{H_0}{aH(a)} \right)^2 \left[ \frac{5\Omega_m}{D^2(a)} - \frac{3\Omega_m}{a} - 2(1 - \Omega_m - \Omega_\Lambda) \right],
\]

Figure 2.2 shows the logarithmic growth rate for three different cosmologies: sCDM model, \( \Lambda \)CDM model, and oCDM model. When universe is flat, matter dominated, \( f = 1 \), and linear growth is slowing down once cosmological constant start to affect the expansion in \( \Lambda \)CDM universe. For oCDM universe, the growth rate is always slower than \( \Lambda \)CDM or sCDM universe.

What about the wave vector \( k \) dependence? We can divided the \( k \)-dependence of linear perturbation by two parts: \( k \)-dependence due to the generation of primordial perturbation from inflation, and \( k \)-dependence due to the subsequent evolution of density perturbation to matter epoch.

Inflation stretches the quantum fluctuation outside of horizon, and generate the primordial curvature perturbation, \( \zeta(k) \), which is conserved outside of horizon even if the equation of state \( w \equiv P/\rho \) changing (Mukhanov et al., 1992). The Bardeen’s potential, a
relativistic generalization of the peculiar gravitational potential\(^1\), \(\Phi_H(k)\) is also conserved outside of horizon, but only for constant \(w\), and for the universe dominated by a perfect fluid whose equation of state is \(w\), it is related to the primordial curvature perturbation \(\zeta(k)\) by

\[
\Phi_H(k) = \frac{3 + 3w}{5 + 3w}\zeta(k). \quad (2.14)
\]

When universe is dominated by radiation or matter, expansion of the universe decelerates, and the wavemodes once stretched outside of horizon by inflation start to re-enter inside of horizon. As \(w = 0\) for matter, the Bardeen’s potential of the mode which re-enter the horizon during matter era is \(\Phi_H(k) = 3/5\zeta(k)\) at horizon crossing time. Inside of horizon, the Bardeen’s potential \(\Phi(k, a)\) is related to the density field\(^2\) by the Poisson equation:

\[
k^2\Phi(k, a) = 4\pi G a^2 \bar{\rho}(a) \delta_1(k, a) = \frac{3}{2} H_0^2 \Omega_m (1 + z) \delta_1(k, a). \quad (2.15)
\]

We denote the time evolution of the peculiar gravitational potential as \(g(z)\), and it is apparent from equation (2.15) that

\[
g(z) = (1 + z) D(z). \quad (2.16)
\]

Then, we rewrite the Bardeen’s potential at large scales as

\[
\Phi(k, a) = g(z) \Phi(k), \quad (2.17)
\]

where \(\Phi(k)\) is the Bardeen’s potential extrapolated at present epoch\(^3\); thus, it is related to the horizon crossing value as \(\Phi(k)/\Phi_H(k) = 1/\left.g(z)\right|_{\text{EAS}} = D_{\Lambda}(a = 1) \approx 0.765\). The numerical value is for cosmological parameters in Table 1 (“WMAP5+BAO+SN”) of Komatsu et al. (2009).

On the other hand, for the wave modes re-enter horizon during radiation era, as perturbation of dominant component (radiation) cannot grow due to its pressure, peculiar gravitational potential decays and matter density contrast can only grow logarithmically. Therefore, the amplitude of sub-horizon perturbations are suppressed relative to the super-horizon perturbations. Plus, at that time baryons were tightly coupled to photon, and could not contribute to the growth of matter fluctuation.

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\(^1\)Note that \(\Phi_H\) has an opposite sign of the Newtonian peculiar gravitational potential \(\phi\) we defined earlier.

\(^2\)To be precise, this equation holds for comoving gauge where \(\delta u = 0\).

\(^3\)Throughout this dissertation, we consistently follow this convention: a dynamical quantity, such as \(\Phi\), \(\delta_1\), \(P_L\), written without explicit time (redshift) dependence denotes the quantity extrapolated to its present value.
In order to take into account these evolution, we need to solve the perturbed Einstein equation and Boltzmann equation for coupled dark matter, photon, baryon, neutrino system. There are many publically available code for calculating such equation systems; among them, CAMB\textsuperscript{4} and CMBFAST\textsuperscript{5} are most widely used in the cosmology community.

These codes calculate so called the ‘transfer function’ $T(k)$. The transfer function $T(k)$ encodes the evolution of density perturbation throughout the matter-radiation equality and CMB last scattering. Since transfer function is defined as the relative changes of small scale modes (which enter horizon earlier) compared to the large scale modes (which enter horizon during matter dominated epoch), the transfer function is unity on large scales: $T(k) = 1$. Therefore, the effects of the retarded growth in the radiation epoch and tight coupling between baryon-photon can be taken into account by multiplying the transfer function to the left hand side of equation (2.15):

$$
\delta_1(k, z) = \frac{2 k^2 T(k)}{3 H_0^2 \Omega_m} D(z) \Phi(k) \equiv M(k) D(z) \Phi(k).
$$

(2.18)

Primordial curvature perturbation predicted by the most inflationary models, and confirmed by observations such as WMAP and SDSS, is characterized by nearly a scale invariant power spectrum. Therefore, we conventionally parametrize the shape of the primordial curvature power spectrum as

$$
P_\zeta(k) = 2\pi^2 \Delta^2_\zeta(k_p) \left( \frac{k}{k_p} \right)^{n_s(k_p) - 4 + \frac{1}{2} \alpha_s \ln \left( \frac{k}{k_p} \right)},
$$

(2.19)

where we use three parameters: amplitude of primordial power spectrum $\Delta^2_\zeta$, spectral tilt $n_s$, and running index $\alpha_s$. Here, $k_p$ is a pivot wavenumber\textsuperscript{6}. Note that the perfectly scale invariant primordial perturbation corresponds to $n_s = 1$, $\alpha_s = 0$.

Combining the primordial power spectrum [Eq (2.19)] and the late time linear evolution [Eq (2.18)], we calculate the linear matter power spectrum as

$$
P_L(k) = \frac{8\pi^2}{25} \frac{D_+(a = 1)^2}{H_0^4 \Omega_m^2} \Delta^2_\zeta(k_p) D^2(z) T^2(k) \left( \frac{k}{k_p} \right)^{n_s(k_p) + \frac{1}{2} \alpha_s \ln \left( \frac{k}{k_p} \right)}.
$$

(2.20)

\textsuperscript{4}http://camb.info

\textsuperscript{5}http://www.cmbfast.org

\textsuperscript{6}Different authors, surveys use different value of $k_p$. Komatsu et al. (2009) uses $k_p \equiv 0.002$ [Mpc$^{-1}$] for WMAP, while Reid et al. (2010) uses $k_p \equiv 0.05$ [Mpc$^{-1}$] for SDSS.
Alternatively, we can also normalize the linear power spectrum by fixing $\sigma_8$, a r.m.s. density fluctuation smoothed by the spherical top-hat filter of radius 8 Mpc/h, whose explicit formula is given by

$$\sigma_8^2 = \int d\ln k \frac{k^3 P_L(k)}{2\pi^2} W^2(kR),$$

where

$$W(kR) = 3 \left[ \frac{\sin(kR)}{k^3 R^3} - \frac{\cos(kR)}{k^2 R^2} \right]$$

with $R = 8$ Mpc/h.

### 2.1.2 Non-linear solution for density field and velocity field

Let us come back to the original non-linear equations. In order to solve these coupled integro-differential equations, we shall expand $\delta_k(\tau)$ and $\theta_k(\tau)$ perturbatively by using the $n$-th power of linear density contrast $\delta_1(k, \tau)$ as a basis:

$$\delta_k(\tau) = \sum_{n=1}^{\infty} \int \frac{d^3 q_1}{(2\pi)^3} \cdots \int \frac{d^3 q_{n-1}}{(2\pi)^3} \int d^3 q_n \delta^D(k - \sum_{i=1}^{n} q_i) \times F_n^{(s)}(q_1, q_2, \cdots, q_n, \tau) \delta_1(q_1, \tau) \cdots \delta_1(q_n, \tau)$$

$$\theta_k(\tau) = -f(\tau)H(\tau) \sum_{n=1}^{\infty} \int \frac{d^3 q_1}{(2\pi)^3} \cdots \int \frac{d^3 q_{n-1}}{(2\pi)^3} \int d^3 q_n \delta^D(k - \sum_{i=1}^{n} q_i) \times G_n^{(s)}(q_1, q_2, \cdots, q_n, \tau) \delta_1(q_1, \tau) \cdots \delta_1(q_n, \tau).$$

Here, $f(\tau) = d\ln D/d\ln a$ with the linear growth factor $D(a)$, and $F_n^{(s)}$ and $G_n^{(s)}$ are symmetrized kernels, which characterize coupling between different wave modes. We introduce $f(\tau)H(\tau)$ factor in equation (2.23) motivated by the linear relation between density contrast and velocity divergence: $\theta_1(k, \tau) = -f(\tau)H(\tau) \delta_1(k, \tau)$. In this definition the first order kernels become unity: $F_1^{(s)} = G_1^{(s)} = 1$.

As we know the evolution of $\delta_1(q, \tau)$ from the linear theory, calculating $F_n^{(s)}$ and $G_n^{(s)}$ will complete the solution. The standard procedure (Goroff et al., 1986; Jain & Bertschinger, 1994) is calculating the un-symmetized kernels $F_n$ and $G_n$ first, then symmetrize them under changing arguments:

$$F_n^{(s)}(q_1, \cdots, q_n) = \frac{1}{n!} \sum_{\sigma} F_n(q_{\sigma_1}, \cdots, q_{\sigma_n})$$

$$G_n^{(s)}(q_1, \cdots, q_n) = \frac{1}{n!} \sum_{\sigma} G_n(q_{\sigma_1}, \cdots, q_{\sigma_n}).$$
Here, sum is taken for all the permutations \( \sigma \equiv (\sigma_1, \ldots, \sigma_n) \) of the set \( \{1, \ldots, n\} \). By substituting the perturbative expansion of equation (2.22) and equation (2.23) back into the original equations, we get the equations of \( F_n^{(s)} \)s and \( G_n^{(s)} \)s. For example, the continuity equation [Eq. (2.4)] becomes

\[
\sum_{n=1}^{\infty} \left( \frac{d^3 q_1}{(2\pi)^3} \cdots \frac{d^3 q_{n-1}}{(2\pi)^3} \int d^3 q_n \delta^D(k - \sum_{i=1}^{n} q_i) \delta_1(q_1, \tau) \cdots \delta_1(q_n, \tau) \right) \\
\times \left[ \frac{\partial F_n^{(s)}(q_1, \ldots, q_n, \tau)}{\partial \tau} + nHfF_n^{(s)}(q_1, \ldots, q_n, \tau) - HfG_n^{(s)}(q_1, \ldots, q_n, \tau) \right] \\
= (2\pi)^3 Hf \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \int d^3 q_{11} \cdots \int d^3 q_{lm} \frac{d^3 q_{21}}{(2\pi)^3} \cdots \frac{d^3 q_{2l}}{(2\pi)^3} \\
\times \frac{k \cdot (q_{11} + \cdots + q_{lm})}{|q_{11} + \cdots + q_{lm}|^2} G_m^{(s)}(q_{11}, \ldots, q_{lm}) F_l^{(s)}(q_{21}, \ldots, q_{2l}) \\
\times \delta^D(k - \sum_{i=1}^{m} q_{1i} - \sum_{j=1}^{l} q_{2j}) \delta_{11}(q_1, \tau) \cdots \delta_{lm}(q_n, \tau) \delta_{21}(q_1, \tau) \cdots \delta_{2l}(q_n, \tau).
\]

Here, we use an identity from linear theory \( \partial \delta_1 / \partial \tau = fH \delta_1 \) to replace the conformal-time \( (\tau) \) derivatives. In order to isolate the equation for the kernels, we have to identify \( q_i \)s to \( q_{1i} \)s and \( q_{2i} \)s. In principle, one can take all possible identifications and symmetrize the equation itself, but we follow the standard approach where (for given \( n \)) indexes are matched as \( (q_{11}, \ldots, q_{lm}) = (q_1, \ldots, q_m) \), and \( (q_{21}, \ldots, q_{2l}) = (q_{m+1}, \ldots, q_n) \). By matching indexes this way, the equations are not manifestly symmetric anymore, and as a result the kernels are not symmetric: we drop the superscript \( (s) \) and symmetrize kernels later by using equation (2.24) and equation (2.25).

Once \( q_i \)s are identified, we can read off the equation for \( F_n \) and \( G_n \) from the equation above as

\[
\frac{1}{f(\tau)H(\tau)} \frac{\partial F_n(k, \tau)}{\partial \tau} + nF_n(k, \tau) - G_n(k, \tau)
= \sum_{m=1}^{n-1} \frac{k \cdot k_2}{k_1^2} G_m(k_1, \tau) F_{n-m}(k_2, \tau),
\]

(2.26)

where we use the short hand notation of \( F_n(k, \tau) \equiv F_n(q_1, \ldots, q_n, \tau) \), and \( G_n(k, \tau) \equiv G_n(q_1, \ldots, q_n, \tau) \) with a constraint \( k = q_1 + \cdots + q_n \). Note that from the matching condition of \( q_i \), \( G_m(k_1, \tau) = G_n(q_1, \ldots, q_m, \tau) \) and \( F_{n-m}(k_2, \tau) = F_{n-m}(q_{m+1}, \ldots, q_n, \tau) \) in the right hand side of equation (2.26). Also, Dirac delta function dictates \( k_1 + k_2 = k \). Similarly,
we find
\[
\frac{1}{f(\tau) \mathcal{H}(\tau)} \frac{\partial G_n(k, \tau)}{\partial \tau} + \left[ \frac{3 \Omega_m(\tau)}{2 f^2(\tau)} + n - 1 \right] G_n(k, \tau) - \frac{3 \Omega_m(\tau)}{2 f^2(\tau)} F_n(k, \tau)
\]
\[
= \sum_{m=1}^{n-1} \frac{k^2(k_1 \cdot k_2)}{2k_1^2 k_2^2} G_m(k_1, \tau) G_{n-m}(k_2, \tau),
\]
from the Euler equation [Eq. (2.5)].

In general, the kernels \( F_n \) and \( G_n \) depend on time, and we have to solve complicated differential equations, [Eq. (2.26)] and [Eq. (2.27)]. However, it is well known that the kernels are extremely insensitive to the underlying cosmology, and the next-to-leading order correction to \( P(k) \) can be correctly modeled as long as one uses the correct growth factor for \( \delta_1(k, \tau) \) (Bernardeau et al., 2002). Therefore, we shall calculate the kernels in the Einstein de-Sitter (spatially flat, matter-dominated) universe. In Einstein de-Sitter Universe, as \( \Omega_m/f^2 = 1 \), \( F_n \) and \( G_n \) are constant in time and equation (2.26) and equation (2.27) reduce to the algebraic equations. Moreover, Takahashi (2008) has calculated the exact solution up to third order in general dark energy model, and has concluded that the difference between the next-to-leading order power spectrum from exact kernels and that from Einstein de-Sitter kernels is extremely small. It is at most sub-percent level at \( z = 0 \), and decreases as redshift increases to \( \sim 10^{-4} \) at \( z = 3 \). As we are mostly interested in the high redshift \( (z > 1) \), we can safely ignore such a small difference in the kernels.

In Einstein de-Sitter Universe, equation (2.26) and equation (2.27) become
\[
nF_n(k) - G_n(k) = \sum_{m=1}^{n-1} \frac{k \cdot k_1}{k_1^2} G_m(k_1) F_{n-m}(k_2),
\]
\[
(2n + 1)G_n(k) - 3F_n(k) = \sum_{m=1}^{n-1} \frac{k^2(k_1 \cdot k_2)}{k_1^2 k_2^2} G_m(k_1) G_{n-m}(k_2).
\]
By solving the algebraic equations, we find the recursion relations
\[
F_n(q_1, \cdots, q_n) = \sum_{m=1}^{n-1} \frac{G_m(q_1, \cdots, q_m)}{(2n+3)(n-1)} \left\{ (2n+1) \frac{k \cdot k_1}{k_1^2} F_{n-m}(q_{m+1}, \cdots, q_n) + \frac{k^2(k_1 \cdot k_2)}{k_1^2 k_2^2} G_{n-m}(q_{m+1}, \cdots, q_n) \right\}
\]
\[
G_n(q_1, \cdots, q_n) = \sum_{m=1}^{n-1} \frac{G_m(q_1, \cdots, q_m)}{(2n+3)(n-1)} \left\{ 3 \frac{k \cdot k_1}{k_1^2} F_{n-m}(q_{m+1}, \cdots, q_n) + \frac{k^2(k_1 \cdot k_2)}{k_1^2 k_2^2} G_{n-m}(q_{m+1}, \cdots, q_n) \right\}.
\]
With the recursion relations and the kernels for linear theory, $F_1 = G_1 = 1$, we can calculate $F_n^{(s)}$, $G_n^{(s)}$ for all order. For example, the second order kernels are

$$F_2^{(s)}(k_1, k_2) = \frac{5}{7} \left( \frac{k_1 \cdot k_2}{k_1^2 k_2^2} + \frac{k_1 \cdot k_2}{2} \left( \frac{1}{k_1^2} + \frac{1}{k_2^2} \right) \right),$$

(2.32)

$$G_2^{(s)}(k_1, k_2) = \frac{3}{7} \left( \frac{k_1 \cdot k_2}{k_1^2 k_2^2} + \frac{k_1 \cdot k_2}{2} \left( \frac{1}{k_1^2} + \frac{1}{k_2^2} \right) \right),$$

(2.33)

and third order kernels can be calculated from second order kernels as

$$F_3^{(s)}(k_1, k_2, k_3) = \frac{2k^2}{54} \left[ \frac{k_1 \cdot k_2}{k_1^2 k_2^2} G_2^{(s)}(k_2, k_3) + (2 \text{ cyclic}) \right] + \frac{7}{54} k \cdot \left[ \frac{k_1}{k_1} G_2^{(s)}(k_1, k_2) + (2 \text{ cyclic}) \right] + \frac{7}{54} k \cdot \left[ \frac{k_1}{k_1} F_2^{(s)}(k_2, k_3) + (2 \text{ cyclic}) \right],$$

(2.34)

and

$$G_3^{(s)}(k_1, k_2, k_3) = \frac{k^2}{9} \left[ \frac{k_1 \cdot k_2}{k_1^2 k_2^2} G_2^{(s)}(k_2, k_3) + (2 \text{ cyclic}) \right] + \frac{1}{18} k \cdot \left[ \frac{k_1}{k_1} G_2^{(s)}(k_1, k_2) + (2 \text{ cyclic}) \right] + \frac{1}{18} k \cdot \left[ \frac{k_1}{k_1} F_2^{(s)}(k_2, k_3) + (2 \text{ cyclic}) \right].$$

(2.35)

In summary, the solution of Eulerian Perturbation Theory consists of the perturbative expansions (Eq. (2.22) and Eq. (2.23)) with kernels calculated by recursion relations (Eq. (2.30) and Eq. (2.31)) followed by symmetrization (Eq. (2.24) and Eq. (2.25)). By using this solution, we can describe the non-linear growth of density field and velocity field of cosmic matter in the quasi-nonlinear regime.

### 2.2 Statistics of the cosmological density field: Gaussian vs. non-Gaussian

In the previous section, we calculate the solution for the non-linear evolution of the density and velocity fields in terms of the linear density field. As the direct observable of the

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7For the compactness of the equation, we adopt the short-hand notation of

$$k_{ijk \cdots} \equiv k_i + k_j + k + \cdots$$

throughout this dissertation.
cosmological observation is the statistical correlation function such as power spectrum and bispectrum, in order to compare the non-linear solution to observation, we have to calculate the statistical correlation of density and velocity field.

Inflationary theories predict that the primordial curvature perturbation obeys a nearly Gaussian statistics; thus, the linear density field $\delta_1(k)$, which evolves linearly from the curvature perturbation [Eq. (2.18)], also obeys the same statistics. Let us first consider the Gaussian statistics.

In order to describe the statistics of a field, we have to introduce a probability functional $P[\delta_1(x)]$, which describes the probability of having a configuration of density field $\delta_1$ whose value is $\delta_1(x)$ at a point $x$. Note that statistical homogeneity dictates $P[\delta_1(x)]$ to be independent of position $x$. For Gaussian case, it is given by (Gabrielli et al., 2005)

$$P[\delta_1(x)] = \frac{1}{Z} \exp \left[ -\frac{1}{2} \int d^3 y \int d^3 z \delta_1(y)K(y,z)\delta_1(z) \right],$$

where $K(y,z)$ is corresponding to the inverse of the covariance matrix $C^{-1}$ in the usual Gaussian statistics of discrete variables, and

$$Z = \int [D\delta_1] \exp \left[ -\frac{1}{2} \int d^3 y \int d^3 z \delta_1(y)K(y,z)\delta_1(z) \right]$$

is the normalization constant, and $[D\delta_1]$ is the integration measure in the Hilbert space.

By using the probability functional, we calculate the $n$-point correlation function of $\delta_1(x)$ by taking the expectation value as:

$$\langle \delta_1(x_1) \cdots \delta_1(x_n) \rangle = \int [D\delta_1(x)] \delta_1(x_1) \cdots \delta_1(x_n)P[\delta_1(x)].$$

A usual technique of calculating the expectation value is by introducing the generating functional

$$Z[J] \equiv \left\langle e^{\int \frac{d^3 x J(x) \delta_1(x)}} \right\rangle,$$

and taking successive functional derivatives with respect to $J(x)$. Finally, we get the correlation function by setting $J = 0$. That is, $n$-point correlation function is

$$\langle \delta_1(x_1) \cdots \delta_1(x_n) \rangle = \frac{1}{i^n} \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \bigg|_{J=0}.$$

Furthermore, we can also calculate the $n$-point connected correlation function by

$$\langle \delta_1(x_1) \cdots \delta_1(x_n) \rangle_c = \frac{1}{i^n} \frac{\delta^n \ln Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \bigg|_{J=0}.$$
For Gaussian case, we can re-write the generating functional in a closed form:

\[ Z[J] = \frac{1}{Z} \int [Df] \exp \left[ -\frac{1}{2} \int d^3y \int d^3zf(y)K(y, z)f(z) + i \int d^3xJ(x)f(x) \right]. \tag{2.42} \]

Then, we make the coordinate transformation of

\[ \delta_1^*(x) = \delta_1(x) - i \int d^3y K^{-1}(x, y)J(y), \]

where \( K^{-1}(x, y) \) is the functional inverse of \( K(x, y) \), which is defined by

\[ \int d^3z K(x, z) K^{-1}(z, y) = \delta^D(x - y). \]

This coordinate transformation does not change the integral measure, as is simple translation in the functional space. Under the coordinate transformation, exponent in equation \(2.42\) becomes

\[ -\frac{1}{2} \int d^3y \int d^3z \delta_1^*(y)K(y, z)\delta_1(z) - \frac{1}{2} \int d^3x \int d^3yJ(x)K^{-1}(x, y)J(y), \tag{2.43} \]

where the functional integration of the first term is unity; thus, generating functional for Gaussian case becomes

\[ Z[J] = \exp \left[ -\frac{1}{2} \int d^3x \int d^3yJ(x)K^{-1}(x, y)J(y) \right]. \tag{2.44} \]

Therefore, by using equation \(2.41\), we calculate the the \( n \)-point correlation function. By taking the functional derivative twice, we find that

\[
\frac{\delta Z[J]}{\delta J(x_1)} = -\frac{1}{2} \left[ \int d^3x \int d^3y \delta^D(x - x_1)K^{-1}(x, y)J(y) \right.
\]

\[ + \int d^3x \int d^3y J(x)K^{-1}(x, y) \delta^D(y - x_1) \] \[ Z[J] \]

\[ = - \left[ \int d^3xK^{-1}(x, x_1)J(x_1) \right] Z[J], \]

and

\[
\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)} = -K^{-1}(x, x_1)Z[J].
\]

By carefully observing this procedure, we find that any odd-number \( n \)-point correlation function has to vanish when setting \( J = 0 \), and any even-number \( n \)-point correlation function is given only by \( K^{-1}(x, y) \). Especially, from the calculation above, we can read that the two point correlation function is given by

\[ \xi(x_1, x_2) \equiv (\delta_1(x_1)\delta_1(x_2)) = K^{-1}(x_1, x_2). \tag{2.45} \]
This result is so called Wick’s theorem, which states that “the average of a product of an even number of δ₁s is the sum over all ways of pairing δ₁s with each other of a product of the average values of the pairs:

\[
\langle \delta_1(x_1)\delta_1(x_2)\cdots \rangle = \sum_{\text{pairings}} \prod_{\text{pairs}} \langle \delta_1\delta_1 \rangle,
\]

with the sum over pairings not distinguishing those which interchange coordinates in a pair, or which merely interchange pairs” (Weinberg, 2008). What about the connected correlation function? From equation (2.41), it is clear that, for Gaussian case, all higher order \(n > 2\) connected correlation functions vanish:

\[
\langle \delta_1(x_1)\cdots\delta_1(x_n) \rangle_c = - \frac{1}{2^n} \delta^{\mu}\delta J(x_1)\cdots\delta J(x_n) \int d^3 x \int d^3 y J(x) K^{-1}(x, y) J(y).
\]

From equation (2.45), \(K(x, y)\) is related to the correlation function by

\[
\int d^3 z \xi(x, z) K(z, y) = \delta^D(x - y).
\]

Let us further investigate on the kernel, \(K(x, y)\). Due to the statistical homogeneity and isotropy of universe, \(\xi(x, z)\) and \(K(x, y)\) have to depend only on the separation, i.e. \(\xi(x, z) = \xi(x - z)\). Therefore,

\[
\int d^3 z \xi_f(x - z) K(z - y) = \delta^D(x - y).
\]

Then, Fourier transform of the equation above leads

\[
K(q) = \frac{1}{P_L(q)},
\]

where \(P_L(q)\) is the linear power spectrum, which is related to the two-point correlation function by

\[
\xi(r) = \int \frac{d^3 q}{(2\pi)^3} P_L(q) e^{i q r}.
\]

That is, \(K(x - y)\) is Fourier transform of the inverse of linear power spectrum \(P_L(q)\). One can also show that directly by using Fourier space representation of the linear density field \(\delta_1(k)\). By Fourier transforming the exponent of equation (2.36), we find

\[
- \frac{1}{2} \int d^3 y \int d^3 z \delta_1(y) K(y - z) \delta_1(z)
= - \frac{1}{2} \int d^3 q_1 \int d^3 q_2 \delta_1(q_1) \left[ \frac{K(q_1)}{(2\pi)^3} \delta^D(q_1 + q_2) \right] \delta_1(q_2).
\]
As the structure of the probability functional is the same as that for the real space, we can simply relate $K(q)$ to the power spectrum by using the analogy. Because the functional inversion of the Fourier space kernel is

$$
\left[ \frac{K(q_1)\delta^D(q_1 + q_2)}{(2\pi)^3} \right]^{-1} = \frac{(2\pi)^3}{K(q_1)} \delta^D(q_1 + q_2),
$$

(2.52)

the power spectrum is given by

$$
\langle \delta_1(q_1)\delta_1(q_2) \rangle \equiv (2\pi)^3 P(q_1)\delta^D(q_1 + q_2) = \frac{(2\pi)^3}{K(q_1)} \delta^D(q_1 + q_2).
$$

(2.53)

By using the second equality, we reproduce equation (2.49). Politzer & Wise (1984) extended this method to calculate the $n$-point correlation function of peaks.

As we have shown here, when the linear density field follows the Gaussian statistics, all higher order statistical properties are solely determined by the two point correlation function, or power spectrum. For non-Gaussian case, however, the connected $n$-point correlation functions do not vanish, and we have to know the probability functional or generating functional in order to calculate the statistical quantities. Conversely, if we know the $n$-point correlation functions to infinite order, then, we can calculate the generating functional by inverting equation (2.41):

$$
\ln Z[J] = \sum_{n=2}^{\infty} \frac{i^n}{n!} \int d^3y_1 \cdots \int d^3y_n \langle \delta_1(y_1) \cdots \delta_1(y_n) \rangle \cdot J(y_1) \cdots J(y_n),
$$

(2.54)

thus fully specify the statistics of the linear density field. Matarrese et al. (1986) derives the $n$-point correlation function of peaks by using this method (MLB formula), and we shall calculate the galaxy bispectrum in the presence of primordial non-Gaussianity in Chapter 5 based on this formula. The full derivation of MLB formula is also shown in Appendix K.

### 2.3 Next-to-leading order power spectrum from the perturbative solution: the theoretical template

In the following sections, we shall calculate the power spectrum of density contrast of matter in real space (Section 2.4), matter in redshift space (Section 2.6), galaxies in real space (Section 2.5), and galaxies in redshift space (Section 2.7). For all cases, we shall calculate corresponding density contrasts as a function of the matter density contrast $\delta_k(\tau)$ and velocity gradient $\theta_k(\tau)$ for which we know the perturbation theory solution.
Therefore, in most general form, we can also write a quantity $X(k, \tau)$, of which we want to calculate power spectrum, as a sum of the perturbative series

$$X(k, \tau) = X^{(1)}(k, \tau) + X^{(2)}(k, \tau) + X^{(3)}(k, \tau) + \cdots$$

$$= \sum_{n=1}^{\infty} \int \frac{d^3q_1}{(2\pi)^3} \cdots \int \frac{d^3q_{n-1}}{(2\pi)^3} \int d^3q_n \delta^D(k - \sum_{i=1}^{n} q_i)$$

$$\times K_n^{(s)}(q_1, q_2, \cdots, q_n) \delta_1(q_1, \tau) \cdots \delta_1(q_n, \tau),$$

(2.55)

with a general symmetrized kernel $K_n^{(s)}(q_1, \cdots, q_n)$, which is given by a combination of $F_n^{(s)}$ and $G_n^{(s)}$. Here, $X^{(n)}(k)$ denote that the quantity is $n$-th order in linear density contrast, $\delta_1(k, \tau)$. For a quantity which can be expanded as equation (2.55), we can calculate next-to-leading order (called one-loop) power spectrum and bispectrum as a function of linear order (called tree-level) quantities. In this section, we present these general formulas which we shall use in the rest of this chapter.

As the power spectrum, $P_X(k, \tau)$, is a quadratic quantity of $X(k, \tau)$ in Fourier space,

$$\langle X(k)X(k') \rangle = (2\pi)^3 P_X(k) \delta^D(k + k'),$$

(2.56)

for the expansion in equation (2.55), the left hand side of equation (2.56) becomes

$$\langle X(k, \tau)X(k', \tau) \rangle$$

$$= \langle X^{(1)}(k, \tau)X^{(1)}(k', \tau) \rangle + 2\langle Re \left[ X^{(1)}(k, \tau)X^{(2)}(k', \tau) \right] \rangle$$

$$+ \langle X^{(2)}(k, \tau)X^{(2)}(k', \tau) \rangle + 2\langle Re \left[ X^{(1)}(k, \tau)X^{(3)}(k', \tau) \right] \rangle + \cdots.$$ 

(2.57)

Here, we use the Hermitianity (Reality) condition,

$$X^{(i)}(k, \tau) = \left[ X^{(i)}(-k, \tau) \right]^* = \left[ X^{(i)}(k', \tau) \right]^*,$$

and $Re$ takes the real part of a complex number. In equation (2.57), we explicitly write down the terms up to order $O(\delta_1^2)$.

### 2.3.1 One-loop Power spectrum with Gaussian linear density field

When linear density contrast, $\delta_1$, is Gaussian, odd power of $\delta_1$ vanishes when taking an ensemble average. Therefore, the next-to-leading order nonlinear power spectrum consists of $\langle X^{(1)}(k, \tau)X^{(1)}(k', \tau) \rangle$, $\langle X^{(2)}(k, \tau)X^{(2)}(k', \tau) \rangle$ and $\langle X^{(1)}(k, \tau)X^{(3)}(k', \tau) \rangle$ which is often
denoted as $P_{X,11}(k, \tau)$, $P_{X,22}(k, \tau)$ and $P_{X,13}(k, \tau)$, respectively. We calculate the next-to-leading order power spectrum by using Wick’s theorem as\(^8\)

$$P_X(k, \tau)$$

$$= P_{X,11}(k, \tau) + P_{X,22}(k, \tau) + 2P_{X,13}(k, \tau)$$

$$= \left[K_1^{(s)}(k)\right]^2 P_L(k, \tau) + 2 \int \frac{d^3q}{(2\pi)^3} P_L(q, \tau) P_L(|k-q|, \tau) \left[K_2^{(s)}(q, k-q)\right]^2$$

$$+ 6K_1^{(s)}(k) P_L(k, \tau) \int \frac{d^3q}{(2\pi)^3} P_L(q, \tau) K_3^{(s)}(q, -q, k),$$

(2.58)

where $P_L(k)$ is the linear power spectrum.

### 2.3.2 One-loop Power spectrum with non-Gaussian linear density field

When linear density contrast follows non-Gaussian distribution, the second term in equation (2.57), $\langle X^{(1)}(k)X^{(2)}(k') \rangle$, is non-zero, and becomes the leading order non-Gaussian correction to the power spectrum,

$$\Delta P_{X,NG}(k, \tau) = 2P_{12}(k, \tau)$$

$$= 2K_1^{(s)}(k) \int \frac{d^3q}{(2\pi)^3} K_2^{(s)}(q, k-q) B_L(-k, q, k-q, \tau).$$

(2.59)

Here, $B_L$ is the bispectrum of linear density field. There also are the non-Gaussian correction term coming from $P_{X,22}(k, \tau)$ and $P_{X,13}(k, \tau)$ proportional to the linear trispectrum, $T_L(k_1, k_2, k_3, k_4)$\(^9\):

$$\Delta P_{X,22}(k, \tau) = \int \frac{d^3q_1}{(2\pi)^3} \int \frac{d^3q_2}{(2\pi)^3} K_2^{(s)}(q_1, k-q_1) K_2^{(s)}(q_2, -k-q_2)$$

$$\times T_L(q_1, k-q_1, q_2, -k-q_2, \tau)$$

(2.60)

$$\Delta P_{X,13}(k, \tau) = K_1^{(s)}(k) \int \frac{d^3q_1}{(2\pi)^3} \int \frac{d^3q_2}{(2\pi)^3} K_3^{(s)}(q_1, q_2, k-q_1-q_2)$$

$$\times T_L(-k, q_1, q_2, k-q_1-q_2, \tau)$$

(2.61)

Therefore, in order to calculate the effect of primordial non-Gaussianity fully in one-loop level of perturbation theory, we need to know both linear bispectrum and the linear trispectrum. However, linear trispectrum ($T_L \sim M^4 \sigma_4^3$) is parametrically smaller

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\(^8\)For the derivation of $P_{X,22}$ and $P_{X,13}$, see Appendix B.1.

\(^9\)For derivation, see Appendix B.2
than linear bispectrum \( B_L \sim M^3 P^L_\phi \) in the quasi-nonlinear regime, as the primordial curvature power spectrum \( P_\phi \) sharply decreases as \( k \) increases, \( P_\phi(k) \sim k^{n_s-4} \). In other words, linear trispectrum arises from the two-loop contribution of the primordial curvature perturbation, we shall neglect \( \Delta P_{X,22} \) and \( \Delta P_{X,13} \) correction terms (Taruya et al., 2008). However, note that the trispectrum term may be important on the very large scales. Indeed \( \Delta P_{X,22} \) term include a term proportional to \( f_{\text{NL}}^2 / (k^4 T^2(k)) \) which sharply peaks on large scales (Desjacques & Seljak, 2010). In the rest of the chapter, we shall use equation (2.59) to calculate the non-Gaussian correction to the one-loop power spectrum.

### 2.3.2.1 Local primordial non-Gaussianity

Although we can apply equation (2.59) to any kind of non-Gaussianities, we shall focus on the local type non-Gaussianity. The local type non-Gaussianity is particularly interesting because of the consistency relation (Maldacena, 2003; Acquaviva et al., 2003; Creminelli & Zaldarriaga, 2004), which states that the coefficient of the squeezed bispectrum [Eq. (2.64)], \( f_{\text{NL}} \), has to satisfy

\[
f_{\text{NL}} = \frac{5}{12} (1 - n_s)
\]

for any kind of single field inflation model. Here, \( n_s \) is the spectral tilt of the primordial power spectrum [Eq. (2.19)], whose current best estimation is \( n_s = 0.963 \pm 0.012 \) (68\% CL) from WMAP7 (Komatsu et al., 2010). Therefore, any significant detection of \( f_{\text{NL}} \) above the value dictated by the consistency relation \( f_{\text{NL}} \sim 0.017 \) will rule out the single field inflationary models. The current limit from the WMAP 7-year data is \( f_{\text{NL}} = 32 \pm 21 \) (68\% CL) (Komatsu et al., 2010), and from the SDSS is \( f_{\text{NL}} = 31^{+16}_{-27} \) (68\% CL) (Slosar et al., 2008). We shall further study the effect of local primordial non-Gaussianity on the galaxy bispectrum and the galaxy-galaxy, galaxy-CMB weak gravitational lensing in Chapter 5 and Chapter 6, respectively.

Local type primordial non-Gaussianity is defined by the Bardeen’s potential \( \Phi(x) \) [Eq. (2.14)] in real space (Salopek & Bond, 1990; Gangui et al., 1994; Verde et al., 2000; Komatsu & Spergel, 2001):

\[
\Phi(x) = \phi(x) + f_{\text{NL}} \left[ \phi^2(x) - \langle \phi^2 \rangle \right] + O(\phi^3),
\]

where \( \phi(x) \) is a Gaussian random field. The primordial bispectrum generated by the local
non-Gaussianity\textsuperscript{10} is

\[
B_\Phi(k_1, k_2, k_3) = 2f_{NL} [P_\phi(k_1)P_\phi(k_2) + (2 \text{ cyclic})].
\] (2.64)

We can find the linear matter bispectrum by linearly evolving the primordial bispectrum [Eq. (2.64)] to present using equation (2.18):

\[
B_L(k_1, k_2, k_3; z) = 2f_{NL}D^3(z)M(k_1)M(k_2)M(k_3) [P_\phi(k_1)P_\phi(k_2) + (2 \text{ cyclic})].
\] (2.65)

Using the linear bispectrum above, we calculate the non-Gaussian correction to the nonlinear power spectrum as

\[
\Delta P_{X,nG}(k, z) = 4f_{NL}D^3(z)K_1^{(s)}(k)M(k) \int \frac{d^3q}{(2\pi)^3}M(q)M(|k-q|)
\]

\[
\times K_2^{(s)}(q, k-q)P_\phi(q) [2P_\phi(k) + P_\phi(|k-q|)].
\] (2.66)

Note that we ignore the nonlinearity in the linear power spectrum generated by equation (2.63), and use a linear approximation as

\[
P_L(k, z) \simeq M^2(k)D^2(z)P_\phi(k).
\] (2.67)

This approximation is valid up to slight rescaling of amplitude and slope of the primordial curvature power spectrum. For more discussion, see, Section II of McDonald (2008).

\subsection{2.4 Nonlinear matter power spectrum in real space}

For a nonlinear matter power spectrum in real space, we can simply use the perturbative solution for $\delta_k(\tau)$ in equation (2.22). That is,

\[
K_1^{(s)}(q_1) = 1
\]

\[
K_2^{(s)}(q_1, q_2) = F_2^{(s)}(q_1, q_2)
\]

\[
K_3^{(s)}(q_1, q_2, q_3) = F_3^{(s)}(q_1, q_2, q_3),
\]

where $F_2^{(s)}$ and $F_3^{(s)}$ are presented in equation (2.32), and equation (2.34), respectively.

\textsuperscript{10}For derivation, see Appendix C.
2.4.1 Gaussian case

By substituting this kernels and equation (2.58), we calculate the matter power spectrum in real space as

\[ P_m(k, z) = D^2(z)P_L(k) + D^4(z) [P_{m,22}(k) + 2P_{m,13}(k)] , \]

where \( D(z) \) is the linear growth factor and

\[ P_{m,22}(k) = \frac{k^3}{98 (2\pi)^2} \int dr P_L(kr) \int_{-1}^{1} dx P_L(k \sqrt{1 + r^2 - 2rx}) \left[ \frac{7x + 3r - 10rx^2}{1 + r^2 - 2rx} \right]^2 \]

\[ P_{m,13}(k) = \frac{k^3}{504 (2\pi)^2} P_L(k) \int dr P_L(kr) \times \left[ \frac{12}{r^2} - 158 + 100r^2 - 42r^4 + \frac{3}{r^3} (r^2 - 1)^3 (7r^2 + 2) \ln \left( \frac{r + 1}{|r - 1|} \right) \right] . \]

Here, \( P_L(k) \) is calculated at present where linear growth factor is normalized to be unity. This form is practical useful as, for given linear power spectrum, we only need to calculate the integration once for a give redshift. Then, the non-linear matter power spectrum for different redshifts can be calculated by simple rescaling of \( P_{11}, P_{22} \) and \( P_{13} \) with corresponding powers of linear growth factor.

We show that, in the quasi-nonlinear regime at high redshift, this analytic expression accurately models the nonlinear evolution of the matter power spectrum from a series of N-body simulations we run in Chapter 3. We also verify the result against the matter power spectrum from Millennium Simulation (Springel et al., 2005) in Section 4.2.

2.4.2 Non-Gaussianity case

We also calculate the leading order non-Gaussian term due to the local type primordial non-Gaussianity from equation (2.66).

\[ \Delta P_{m,G}(k, z) = \frac{3}{7} f_{NL} H_0^2 \Omega_m D^3(z) \frac{k}{(2\pi)^2} \int dr P_L(kr) \int_{-1}^{1} dx \frac{T(k\sqrt{1 + r^2 - 2rx})}{T(k)} \left( \frac{7x + 3r - 10rx^2}{1 + r^2 - 2rx} \right) \]

\[ \times \left[ 2P_L(k)(1 + r^2 - 2rx) \frac{T(k\sqrt{1 + r^2 - 2rx})}{T(k)} + P_L(k) \frac{T(k)}{1 + r^2 - 2rx} \right] \]

This equation is first derived from Taruya et al. (2008), and they find that non-Gaussianity signal in matter power spectrum is so tiny that gigantic space based survey with survey volume of 100 [Gpc$^3$/h$^3$] only detect with large uncertainty (\( \Delta f_{NL} \approx 300 \)).
2.5 Nonlinear galaxy power spectrum in real space

In galaxy surveys, what we observe are galaxies, not a matter fluctuation. Since galaxies are the biased tracers of the underlying matter fluctuation, we have to understand how galaxy distribution and the matter fluctuation are related. This relation is known to be very complicated, because we have to understand the complex galaxy formation processes as well as the dark matter halo formation processes for given matter fluctuation in order to calculate the relation from the first principle. Both of which are the subject of the forefront research and need to be investigated further.

We simplify the situation by assuming that the galaxy formation and halo formation are local processes. This assumption is valid on large enough scale, which may include the quasi nonlinear scale where PT models the nonlinear evolution very well. Then, the galaxy over/under density at a given position depends only on the matter fluctuation at the same position. Therefore, the galaxy density contrast $\delta_g(x)$ can be Taylor-expanded with respect to the smoothed matter density contrast

$$\delta_R(x) = \int d^3 y \delta(y) W_R(x - y),$$

as

$$\delta_g(x) = \epsilon + c_1 \delta_R(x) + \frac{c_2}{2} \left[ \delta_R^2(x) - \langle \delta_R^2 \rangle \right] + \frac{c_3}{6} \delta_R^3(x) + \cdots,$$

where $\langle \delta^2 \rangle$ is subtracted in order to ensure $\langle \delta_g \rangle = 0$ (McDonald, 2006). Here, $W_R(r)$ is the smoothing (filtering) function, and $\tilde{W}_R(k)$ is its Fourier transform\(^\text{11}\). We also introduce the stochastic parameter $\epsilon$ which quantifies the “stochasticity” of galaxy bias, i.e. the relation between $\delta_g(x)$ and $\delta_R(x)$ is not completely deterministic, but contains some noise with zero mean, $\langle \epsilon \rangle = 0$ (e.g., Yoshikawa et al. (2001), and reference therein). We further assume that the stochasticity is a white noise, and is uncorrelated with the density fluctuations i.e., $\langle \epsilon^2(k) \rangle \equiv c_0^2$, $\langle \epsilon \delta_R \rangle = 0$. The coefficients of expansion, $c_n$’s, encode the detailed formation history of galaxies, and may vary for different morphological types, colors, flux limits, etc.

By using a convolution theorem, we calculate the Fourier transform of the local bias expansion of equation (2.72)

$$\delta_g(k) = \epsilon(k) + c_1 \delta_R(k) + \frac{c_2}{2} \int \frac{d^3 q_1}{(2\pi)^3} \int d^3 q_2 \delta_R(q_1) \delta_R(q_2) \delta_D(k - q_{12}),$$

$$+ \frac{c_3}{6} \int \frac{d^3 q_1}{(2\pi)^3} \int \frac{d^3 q_2}{(2\pi)^3} \int d^3 q_3 \delta_R(q_1) \delta_R(q_2) \delta_R(q_3) \delta_D(k - q_{123}),$$

\(^\text{11}\)For the notational simplicity, we shall drop the tilde, but it should be clear from the argument whether the filtering function is defined in real space or Fourier space.
in terms of the smoothed non-linear density field $\delta_R(k)$:

$$
\delta_R(k) \equiv \mathcal{W}_R(k) \left[ \delta^{(1)}(k) + \delta^{(2)}(k) + \delta^{(3)}(k) + \ldots \right].
$$

(2.74)

Here, $\delta^{(n)}(k)$ denotes the $n$-th order perturbation theory solution in equation (2.22). We find the kernel for the real space galaxy density contrast by substituting equation (2.74) into equation (2.73).

$$
K_1^{(s)}(q_1) = c_1 \mathcal{W}_R(q_1)
$$
$$
K_2^{(s)}(q_1, q_2) = \frac{c_2}{2} \mathcal{W}_R(q_1) \mathcal{W}_R(q_2) + c_1 F^{(s)}_2(q_1, q_2) \mathcal{W}_R(q_{12})
$$
$$
K_3^{(s)}(q_1, q_2, q_3) = \frac{c_3}{6} \mathcal{W}_R(q_1) \mathcal{W}_R(q_2) \mathcal{W}_R(q_3) + c_1 F^{(s)}_3(q_1, q_2, q_3) \mathcal{W}_R(q_{123})
$$
$$
+ \frac{c_2}{3} \left[ F^{(s)}_2(q_1, q_2) \mathcal{W}_R(q_3) \mathcal{W}_R(q_{12}) + (2 \text{ cyclic}) \right].
$$

2.5.1 Gaussian case

As we assume that the stochastic parameter $\epsilon(k)$ is not correlated with the density field, we calculate the real space galaxy power spectrum in Gaussian case as

$$
P_g(k, z) = \langle \epsilon^2 \rangle + D^2(z) P_{g,11}(k) + D^4(z) [P_{g,22}(k) + 2P_{g,13}(k)],
$$

(2.75)

where

$$
P_{g,11}(k) = c_1^2 \mathcal{W}_R^2(k) P_L(k)
$$

(2.76)

is the linear bias term with linear matter power spectrum and $P_{g,22}$ and $P_{g,13}$ include the non-linear bias as well as the non-linear growth of the matter density field:

$$
P_{g,22}(k) = \frac{c_2^2}{2} \int \frac{d^3q}{(2\pi)^3} \mathcal{W}_R^2(q) P_L(q) \mathcal{W}_R(|k - q|) P_L(|k - q|)
$$
$$
+ 2c_1c_2 \mathcal{W}_R(k) \int \frac{d^3q}{(2\pi)^3} \mathcal{W}_R(q) P_L(q) \mathcal{W}_R(|k - q|) P_L(|k - q|) F_2^{(s)}(q, k - q)
$$
$$
+ 2c_1^2 \mathcal{W}_R^2(k) \int \frac{d^3q}{(2\pi)^3} P_L(q) P_L(|k - q|) \left[ F_2^{(s)}(q, k - q) \right]^2
$$

(2.77)

$$
P_{g,13}(k) = \frac{1}{2} c_1 c_3 \mathcal{W}_R^2(k) P_L(k) \sigma_R^2 + 3c_1^2 \mathcal{W}_R^2(k) P_L(k) \int \frac{d^3q}{(2\pi)^3} P_L(q) F_3^{(s)}(q, -q, k)
$$
$$
+ 2c_1 c_2 \mathcal{W}_R(k) P_L(k) \int \frac{d^3q}{(2\pi)^3} P_L(q) \mathcal{W}_R(q) \mathcal{W}_R(|k - q|) F_2(k, -q).
$$

(2.78)
Here,

\[ \sigma^2_R = \int \frac{d^3q}{(2\pi)^3} P_L(q) |\hat{W}_R(q)|^2 \]

is the root-mean-squared (r.m.s.) value of the smoothed linear density contrast at \( z = 0 \).

This equation is first derived in Smith et al. (2007) in the context of HaloPT, but they found the poor agreement between equation (2.75) and the halo power spectrum directly calculated from N-body simulation. However, it does not necessarily mean that the local bias ansatz of equation (2.72) is wrong. We rather attribute the failure of their modeling to the inaccurate modeling of the bias parameters (\( c_1, c_2 \) and \( c_3 \)) from the halo model. For example, the halo/galaxy power spectrum driven from local bias successfully models the halo/galaxy power spectrum from Millennium Simulation when fitting nonlinear bias parameters in Chapter 4.

Instead of using the bias parameters from the halo model, we shall treat the bias parameters as free parameters, and fit them to the observed galaxy power spectrum\(^{12}\). In order to convert equation (2.75) into the practically useful form for fitting, we need to re-parametrize the bias parameters. It is because the theoretical template for fitting galaxy power spectrum shown in equation (2.75) has a few problems, as it was first pointed out by McDonald (2006). First, \( P_{g,13}(k) \) in equation (2.75) contains \( \sigma^2_R \), which diverges, or is sensitive to the details of the small scale treatment, e.g. imposing a cut-off scale, choosing particular smoothing function, etc. Second, the first term in equation (2.77), one proportional to \( c_2^2 \), approaches to a constant value on large scale limit i.e., \( k \to 0 \). The constant value can be large depending on the spectral index, or, again, sensitive to the small scale treatment.

In order to avoid these problems, we re-define the nonlinear bias parameters such that all terms sensitive to the small-scale treatment are absorbed into the parametrization. In other words, as we are interested in the power spectrum on sufficiently large scales, \( k \ll 1/R \), we want to make the effect of small scale smoothing to be shown up only through the value of the bias parameters. On such large scales, we could approximate \( \hat{W}_R(k) = 1 \), and the last term of \( P_{g,13}(k) \) (Eq. [2.78]) is simply proportional to the linear power spectrum, and the proportionality constant depends only on the smoothing scale \( R \). That is, if we

---

\(^{12}\)For the goodness of the fitting method including the effect of fitting to extracting the cosmological parameters, see Chapter 4.
then, \( S_R(k) \) approaches to the constant value
\[
S_R(k) \rightarrow \frac{13}{84} + \frac{1}{4\sigma_R^2} \int \frac{d^3q}{(2\pi)^3} P_L(q) W_R(q) \frac{\sin(qR)}{qR},
\]
(2.80)
as \( k \to 0 \). See Figure 5.7 for the shape of \( S_R(k) \) for \( R = 1, 2, 5 \) and 10 Mpc/h. We also show the large scale asymptotic value of \( S_R(0) \) as a function of \( R \) in Figure 5.8. With the definition of \( S_R(k) \) in equation (2.79) and the non-linear matter power spectrum in equation (2.68), we rewrite the equation (2.75) as
\[
P_g(k, z) = \langle \epsilon^2 \rangle + c_1^2 W_R^2(k) P_m(k, z) + D^2(z) \left[ c_1 c_3 \sigma_R^2 + 8 c_1 c_2 \sigma_R^2 S_R(k) \right] W_R^2(k) P_L(k, z) + \frac{c_2^2}{2} \int \frac{d^3q}{(2\pi)^3} W_R^2(q) P_L(q, z) W_R^2(|k-q|) P_L(|k-q|, z) + 2 c_1 c_2 W_R(k) \int \frac{d^3q}{(2\pi)^3} W_R(q) P_L(q, z) \times W_R(|k-q|) P_L(|k-q|, z) F_2^{(s)}(q, k-q).
\]
(2.82)
We re-parametrize the nonlinear bias parameters as,
\[
P_0 = \langle \epsilon^2 \rangle + D^4(z) \frac{c_2^2}{2} \int \frac{d^3q}{(2\pi)^3} q^2 \left[ P_L(q) W_R^2(q) \right]^2,
\]
(2.83)
\[
b_1^2 = \langle \epsilon^2 \rangle + D^2(z) \left[ c_1 c_3 \sigma_R^2 + 8 c_1 c_2 \sigma_R^2 S_R(k) \sigma_R^2 \right],
\]
(2.84)
\[
b_2 = \frac{c_2}{b_1},
\]
(2.85)
then, the galaxy power spectrum becomes
\[
P_g(k, z) = P_0 + b_1^2 \left[ W_R^2(k) P_m(k, z) + b_2 D^4(z) P_{b2}(k) + b_2^2 D^4(z) P_{b22}(k) \right],
\]
(2.86)
\[\text{For general window function } W_R(k), \text{ as } k \to 0,
\]
\[
\int \frac{d^3q}{(2\pi)^3} P_L(q) W_R(q) W_R(|k-q|) F_2^{(s)}(k-q) \to \frac{17}{21} \sigma_R^2 + \frac{1}{6} \int \frac{d^3q}{(2\pi)^3} q^2 P_L(q) \frac{d^3W_R(q)}{dq}.
\]
(2.81)
Therefore, if we do not employ the smoothing function, i.e. \( W_R(k) = \), the integration becomes \( 17/21 \sigma_R^2 \), and hence, the last term of \( P_{g,13}(k) \) is simply \( 34/21 c_1 c_2 \sigma_R^2 P_L(k) \). This result coincides with McDonald (2006).
where $P_{b^2}(k)$ and $P_{b^22}(k)$ are given by

$$P_{b^2}(k) = 2W_R(k) \frac{d^3q}{(2\pi)^3} W_R(q) P_L(q) W_R(|k-q|) P_L(|k-q|) F^s_2(q, k - q)$$  \hspace{1cm} (2.87)$$

$$P_{b^22}(k) = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} W^0_R(q) P_L(q) \left[ W^2_R(|k-q|) P_L(|k-q|) - W^2_R(q) P_L(q) \right].$$  \hspace{1cm} (2.88)$$

Note that equation (2.86) is the same as the original equation up to next-to-leading order, e.g. $\sigma_R^2 P_m \simeq \sigma_R^2 P_L + O(P^3_L)$. As we have desired, the terms depending on the smoothing scale $R$ are absorbed into the newly defined bias parameters $P_0$, $b_1$ and $b_2$, and $P_m(k)$, $P_{b^2}(k)$, $P_{b^22}(k)$ are independent of the smoothing scale on large scales, $k \gg 1/R$.

Note that $b_1$ we defined here reduces to the ‘effective bias’ of Heavens et al. (1998) in the $R \rightarrow 0$ limit, and in $k \rightarrow 0$ limit, equation (2.86) approaches to

$$P_g(k) \rightarrow P_0 + b_1^2 P_m(k),$$

the usual linear bias model plus a constant.

The ‘re-parametrized’ nonlinear bias parameters, $P_0$, $b_1$, $b_2$, encode the complex galaxy formation processes, which will be very hard to model from the first principle (Smith et al., 2007). Nevertheless, the nonlinear galaxy power spectrum we calculate here has to be the ‘right’ prescription as long as the locality of bias assumption is correct in the quasi-nonlinear regime. In Chapter 4, we tested the nonlinear bias model in equation (2.86) against the halos/galaxies power spectrum of the Millennium Simulation (Springel et al., 2005). In order to test the prescription itself, we set $P_0$, $b_1$ and $b_2$ as free parameters, and fit the measured power spectrum from Millennium simulation with equation (2.86). We found that nonlinear bias model provides a significantly better fitting than the linear bias model. In addition to that, we could reproduce the correct distance scales within a statistical error-bar, when marginalizing over three free nonlinear bias parameters.

### 2.5.2 Galaxy-matter cross power spectrum

As we shall marginalize over the bias parameters, the more do we add information about bias parameters, the better can we estimate the other cosmological parameters. The galaxy-matter cross power spectrum at high redshift can be a source of such an additional information, as it is proportional to the galaxy density contrast; thus, it also depends on the bias parameter. We can measure the galaxy-matter cross power spectrum from the galaxy-galaxy, and galaxy-CMB weak lensing measurements$^{14}$.

---

$^{14}$We study the galaxy-galaxy, and galaxy-CMB weak lensing on large scales in Chapter 6.
The fastest way of calculating the galaxy-matter cross power spectrum is using the calculation of galaxy-galaxy power spectrum. Let us abbreviate equation (2.73) as

$$\delta_g(k) = \epsilon(k) + c_1 \delta_R(k) + c_2 \delta_g^{(2)}(k) + c_3 \delta_g^{(3)}(k). \quad (2.89)$$

Then, we can think of calculating the galaxy-galaxy power spectrum as

$$\langle \delta_g(k) \delta_g(k') \rangle = \langle \epsilon^2 \rangle + c_1 \langle \delta_R(k) \delta_g(k') \rangle + \left( c_2 \delta_g^{(2)}(k) + c_3 \delta_g^{(3)}(k) \right) \delta_g(k')$$

From equation (2.90), it is clear that adding up the terms proportional to $c_1$ in $P_g(k)$ are the same as $c_1 [2P_{gm}(k) - c_1 P_m(k)]$. Therefore, the nonlinear galaxy-matter cross correlation function is

$$P_{gm}(k, z) = c_1 W_R^2(k) P_m(k, z) + D^2(z) \left[ \frac{c_3}{2} \sigma_R^2 + 4c_2 \sigma_R^2 \bar{\sigma}_R(k) \right] W_R^2(k) P_L(k, z)$$

$$+ c_2 W_R(k) \int \frac{d^3q}{(2\pi)^3} W_R(q) P_L(q, z)$$

$$\times W_R(\|k - q\|) P_L(\|k - q\|, z) F_q^{(s)}(q, k - q). \quad (2.91)$$

We also re-parametrize the bias for this case,

$$\bar{b}_1 = c_1 + D^2(z) \left[ \frac{c_3}{2} \sigma_R^2 + 4c_2 \sigma_R^2 \bar{\sigma}_R(k) \right] \quad (2.92)$$

$$\bar{b}_2 = \frac{c_2}{\bar{b}_1}, \quad (2.93)$$

so that the galaxy-matter cross power becomes

$$P_{gm}(k, z) = \bar{b}_1 \left[ W_R^2(k) P_m(k, z) + \frac{\bar{b}_2}{2} D^4(z) P_{b2}(k) \right], \quad (2.94)$$

where $P_{b2}(k)$ is defined in equation (2.87). Note that when $\sigma_R \ll 1$, $\bar{b}_1 \sim b_1$ and $\bar{b}_2 \sim b_2$. 

32
2.5.3 non-Gaussian case

What about the non-Gaussian correction term? We calculate the non-Gaussian term by substituting the real space galaxy kernels into equation (2.66).

\[ \Delta P_{g,nG}(k, z) = 2c_1 f_{NL} D^3(z) \mathcal{W}_R(k) \mathcal{M}(k) \int \frac{d^3q}{(2\pi)^3} \mathcal{M}(q) \mathcal{M}(|k - q|) \mathcal{P}_\phi(q) \left[ 2\mathcal{P}_\phi(k) + \mathcal{P}_\phi(|k - q|) \right] \]

\[ \times \left[ c_2 \mathcal{W}_R(q) \mathcal{W}_R(|k - q|) + 2c_1 \mathcal{W}_R(k) P^2(k) \mathcal{W}_R(q, k - q) \right] \]

\[ = c_1^2 \mathcal{W}_R^2(k) \Delta P_{m,nG}(k, z) + 4c_1 c_2 D^3(z) \sigma^2_{NL} \mathcal{W}_R(k) \mathcal{F}_R(k) \frac{P_L(k)}{M(k)} \]

(2.95)

Here, \( \Delta P_{m,nG} \) is the non-Gaussian correction to the matter power spectrum, and

\[ \mathcal{F}_R(k) \equiv \frac{1}{2\sigma^2_R} \int \frac{d^3q}{(2\pi)^3} \mathcal{M}_R(q) \mathcal{M}_R(|k - q|) \mathcal{P}_\phi(q) \left[ \frac{P_\phi(|k - q|)}{P_\phi(k)} + 2 \right] \]

(2.96)

is a function which is unity on large scales \( k \ll 1/R \), See, e.g., Matarrese & Verde, 2008). See Figure 5.6 for the shape of \( \mathcal{F}_R(k) \) for \( R = 1, 2, 5 \) and 10 Mpc/h.

The first term in equation (2.95) is simply the non-Gaussian matter power spectrum multiplied by the linear bias factor. The second term in Eq. (2.95) is the non-Gaussianity term generated by non-linear bias, and shows the same behavior as the scale dependent bias from local type primordial non-Gaussianity\(^{15} \). In fact, this term reduces to the result of MLB formula (Matarrese et al., 1986; Matarrese & Verde, 2008, see, Appendix K.2 for derivation)

\[ \Delta P_{g,nG}(k, z) = 4c_1 (c_1 - 1) \delta_c D(z) \alpha f_{NL} \mathcal{W}_R(k) \mathcal{F}_R(k) \frac{P_L(k)}{M(k)} \]

(2.97)

in the linear regime and for the high-peak limit of the halo model\(^{16} \). It is sufficient to show that \( c_1 c_2 D^2(z) \sigma^2_{NL} \) becomes \( \alpha c_1 (c_1 - 1) \delta_c \) in the high-peak limit. Consider the bias parameters from the halo model: (Scoccimarro et al., 2001b)

\[ c_1 = 1 + \frac{\alpha \nu^2 - 1}{\delta_c} + \frac{2p/\delta_c}{1 + (\alpha \nu^2)^p} \]

(2.98)

\[ c_2 = \frac{8}{21} (c_1 - 1) + \frac{\alpha \nu^2}{\delta_c^2} (\alpha \nu^2 - 3) + \frac{2p/\delta_c}{1 + (\alpha \nu^2)^p} \left( \frac{1 + 2p}{\delta_c} + 2 \frac{\alpha \nu^2 - 1}{\delta_c} \right), \]

(2.99)

where \( \delta_c \simeq 1.686 \) is the critical overdensity above which halo forms, and \( \nu \equiv \delta_c/(D^2(z) \sigma^2_{NL}) \).

For Press-Schechter mass function (spherical collapse, Press & Schechter, 1974; Mo & White,

\(^{15}\)For the scale dependent bias, see the introduction in Chapter 5, and Appendix I.3.

\(^{16}\)\( \alpha \) here is the same as \( q \) in Carbone et al. (2008). We reserve \( q \) for the Fourier space measure.
\[ c_1 c_2 D^2(z) \sigma_R^2 \approx c_1 \alpha^2 \nu^2 \left( \frac{D^2(z) \sigma_R^2}{\delta_c^2} \right) = c_1 \alpha^2 \nu^2 \approx \alpha c_1 (c_1 - 1) \delta_c. \] (2.100)

This relation motivate us to define a new bias parameter \( \tilde{b}_2 \equiv \sigma_R^2 D^2(z) c_2 / c_1 \), which approaches \( \tilde{b}_2 \to \alpha \delta_c \) for the high peak limit of the halo model.

By using a re-parametrized bias, \( b_1 \) and \( \tilde{b}_2 \), the non-Gaussian correction term becomes

\[
\Delta P_{g,nG}(k, z) = b_1^2 \left[ W_R^2(k) \Delta P_{m,ng}(k, z) + 6 \tilde{b}_2 f_{NL} D(z) W_R(k) F_R(k) \frac{H_0^2 \Omega_m P_L(k)}{k^2 T(k)} \right].
\] (2.101)

and on large scales (\( k \ll 1/R \)), for high-peak, the formula reduces to the usual form in the literature (Dalal et al., 2008; Matarrese & Verde, 2008; Slosar et al., 2008; Afshordi & Tolley, 2008; Taruya et al., 2008; McDonald, 2008; Sefusatti, 2009):

\[
\Delta P_{g,nG}(k, z) = b_1^2 \left[ \Delta P_{m,ng}(k, z) + 6 \delta_c f_{NL} D(z) \frac{H_0^2 \Omega_m P_L(k)}{k^2 T(k)} \right].
\] (2.102)

Although equation (2.101) coincides with equation (2.102) for high peak limit, it may not be the dominant contribution of scale-dependent bias for intermediate size peaks where the nonlinear bias \( b_2 \) is actually small. Recent study based on Peak Background Split method (Giannantonio & Porciani, 2010) suggests that for non-Gaussian case, the local ansatz [Eq. (2.72)] has to be modified to include the effect of Gaussian piece of gravitational potential \( \phi(x) \) directly as

\[
\delta_f(\mathbf{x}) = b_{10} \delta(\mathbf{x}) + b_{01} \phi(\mathbf{x}) + \frac{1}{2!} \left( b_{20} \delta^2(\mathbf{x}) + 2 b_{11} \delta(\mathbf{x}) \phi(\mathbf{x}) + b_{02} \phi^2(\mathbf{x}) + \cdots \right),
\] (2.103)

where \( b_{ij} \)s are bias parameters. If this holds, the non-Gaussianity signal from the power spectrum of very massive clusters (where \( \tilde{b}_2 \) is indeed close to \( \alpha \delta_c \)) is expected to be twice as high as the scale dependent bias in equation (2.102).

### 2.6 Nonlinear matter power spectrum in redshift space

In the previous sections, we have calculated the matter power spectrum and the galaxy power spectrum in real space. By real space, we mean an idealistic universe where
we can observe the true distance of the galaxies (or matter particles) relative to us. With
galaxy survey alone, however, we cannot measure the true distance to the galaxies, as we
infer the distance to a galaxy from the galaxy’s spectral line shift by assuming the Hubble
law. The problem here is that the observed spectral shift depends not only on the position
of the galaxy (as a result of the expansion of the Universe), but also on the peculiar velocity
of the galaxy. As a result, the three-dimensional map of the galaxies generated from galaxy
surveys is different from the real space galaxy distribution. In contrast to the real space, we
call the observed coordinate of galaxies the redshift space, and the radial distortion in the
redshift space due to the peculiar velocity is called redshift space distortion.

We formulate the redshift space position vector $s$ as follow:

$$s = x + (1 + z) \frac{v_r(x)}{H(z)} \hat{r}.$$  \hspace{1cm} (2.104)

Here, $x$ denotes the real space comoving position vector, and $z$ denotes the redshift of galaxy
without peculiar velocity, $H(z)$ is the Hubble parameter at that redshift, and $v_r$ denotes
the line-of-sight directional peculiar velocity. As redshift space distortion is due to the
peculiar velocity, we can model it by using the peculiar velocity solution $\theta_k(\tau)$ (Eq. [2.23])
of perturbation theory. In this section and the next section, we calculate the matter power
spectrum and the galaxy power spectrum in redshift space, respectively.

How does the real space power spectrum changed under the redshift space distortion? In order to simplify the analysis, we make the plane parallel approximation that the
galaxies are so far away that the radial direction is parallel to the $\hat{z}$ direction\(^\text{17}\). Also, we
define the reduced velocity field $u \equiv v/(f\Omega_c)$ so that equation (2.104) becomes

$$s = x + f u_z(x) \hat{z}.$$  \hspace{1cm} (2.105)

As $u(k) \equiv v(k)/(f\Omega_c) = -i k \theta_k(\tau)/(k^2 f\Omega_c)$, the Fourier transform of $u_z(x)$ becomes

$$u_z(k, \tau) = \frac{i \mu}{k} \sum_{n=1}^{\infty} \int \frac{d^3 q_1}{(2\pi)^3} \cdots \int \frac{d^3 q_n}{(2\pi)^3} \int d^3 q_{n-1} \delta^D(k - \sum_{i=1}^{n} q_i)$$

$$\times G_n^{(s)}(q_1, q_2, \ldots, q_n) \delta_1(q_1, \tau) \cdots \delta_1(q_n, \tau) \equiv \frac{i \mu}{k} \eta(k, \tau),$$  \hspace{1cm} (2.106)

where $\mu = k \cdot \hat{z}/k$ is the directional cosine between the wavenumber vector $\hat{k}$ and the line of
sight direction $\hat{z}$. Note that the time evolution of the new variable $u_z(k)$ only comes from
the linear density contrast.

\textsuperscript{17}For a redshift space distortion including a light-cone effect, see, e.g. de Lai & Starkman (1998); Ya-
mamoto et al. (1999); Nishioka & Yamamoto (2000); Wagner et al. (2008).
Let us denote the real space over-density as $\delta_r(\mathbf{x})$, and the redshift space over-density as $\delta_s(\mathbf{s})$. The mass conservation relates the measure in real space $d^3r$ and that in redshift space $d^3s$ as

$$(1 + \delta_s(\mathbf{s}))d^3s = (1 + \delta_r(\mathbf{x}))d^3x.$$  \hfill (2.107)

By using this relation, we find the *exact* relation between two over-densities in Fourier space (Scoccimarro, 2004; Matsubara, 2008).

$$\delta_s(\mathbf{k}) = \int d^3s \left[ 1 + \delta_s(\mathbf{s}) \right] e^{-i\mathbf{k} \cdot \mathbf{s}} - \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}}$$

$$= \int d^3x \left[ 1 + \delta_r(\mathbf{x}) \right] e^{-i\mathbf{k} \cdot \mathbf{x} + f u_z(x) x} - \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}}$$

$$= \delta_r(\mathbf{k}) + \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}} \left( e^{-i f u_z(x)} - 1 \right) \left[ 1 + \delta_r(\mathbf{x}) \right]$$  \hfill (2.108)

In order to calculate the 3rd order power spectrum, we Taylor-expand the exponential function up to 3rd order:

$$\delta_s(\mathbf{k}) = \delta_r(\mathbf{k}) + f \mu^2 \eta(\mathbf{k}) - \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}}$$

$$\times \left[ ik_z f u_z(x) \delta_r(\mathbf{x}) + \frac{1}{2} k_z^2 f^2 u_z^2(x) + \frac{1}{2} k_z^2 f^2 u_z^2(x) \delta_r(\mathbf{x}) - \frac{i}{6} k_z^3 f^3 u_z^3(x) \right].$$  \hfill (2.109)

We calculate the 3rd order nonlinear matter kernels in redshift space from equation (2.109) and using the convolution theorem:

$$K_1^{(s)}(\mathbf{k}) = 1 + f \mu^2$$  \hfill (2.110)

$$K_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) = F_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) + f \mu^2 G_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2)$$

$$+ \frac{f k \mu}{2} \left( \frac{q_1 z}{q_1^2} + \frac{q_2 z}{q_2^2} \right) + \frac{(f k \mu)^2}{2} \frac{q_1 z q_2 z}{q_1^2 q_2^2}$$  \hfill (2.111)

$$K_3^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = F_3^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) + f \mu^2 G_3^{(s)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$$

$$+ \frac{(f k \mu)^2}{6} \left( \frac{q_1 z q_2 z}{q_1^2 q_2^2} + \frac{q_2 z q_3 z}{q_2^2 q_3^2} + \frac{q_3 z q_1 z}{q_3^2 q_1^2} \right) + \frac{(f k \mu)^3}{6} \frac{q_1 z q_2 z q_3 z}{q_1^2 q_2^2 q_3^2}$$

$$+ \frac{f k \mu}{3} \left\{ F_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) \frac{q_3 z}{q_3^2} + (2 \text{ cyclic}) \right\}$$

$$+ \frac{f k \mu}{3} \left\{ G_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) \frac{q_3 z (q_1 + q_2 z)}{|q_1 + q_2|^2} + (2 \text{ cyclic}) \right\}$$

$$+ \frac{f k \mu}{3} \left\{ G_2^{(s)}(\mathbf{q}_1, \mathbf{q}_2) \frac{q_3 z (q_1 + q_2 z)}{|q_1 + q_2|^2} + (2 \text{ cyclic}) \right\}.$$  \hfill (2.112)
Note that the kernels we present here coincide those in equation (13) of Heavens et al. (1998) when setting \( b_1 = 1 \) and \( b_2 = 0 \).

Before calculating the power spectrum, it is instructive to compare the result here with other formulas in the literature. In linear regime, equation (2.109) reduces to

\[
\delta_s(k) = (1 + f \mu^2) \delta^{(1)}(k),
\]

and the redshift space matter power spectrum becomes

\[
P_s(k, \mu, z) = \left(1 + f \mu^2\right)^2 D^2(z) P_L(k),
\]

as shown in Kaiser (1987). That is, as a result of non-linear mapping between real and redshift space, the redshift space matter power spectrum is no longer spherically symmetric. This is called ‘Kaiser effect’ in the literature. Note that the redshift space distortion effect is larger for the line of sight direction (\( \mu = 1 \)), and it does not change the power spectrum along the direction perpendicular to the line of sight (\( \mu = 0 \)).

If we pick up the linear terms in equation (2.109)

\[
\delta_s(k) = \delta_r(k) + f \mu^2 \eta(k),
\]

and use the third order solution of \( \delta_r(k) \) and \( \eta(k) \), the redshift space matter power spectrum reduces to the formula given in Scoccimarro (2004):

\[
P_s(k, \mu, z) = P_{\delta\delta}(k, z) + 2f \mu^2 P_{\delta\theta}(k, z) + f^2 \mu^4 P_{\theta\theta}(k, z).
\]

Here, \( P_{\delta\delta}(k, z) \) is the same as the non-linear matter power spectrum \( P_m(k, z) \) in real space [Eq. (2.68)], \( P_{\delta\theta}(k, z) \) is the non-linear density-velocity cross power spectrum

\[
P_{\delta\theta}(k, z) = D^2(z) P_L(k) + D^4(z) \left[ P_{\delta\theta, 22}(k) + P_{\delta\theta, 13}(k) + P_{\theta\theta, 13}(k) \right],
\]

and \( P_{\theta\theta}(k, z) \) is the non-linear velocity power spectrum

\[
P_{\theta\theta}(k, z) = D^2(z) P_L(k) + D^4(z) \left[ P_{\theta\theta, 22}(k) + 2 P_{\theta\theta, 13}(k) \right],
\]
where $P_{56,13}(k)$ is shown in equation (2.70), and

$$
P_{66,22}(k) = 2 \int \frac{d^3 q}{(2\pi)^3} P_L(q) P_L(|k - q|) \hat{F}^4_2(q, k - q) \hat{G}^4_2(q, k - q),
$$

$$
= \frac{1}{98 (2\pi)^2} \int dr P_L(kr) \int_{-1}^{1} dx P_L(k \sqrt{1 + r^2 - 2rx}) \times \left( \frac{7x + 3r - 10rx^2}{(1 + r^2 - 2rx)^2} \right),
$$

$$
(2.119)
$$

$$
P_{66,22}(k) = 2 \int \frac{d^3 q}{(2\pi)^3} P_L(q) P_L(|k - q|) \left[ \hat{G}^4_2(q, k - q) \right]^2
$$

$$
= \frac{1}{98 (2\pi)^2} \int dr P_L(kr) \int_{-1}^{1} dx P_L(k \sqrt{1 + r^2 - 2rx}) \left[ \frac{7x - r - 6rx^2}{1 + r^2 - 2rx} \right]^2,
$$

$$
(2.120)
$$

$$
P_{66,13}(k) = 3 P_L(k) \int \frac{d^3 q}{(2\pi)^3} P_L(q) \hat{G}^4_3(q, k - q)
$$

$$
= \frac{1}{504 (2\pi)^2} P_L(k) \int dr P_L(kr) \times \left[ \frac{36}{r^2} - 246 + 12r^2 - 18r^4 + \frac{9}{r^3}(r^2 - 1)^3(r^2 + 2) \ln \left( \frac{1 + r}{|1 - r|} \right) \right].
$$

$$
(2.121)
$$

### 2.6.1 Gaussian case

Substituting these kernels into equation (2.58), we find the 3rd order nonlinear redshift space matter power spectrum. The integration becomes particularly easier when we align $\hat{q}$ or $\hat{\phi}_q$ in spherical polar coordinate parallel to the direction of the wavenumber vector, $\hat{k}$. With this coordinate choice, the line of sight component of $q$ can be written as

$$
q_z = q(\mu x - \sqrt{1 - \mu^2} \sqrt{1 - x^2} \cos \phi_q),
$$

where $x$ denotes the directional cosine between $q$ and $k$, and we can integrate some part of angular integration analytically (Matsubara, 2008). After the analytical angular integration, the 3rd order matter power spectrum in redshift space becomes

$$
P_s(k, \mu, z) = D^2(z) P_{s,11}(k, \mu) + D^4(z) \left[ P_{s,22}(k, \mu) + 2 P_{s,13}(k, \mu) \right],
$$

$$
(2.122)
$$
where

\[ P_{s,11}(k, \mu) = (1 + f \mu^2)^2 P_L(k), \]  

\[ P_{s,22}(k, \mu) = \sum_{n,m} \mu^{2n} f^m k^2 \int_0^\infty dr P_L(kr) \int_{-1}^1 dx \times P_L \left( k(1 + r^2 - 2rx)^{1/2} \right) \frac{A_{nm}(r, x)}{(1 + r^2 - 2rx)^2}, \]  

\[ P_{s,13}(k, \mu) = (1 + f \mu^2) P_L(k) \sum_{n,m} \mu^{2n} f^m k^2 \int dr P_L(kr) B_{nm}(r), \]  

with

\[ A_{00} = \frac{1}{98} (3r + 7x - 10rx^2)^2 \]  

\[ A_{11} = \frac{2}{49} (3r + 7x - 10rx^2)^2 \]  

\[ A_{12} = \frac{1}{28} (1 - x^2)(7 - 6r^2 - 42rx + 48r^2x^2) \]  

\[ A_{22} = \frac{1}{196} (-49 + 114r^2 + 714rx + 637x^2 - 942r^2x^2 - 1890rx^3 + 1416r^2x^4) \]  

\[ A_{23} = \frac{1}{14} (1 - x^2)(7 - 6r^2 - 42rx + 48r^2x^2) \]  

\[ A_{24} = \frac{3}{16} r^2(x - 1)^2(x + 1)^2 \]  

\[ A_{33} = \frac{1}{14} (-7 + 6r^2 + 54rx + 35x^2 - 66r^2x^2 - 110rx^3 + 88r^2x^4) \]  

\[ A_{34} = \frac{1}{8} (1 - x^2)(2 - 3r^2 - 12rx + 15r^2x^2) \]  

\[ A_{44} = \frac{1}{16} (-4 + 3r^2 + 24rx + 12x^2 - 30r^2x^2 - 40rx^3 + 35r^2x^4) \]

and

\[ B_{00} = \frac{1}{504} \left[ \frac{2}{r^2} (6 - 79r^2 + 50r^4 - 21r^6) + \frac{3}{r^2} (r^2 - 1)^2 (2 + 7r^2) \ln \left( \frac{1 + r}{1 - r} \right) \right] \]  

\[ B_{11} = 3B_{00} \]  

\[ B_{12} = \frac{1}{336} \left[ \frac{2}{r^2} (9 - 89r^2 - 33r^4 + 9r^6) - \frac{9}{r^2} (r^2 - 1)^4 \ln \left( \frac{1 + r}{1 - r} \right) \right] \]  

\[ B_{22} = \frac{1}{336} \left[ \frac{2}{r^2} (9 - 109r^2 + 63r^4 - 27r^6) + \frac{9}{r^2} (r^2 - 1)^2 (1 + 3r^2) \ln \left( \frac{1 + r}{1 - r} \right) \right] \]  

\[ B_{23} = -\frac{1}{3}. \]

This non-linear redshift space power spectrum shows more complicated angular dependence than Kaiser effect.
In Section 3.3, we compare the redshift power spectrum in equation (2.122) with redshift space matter power spectrum we measured from a series of N-body simulations.

2.6.2 non-Gaussian case

We calculate the non-Gaussian correction to the matter power spectrum in redshift space by substituting the kernels in equation (2.110) and equation (2.111) into equation (2.66):

\[
\Delta P_s(k, \mu, z) = 4f_{\text{NL}}D^3(z)(1 + f\mu^2)M(k) \int \frac{d^3q}{(2\pi)^3} \mathcal{M}(q) \mathcal{M}(|k - q|) 
\times P_\phi(q) [2P_\phi(k) + P_\phi(|k - q|)] \left[ F_2^{(s)}(q, k - q) + f\mu^2 G_2^{(s)}(q, k - q) \right]
+ \frac{f\mu}{2} \left[ \frac{kq - q_z}{q^2} + \frac{2}{k - q^2} \right] \left[ \frac{(f\mu)^2 q_z(k\mu - q_z)}{q^2|k - q|^2} \right].
\] (2.126)

As is the case for the non-Gaussianity correction to the matter power spectrum in real space, equation (2.126) is parametrically smaller than the non-linear terms: \( P_{s,22}(k, \mu) \) and \( P_{s,13}(k, \mu) \).

2.7 Nonlinear galaxy power spectrum in redshift space

Finally, in this section, we combine all three non-linear effects on the galaxy power spectrum we shall measure from galaxy surveys: the galaxy power spectrum in redshift space. As real to redshift mapping is the same for the total matter and the galaxies, the redshift space density contrast is also given by equation (2.109), but changing the real space density contrast \( \delta_g(k) \) to the real space galaxy density contrast \( \delta_g(k) \):

\[
\delta_{gs}(k) = \delta_g(k) + f\mu^2 \eta(k) - \int d^3 xe^{-ikx} \times \left[ ik_z f u_z(x) \delta_g(x) + \frac{1}{2} k_z^2 f^2 u_z^2(x) + \frac{1}{2} k_z^2 f^2 u_z^2(x) \delta_g(x) - \frac{i}{6} k_z^3 f^3 u_z^3(x) \right].
\] (2.127)

again, \( \mu \equiv k_z/k \) is the cosine between wave vector \( k \) and the line of sight direction, and \( \eta(k) \) is defined as \( \bar{u}_z(k) \equiv i \mu \eta(k)/k \) (see, equation (2.106)). Note that the first two terms in equation (2.127) lead the linear (Kaiser, 1984) and nonlinear (Scoccimarro, 2004) redshift space power spectrum which has been studied before.
We find the 3rd order kernels of the galaxy density contrast in redshift space by substituting equation (2.73) into equation (2.127). The kernels are

\[
K_1^{(s)}(k) = c_1 W_R(k) + f \mu^2. \tag{2.128}
\]

\[
K_2^{(s)}(q_1, q_2) = \frac{c_2}{2} W_R(q_1) W_R(q_2) + c_1 W_R(q_1 q_2) F_2^{(s)}(q_1, q_2) + f \mu^2 G_2^{(s)}(q_1, q_2)
\]

\[
+ c_1 \frac{f k \mu}{2} \left[ \frac{q_1 z}{q_1^2} W_R(q_2) + \frac{q_2 z}{q_2^2} W_R(q_1) \right] + \frac{(f k \mu)^2}{2} \frac{q_1 z q_2 z}{q_1^2 q_2^2} \tag{2.129}
\]

\[
K_3^{(s)}(q, -q, k) = \frac{c_3}{6} W_R^2(q) W_R(k) + c_1 W_R(k) F_3^{(s)}(q, -q, k)
\]

\[
+ \frac{c_2}{3} W_R(q) \left[ W_R(|k + q|) F_2^{(s)}(q, k) + W_R(|k - q|) F_2^{(s)}(-q, k) \right]
\]

\[
+ f \mu^2 \left[ G_3^{(s)}(q, -q, k) + \frac{c_2}{6} W_R^2(q) \right]
\]

\[
+ c_1 \frac{f k \mu}{3} \left[ W_R(|k - q|) F_2^{(s)}(-q, k) q_z q_z - W_R(|k + q|) F_2^{(s)}(q, k) q_z q_z \right]
\]

\[
+ W_R(q) \left\{ G_3^{(s)}(-q, k) \frac{k_z - q_z}{|k - q|^2} + G_2^{(s)}(q, k) \frac{k_z + q_z}{|k + q|^2} \right\}
\]

\[
+ \frac{(f k \mu)^2}{3} \left[ \frac{c_1 q_z^2}{2} \frac{k_z - q_z}{q^4} W_R(k) + G_2^{(s)}(-q, k) \frac{q_z (k_z - q_z)}{q^2 |k - q|^2} \right.
\]

\[
\left. - G_2^{(s)}(q, k) \frac{q_z (k_z + q_z)}{q^2 |k + q|^2} \right] - \frac{(f k \mu)^3}{6} \frac{q_z^2 k_z}{q^4 k^2} \tag{2.130}
\]

Note that we only show \(K_3^{(s)}(q, -q, k)\), as it is what we need to calculate the third order power spectrum. The kernels in equation (2.128)~(2.130), when setting \(W_R \equiv 1\), coincide those shown in Heavens et al. (1998)\(^{18}\), and also reduces to the kernels for the redshift space matter density contrast, equation (2.110)~(2.112) when setting \(c_1 = 1\) and \(c_2 = c_3 = 0\). For \(\mu = 0\), it reduces to the kernels for the galaxy density contrast in real space, as redshift space distortion does not affect the perpendicular directional wave modes.

As is the case for the matter power spectrum, we can reproduce the formulas widely used in the literature by taking the linear terms in equation (2.127):

\[
\delta_{gs}(k) = \delta_g(k) + f \mu^2 \eta(k). \tag{2.131}
\]

\(^{18}\)Except that the real space kernels, \(J_3\) and \(K_3\), presented in Heavens et al. (1998) have to be replaced by \(K_3^{(s)}\)s and \(G_3^{(s)}\)s in equation (2.34) and equation (2.35), respectively.
By keeping only linear order of $\delta_g(k)$ and $\eta(k)$, and setting $W_R(k) = 1$, we find the linear
galaxy power spectrum in redshift space (Kaiser, 1987)

$$P_{gs}(k, \mu, z) = b^2 (1 + \beta \mu^2) 2D^2(z) P_L(k),$$  \hspace{1cm} (2.132)

where $b$ is the linear bias parameter and $\beta \equiv f/b$. Also, including third order density field
and velocity field, while keeping bias linear, we find the formula used in Shoji et al. (2009):

$$P_{gs}(k, \mu, z) = b^2 \left[ P_{\delta \delta}(k) + 2\beta \mu^2 P_{\delta \theta}(k) + \beta^2 \mu^4 P_{\theta \theta}(k) \right],$$  \hspace{1cm} (2.133)

where $P_{\delta \delta}(k)$, $P_{\delta \theta}(k)$ and $P_{\theta \theta}(k)$ are shown in equation (2.68), equation (2.117) and equation
(2.118), respectively.

### 2.7.1 Gaussian case

Finally, we obtain the 3rd order galaxy power spectrum in redshift space by substituting these kernels into equation (2.58):

$$P_{gs}(k, z) = \langle \epsilon^2 \rangle + D^2(z) P_{gs,11}(k) + D^4(z) \left[ P_{gs,22}(k) + 2P_{gs,13}(k) \right],$$  \hspace{1cm} (2.134)

where

$$P_{gs,11}(k) = (c_1 W_R(k) + f \mu^2)^2 P_L(k)$$  \hspace{1cm} (2.135)

is the same as the linear galaxy power spectrum in the redshift space (linear redshift space
distortion with linear bias, Kaiser, 1987), and non-linear terms are

$$P_{gs,22}(k) = 2 \int \frac{d^3 q}{(2\pi)^3} P_L(q) P_L(|k-q|) \left[ K^{(s)}_2(q, k-q) \right]^2$$

$$= 2 \int \frac{d^3 q}{(2\pi)^3} P_L(q) P_L(|k-q|) \left[ (f \mu^2 G_2^{(s)}(q, k-q))^2 \right.$$

$$+ \left( \frac{c_2}{2} W_R(q) W_R(|k-q|) + c_1 F_2^{(s)}(q, k-q) W_R(k) \right)^2$$

$$+ 2c_1 f \mu^2 W_R(k) F_2^{(s)}(q, k-q) G_2^{(s)}(q, k-q)$$

$$+ F^{(rest)}_{gs,22}(k, \mu; c_1, c_2) \right],$$  \hspace{1cm} (2.136)
\[ P_{gs,13}(k, \mu) = 3(c_1 W_R(k) + f \mu^2 P_L(k)) \int \frac{d^3q}{(2\pi)^3} P_L(q) K_3^{(s)}(q, -q, k) \]
\[ = 3(c_1 W_R(k) + f \mu^2 P_L(k)) \int \frac{d^3q}{(2\pi)^3} P_L(q) \]
\[ \times \left[ \frac{c_3}{6} W_R^2(q) W_R(k) + c_1 W_R(k) F_3^{(s)}(q, -q, k) \right. \]
\[ + \frac{c_2}{3} W_R(q) \left\{ W_R(|k + q|) F_2^{(s)}(q, k) + W_R(|k - q|) F_2^{(s)}(-q, k) \right\} \]
\[ + f \mu^2 \left\{ G_3^{(s)}(q, -q, k) + \frac{c_2}{6} W_R^2(q) \right\} \left[ P_{gs,13}^{(rest)}(k, \mu; c_1) \right]. \tag{2.137} \]

Here, we only show terms which we can be explicitly identified as terms from non-linear evolution of matter \((P_{m,22}(k) \text{ or } P_{m,13}(k))\), from non-linear bias \((P_{b2}(k) \text{ or } P_{b22}(k))\), or from non-linear redshift space distortion \((P_{q0}(k) \text{ or } P_{q0}(k))\). Terms which cannot fall into those categories are called \(P_{gs,22}^{(rest)}(k, \mu; c_1, c_2)\) and \(P_{gs,13}^{(rest)}(k, \mu; c_1)\).

Combining equation (2.135) to (2.137), we calculate the galaxy power spectrum in redshift space

\[ P_{gs}(k, \mu, z) \]
\[ = \langle \epsilon^2 \rangle + c_2 W_R^2(k) P_{\delta\delta}(k, z) + D^2(z) \left[ c_1 c_3 \sigma_R^2 + 8 c_1 c_2 \sigma_R^2 \delta_R(k) \right] W_R^2(k) + D^4(z) \left( \frac{c_2}{2} \int \frac{d^3q}{(2\pi)^3} W_R^2(q) P_L^2(q) + c_1 c_2 D^4(z) P_{b2}(k) \right. \]
\[ + 2 f \mu^2 \left[ c_1 W_R(k) P_{\delta\delta}(k, z) + \frac{1}{2} D^2(z) \left\{ c_3 \sigma_R^2 + 8 c_2 \sigma_R^2 \delta_R(k) \right\} W_R(k) P_L(k) \right) \]
\[ + \frac{1}{2} D^2(z) c_1 c_2 \sigma_R^2 W_R(k) P_L(k) \right] + f \mu^2 \left[ P_{\delta\delta}(k, z) + c_2 D^2(z) \sigma_R^2 P_L(k, z) \right] \]
\[ + D^4(z) \left[ P_{gs,22}^{(rest)}(k, \mu; c_1, c_2) + 2 P_{gs,13}^{(rest)}(k, \mu; c_1) \right]. \tag{2.138} \]

where, \(P_{\delta\delta}(k), P_{b2}(k), P_{b22}(k), P_{\delta\delta}(k), P_{\delta\delta}(k)\) are defined in equation (2.68), equation (2.87), equation (2.88), equation (2.117), and equation (2.118), respectively.

Following the discussion in Section 2.5, we shall absorb the small scale dependent quantities in equation (2.138) by re-defining the bias parameters: \(P_0, b_1 \text{ and } b_2\) as is in equation (2.83), equation (2.84) and equation (2.85), respectively. In addition, we also introduce the new bias parameter \(b_0\):

\[ b_0^2 = 1 + c_2 D^2(z) \sigma_R^2, \tag{2.139} \]

so that additional \(\sigma_R^2\) term can be absorbed. By using the re-defined bias parameters, the
galaxy power spectrum in redshift space becomes

\[ P_{gs}(k, \mu, z) = P_0 + b_1^2 \left( W_R^2(k) P_m(k, z) + b_2 D^4(z) P_{b2}(k) + b_2^2 D^4(z) P_{b22}(k) \right) 
+ 2f \mu^2 b_1 b_0 W_R(k) P_{b0}(k, z) + f^2 \mu^4 b_0^2 P_{b0}(k, z) 
+ D^4(z) \left[ P_{gs,22}^{(\text{rest})}(k, \mu; b_1, b_2) + 2P_{gs,13}^{(\text{rest})}(k, \mu; b_1) \right]. \]  

(2.140)

In quasi-nonlinear regime, r.m.s. density fluctuation is small \( \sigma_R^2 \ll 1 \); thus we make following approximations

\[ b_1 \simeq c_1 + \frac{1}{2} D^2(z) \left[ c_3 \sigma_R^2 + 8c_2 \sigma_R^2 S_R(k) \right] \]  

(2.141)

\[ b_0 \simeq 1 + \frac{c_2}{2} D^2(z) \sigma_R^2, \]  

(2.142)

and

\[ \sigma_R^2 P_L \simeq \sigma_R^2 P_{b0}(k) \simeq \sigma_R^2 P_{\theta\theta}(k) \simeq \sigma_R^2 P_{b\theta}(k), \]

which are true up to the fourth order in the linear density contrast. We also replace \( c_1 \) in the rest terms, \( P_{gs,22}^{(\text{rest})}(k, \mu, f; c_1, c_2) \) and \( P_{gs,13}^{(\text{rest})}(k, \mu, f; c_1) \), to \( b_1 \), as it is consistent up to the same order. However, we set \( c_2 \) to be free, as this may allow us to check the consistency: \( c_2 = b_1 b_2 \).

Note that the new bias parameter \( b_0 \) is multiplied to the velocity divergence field, \( \theta_k \), thus we call it velocity bias. However, it does not mean that the velocity field itself is biased. The velocity bias is the bias introduced when ignoring the coupling between the density field and the velocity field in equation (2.127). For example, if one uses equation (2.133) as an estimator for the velocity power spectrum, then the measured velocity will be biased by at least a factor of \( 1 + b_0 \), and the bias factor will increase when we also take ‘rest’ terms into account.

The ‘rest’ terms \( P_{gs,22}^{(\text{rest})}(k, \mu, f; c_1, c_2) \) and \( P_{gs,13}^{(\text{rest})}(k, \mu, f; c_1) \) are defined as the collection of terms which cannot be simplified as either previously known terms or cannot be absorbed in the re-defined bias parameters. We show the explicit functional formula for the rest terms in Appendix D.
2.7.2 non-Gaussian case

We calculate the non-Gaussian correction to the galaxy power spectrum in redshift space as

\[
\Delta P_g^s(k, \mu, z) = 2f_{\text{NL}} D^3(z)(c_1 W_R(k) + f \mu^2) M(k)
\]

\[
\times \int \frac{d^3 q}{(2\pi)^3} \mathcal{M}(q) \mathcal{M}(|k - q|) P_\phi(q) \left[ 2P_\phi(k) + P_\phi(|k - q|) \right] c_2 W_R(q) W_R(|k - q|)
\]

\[
+ 2c_1 W_R(k) F_2^{(s)}(q, k - q) + 2f \mu^2 G_2^{(s)}(q, k - q) + (f k \mu) \frac{q_z}{|k - q|^2}
\]

\[
+ b_1 (f k \mu) \left\{ W_R(|k - q|) \frac{q_z}{q^2} + W_R(q) \frac{k \mu - q_z}{|k - q|^2} \right\}.
\]

(2.143)

We, again, find that the non-Gaussianity term comes from the non-linear redshift space mapping is parametrically small, and the dominant term is

\[
\Delta P_g^s(k, \mu, z) \approx 2f_{\text{NL}} D^3(z)(c_1 W_R(k) + f \mu^2) M(k)
\]

\[
\times c_2 \int \frac{d^3 q}{(2\pi)^3} \mathcal{M}(q) \mathcal{M}(|k - q|) P_\phi(q) \left[ 2P_\phi(k) + P_\phi(|k - q|) \right]
\]

\[
= 4f_{\text{NL}} D^3(z)(c_1 W_R(k) + f \mu^2) c_2 \sigma_R^2 \mathcal{F}_R(k) \frac{P_L(k)}{M(k)}.
\]

(2.144)

By using \( \tilde{b}_2 = \sigma^2_R D(z)c_2/c_1 \) we define in Section 2.5.3, the correction term becomes

\[
\Delta P_g^s(k, \mu, z) = 4f_{\text{NL}} D(z)c_1 (c_1 W_R(k) + f \mu^2) \tilde{b}_2 \mathcal{F}_R(k) \frac{P_L(k)}{M(k)}.
\]

(2.145)

On large scales \( k \ll 1/R \), \( \mathcal{F}_R(k) = W_R(k) = 1 \), and for the highly biased tracers, \( \tilde{b}_2 \approx \alpha \delta_c^{19} \), we can rewrite the non-Gaussian term as

\[
\Delta P_g^s(k, \mu, z) = 6\alpha \delta_c f_{\text{NL}} D(z)c_1 (c_1 + f \mu^2) \frac{H^2_0 \Omega_m P_L(k)}{k^2 T(k)}.
\]

(2.146)

Therefore, on large scale, the non-Gaussian galaxy power spectrum in redshift space is given by

\[
P_{gs}(k, \mu, z) = \left[ b_1 + f \mu^2 \right] P_L(k) + 6\alpha f_{\text{NL}} \delta_c b_1 (b_1 + f \mu^2) \frac{H^2_0 \Omega_m P_L(k)}{k^2 T(k)} + P_0.
\]

(2.147)

Here, we approximate \( b_1 \approx c_1 \) as the equation is written in the linear order.

\(^{19}\alpha = 1\) for Press-Schechter mass function, and \( \alpha = 0.75 \) for Sheth-Tormen mass function. For more discussion, see Section 2.5.3.
2.8 Summary

In this chapter, we calculate the nonlinear galaxy power spectrum in redshift space within Eulerian perturbation theory framework. After defining three conditions in the quasi non-linear regime, 1) sub-Horizon, 2) pressureless, 3) curl-free, we find the perturbative solutions for the non-linear evolution of the density field and velocity field of cosmic matter field. By using the perturbation theory solution, we calculate the matter power spectrum and the galaxy power spectrum in both real and redshift space. For each case, we also calculate the leading order correction to the power spectrum from the local primordial non-Gaussianity.

2.8.1 On the smoothing of density field

When applying the locality of bias assumption, we assume that the galaxy density contrast is a local function of the smoothed matter density contrast. While it is not certain if locality of bias works for the real density contrast or the smoothed density contrast, we intentionally adopt the smoothing function as it facilitates to access the effect of the small scale smoothing on the large scale power spectrum. As we are mainly interested in the power spectrum on sufficiently large scales, \( k \ll 1/R \), we want to absorb the effect of small scale smoothing into the value of the free bias parameters; thus, our theoretical template of the galaxy power spectrum in redshift space should not depend on the smoothing scale \( R \). We test that every term in equation (2.140) is indeed independent on the smoothing scale as we desire.

In short, in the light of the re-definition of bias parameters, the smoothing does not affect the power spectrum on large scales; smoothing scale (or halo mass) will only changes the exact value of the bias parameters. Therefore, we show the unsmoothed equations in the summary below:

2.8.2 Summary of equations

The galaxy power spectrum in redshift space on the quasi-nonlinear scales is given by

\[
P_{gs}(k, \mu, z) = P_0 + b_1^2 \left[ P_m(k, z) + b_2 D^4(z) P_{b2}(k) + b_3^2 D^4(z) P_{b22}(k) \right]
+ 2 \hat{f} \mu^2 b_1 b_8 P_{b0}(k, z) + f^2 \mu^4 b_0^2 P_{b0}(k, z)
+ D^4(z) \left[ P_{g,33}^{(rest)}(k, \mu; b_1, b_2) + 2 P_{g,13}^{(rest)}(k, \mu; b_1) \right],
\]  

(2.148)

46
where $P_0, b_1, b_2, b_0, c_2$ are free bias parameters, and the components of nonlinear power spectrum are listed as follow.

\[
P_m(k, z) = D^2(z) \left[ P_L(k) + D^2(z) \{ P_{m,22}(k) + 2P_{m,13}(k) \} \right]
\]

\[
P_{b2}(k) = 2 \int \frac{d^3q}{(2\pi)^3} P_L(q) P_L(|k-q|) F_2^{(s)}(q, k-q)
\]

\[
P_{b22}(k) = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} P_L(q) [P_L(|k-q|) - P_L(q)]
\]

\[
P_{50}(k, z) = D^2(z) \left[ P_L(k) + D^2(z) \{ P_{50,22}(k) + P_{50,13}(k) \} \right]
\]

\[
P_{50}(k, z) = D^2(z) \left[ P_L(k) + D^2(z) \{ P_{50,22}(k) + 2P_{50,13}(k) \} \right]
\]

The $P_{22}$ and $P_{13}$ terms in $P_m, P_{50}, P_{50}$ are

\[
P_{m,22}(k) = \frac{1}{98} \frac{k^3}{(2\pi)^2} \int d\tau P_L(kr) \int_{-1}^{1} dx P_L(k\sqrt{1+r^2-2rx}) \left[ \frac{7x + 3r - 10rx^2}{1 + r^2 - 2rx} \right]^2
\]

\[
P_{m,13}(k) = \frac{1}{504} \frac{k^3}{(2\pi)^2} P_L(k) \int d\tau P_L(kr) \left[ \frac{12}{r^2} - 158 + 100r^2 - 42r^4 + \frac{3}{r^3} (r^2 - 1)^3 (7r^2 + 2) \ln \left( \frac{r + 1}{|r - 1|} \right) \right]
\]

\[
P_{\delta0,22}(k) = \frac{1}{98} \frac{k^3}{(2\pi)^2} \int d\tau P_L(kr) \int_{-1}^{1} dx P_L(k\sqrt{1+r^2-2rx}) \left[ \frac{7x - r - 6rx^2}{1 + r^2 - 2rx} \right]^2
\]

\[
P_{\delta0,13}(k) = \frac{1}{504} \frac{k^3}{(2\pi)^2} P_L(k) \int d\tau P_L(kr) \left[ \frac{36}{r^2} - 246 + 12r^2 - 18r^4 + \frac{9}{r^3} (r^2 - 1)^3 (r^2 + 2) \ln \left( \frac{1 + r}{|r - 1|} \right) \right]
\]

and the ‘rest’ terms $P_{gs,22}^{(rest)}(k, \mu; f_1; b_1, c_2) P_{gs,13}^{(rest)}(k, \mu; f_1; b_1)$ are shown in Appendix D (except that now we set all smoothing function to be unity; $W_R = 1$).

By setting appropriate bias parameters and an angle parameter $\mu$, one can reproduce the result for different cases we study in this chapter. If setting $P_0 = 0$ $b_1 = 1$, $b_2 = 0$, $b_0 = 0$, equation (2.148) reduces to the non-linear matter power spectrum in redshift space, whose $\mu = 0$ slice is the non-linear matter power spectrum in real space, and, for $b_0 = 0$, $\mu = 0$, equation (2.148) reduces to the non-linear galaxy power spectrum in real space.
Finally, on large scales, the dominant correction of galaxy power spectrum due to primordial local non-Gaussianity is given by

$$\Delta P^{s}(k, \mu, z) = 6\alpha \delta_{c} f_{NL} D(z) b_{1}(b_{1} + f_{NL}^{2}) \frac{H_{0}^{2} \Omega_{m} P_{L}(k)}{k^{2} T(k)}.$$

(2.159)