Appendix K

The three-point correlation function (bispectrum) of density peaks

Consider the smoothed density field, \( \rho_R(x) = \bar{\rho} [1 + \delta_R(x)] \), with a general smoothing kernel \( W(x) \)

\[
\delta_R(x) = \int d^3y W_R(x - y) \delta(y).
\]  

(K.1)

We define the peaks as regions in the space where the smoothed density contrast exceed a certain threshold value, \( \delta_c \):

\[
n_p(x) = \frac{1}{V_R} \theta [\delta_R(x) - \delta_c],
\]  

(K.2)

where \( \theta(x) \) is a step-function, and \( V_R \) is the volume of the smoothed region\(^1\). In this chapter, we shall calculate the three-point correlation function of peaks when the density field follows general probability distribution. We shall use the functional integration method adopted by Matarrese et al. (1986), but, instead of presenting the general formula, we explicitly calculate the two point correlation function and the three point correlation function.

Let us denote the density contrast of the peaks as \( \delta_p(x) \equiv n_p(x)/\langle n_p \rangle - 1 \). The probability \( P_2(x_1, x_2) \) of finding two peaks at two different locations \( x_1 \) and \( x_2 \), and the probability \( P_3(x_1, x_2, x_3) \) of finding three peaks at three different locations \( x_1, x_2 \) and \( x_3 \) are related to the two-point correlation function \( \xi_p(x_1, x_2) \), and the three-point correlation

\(^1\)For a spherical top-hat filter

\[
W_R(x) = \frac{1}{V_R} \left\{ \begin{array}{ll}
1, & |x| < R \\
0, & \text{otherwise}
\end{array} \right.
\]

the volume of the smoothed region is \( V_R = 4\pi R^3/3 \). However, \( V_R \) may not be well-defined for different filters, e.g. Gaussian filter, where smoothing function is extended to infinity. Fortunately, it is not important to calculate the correlation function from the method we use here, as mean number density cancels out in equation (K.3) and (K.4).
function $\zeta_p(x_1, x_2, x_3)$ of density contrast of peaks as (Peebles, 1980):

$$P_2(x_1, x_2) = \frac{\langle n_p(x_1)n_p(x_2) \rangle}{\langle n_p \rangle^2} = \langle (1 + \delta_p(x_1))(1 + \delta_p(x_2)) \rangle = 1 + \xi_p(x_1, x_2) \quad (K.3)$$

$$P_3(x_1, x_2, x_3) = \frac{\langle n_p(x_1)n_p(x_2)n_p(x_3) \rangle}{\langle n_p \rangle^3} = \langle (1 + \delta_p(x_1))(1 + \delta_p(x_2))(1 + \delta_p(x_3)) \rangle = 1 + \xi_p(x_1, x_2) + \xi_p(x_2, x_3) + \xi_p(x_3, x_1) + \xi_p(x_1, x_2, x_3) \quad (K.4)$$

Therefore, in order to calculate the two-point correlation function and the three-point correlation function, we have to calculate $P_N$ up to $N = 3$.

In Appendix K.1 we calculate the probability $P_N$: starting from the general functional integration method for calculating $P_N$, we explicitly calculate $P_1$ (Appendix K.1.1), $P_2$ (Appendix K.1.2) and $P_3$ (Appendix K.1.3). Then, in Appendix K.2, we calculate the two-point correlation function (power spectrum) of peaks by using equation (K.3). Finally, we calculate the three-point correlation function (bispectrum) of peaks in Appendix K.3 by using equation (K.4).

### K.1 Probability of finding $N$ distinct peaks

The *probability functional* $\mathcal{P}[\delta(x)]$ is the probability distribution function of $\delta(x)$ at all point $x$ in space. It is normalized to be

$$\int [D\delta] \mathcal{P} [\delta(x)] = 1,$$

with a suitable measure $[D\delta]$. The probability distribution function of the density field $\delta_0$ as a specific position $x_0$ can be calculated as

$$P(\delta_0) = \int [D\delta] \mathcal{P} [\delta(x)] \delta^D(\delta(x_0) - \delta_0), \quad (K.5)$$

which means that fixing the density at $x_0$ to be $\delta_0$ and marginalize over all other points.

As defined in equation (K.2), the *peaks* are the regions where the smoothed density field [Eq. (K.1)] exceeds $\delta_c$. By using a $\mathcal{P}[\delta(x)]$ we can formulate the probability
\[ P_N(x_1, \cdots, x_N), \text{ which is the probability of finding } N \text{ peaks at } x_1, \cdots, x_N, \text{ as} \]
\[ P_N(x_1, \cdots, x_N) = \int_{\delta_c}^{\infty} d\alpha_1 \cdots \int_{\delta_c}^{\infty} d\alpha_N \int [\mathcal{D}\delta] \mathcal{P}[\delta(x)] \prod_{r=1}^{N} \delta^D(\delta_R(x_r) - \alpha_r) \]
\[ = \int [\mathcal{D}\delta] \mathcal{P}[\delta(x)] \prod_{r=1}^{N} \frac{d\alpha_r}{2\pi} \int_{\nu \sigma\delta}^{\infty} d\alpha_r e^{i\phi_r \int d^3 y W_R(x_r - y) \delta(y) - \alpha_r}. \quad (K.6) \]
Here, in the second equality we use \( \int \delta^D(x) \equiv 1/(2\pi) \int d\phi e^{i\phi x} \) representation of the Dirac delta function and the definition of the smoothed density field. In order to simplify the notation later, we define a variable \( \nu \equiv \delta_c / \sigma_R \) which quantify the density threshold in a unit of the root-mean-squared (r.m.s.) value of the smoothed density contrast.

Let us define the partition (generating) functional \( Z[J] \) as
\[ Z[J] = \int [\mathcal{D}\delta] \mathcal{P}[\delta(x)] e^{i \int d^3 y \delta(y) \sum_{r=1}^{N} \phi_r W_R(x_r - y)} \]
\[ = \left\langle \exp \left[ i \int d^3 y \delta(y) J(y) \right] \right\rangle, \quad (K.7) \]
with following source function:
\[ J(y) \equiv \sum_{r=1}^{N} \phi_r W_R(x_r - y). \quad (K.8) \]
Then, from equation (K.7), we can define the \( n \)-point connected correlation function as
\[ \xi^{(n)}(y_1, \cdots, y_n) \equiv \langle \delta(y_1) \cdots \delta(y_n) \rangle_c \equiv \frac{\delta^n \ln Z[J]}{i^n \delta J(y_1) \cdots \delta J(y_n)} \bigg|_{J=0}. \quad (K.9) \]
In other words, if we know all \( n \)-point correlation functions, we can reconstruct the partition function as a Taylor expansion:
\[ \ln Z[J] = \sum_{n=2}^{\infty} \frac{i^n}{n!} \int d^3 y_1 \cdots \int d^3 y_n \xi^{(n)}(y_1, \cdots, y_n) J(y_1) \cdots J(y_n). \quad (K.10) \]
By substituting the source function in equation (K.8), the partition functional becomes
\[ \ln Z[J] = \sum_{n=2}^{\infty} \frac{i^n}{n!} \int d^3 y_1 \cdots \int d^3 y_n \xi^{(n)}(y_1, \cdots, y_n) \]
\[ \times \left[ \sum_{r_1=1}^{N} \phi_{r_1} W_R(x_{r_1} - y_1) \right] \cdots \left[ \sum_{r_n=1}^{N} \phi_{r_n} W_R(x_{r_n} - y_n) \right]. \quad (K.11) \]
The strategy of calculating $P_N$ is following. For given $n$-point correlation functions of density field, we can calculate the partition functional from equation (K.11). By using the partition functional, $P_N$ becomes

$$P_N(x_1, \ldots, x_N) = \int_{\nu} \cdots \int_{\nu} d\alpha_1 \cdots d\alpha_N \int_{-\infty}^{\infty} d\phi_1 \cdots \int_{-\infty}^{\infty} d\phi_N e^{-i \sum_{r=1}^{N} \alpha_r \phi_r} Z[J],$$

which involves only ordinary integration.

Matarrese et al. (1986) provide the general solution for $P_N$, from there the authors reach the general formula for the $N$-point correlation function of peaks. We however find that the formula in Matarrese et al. (1986) is too abstract to be directly adopted without justification from the explicit calculation. Therefore, we shall explicitly show the solution for $N = 1, 2$ and $3$ in the following sections. For the notational simplicity we denote the smoothed $n$-th order correlation function as $\xi_R^{(n)}$:

$$\xi_R^{(n)}(x_1, \ldots, x_n) = \left\{ \prod_{r=1}^{n} \int d^3y_r W_R(x_r - y_r) \right\} \xi^{(n)}(y_1, \ldots, y_n).$$

K.1.1 Calculation of $P_1$

Let’s consider $N = 1$ case. The generating functional becomes

$$\ln Z[J] = \sum_{n=2}^{\infty} \frac{i^n}{n!} \int d^3y_1 \cdots \int d^3y_n \xi^{(n)}(y_1, \ldots, y_n) \prod_{r=1}^{n} \phi_1 W_R(x_1 - y_r),$$

and the integration over $y_i$s are simply a smoothed correlation function:

$$\int d^3y_1 \cdots \int d^3y_n \xi^{(n)}(y_1, \ldots, y_n) \prod_{r=1}^{n} W_R(x_1 - y_r) = \xi_R^{(n)}(x_1, \ldots, x_n, n \text{ times}).$$

As we have only one argument in the smoothed correlation function, we simply denote it as $\xi_R^{(n)}$ without arguments. Then, the generating functional is simplified as

$$\ln Z[J] = \sum_{n=2}^{\infty} \frac{i^n}{n!} \phi_1^n \xi_R^{(n)},$$
and we can calculate $P_1$ from equation (K.12).

$$P_1(x_1) = \frac{1}{(2\pi)^{\nu}} \int_0^{\infty} d\alpha_1 \int_{-\infty}^\infty d\phi_1 \exp \left[ -i\alpha_1 \phi_1 + \sum_{n=2}^{\infty} \frac{i^n}{n!} \phi_1^n \xi^{(n)}_{R} \right].$$  \hspace{1cm} (K.17)

Let us first consider the $\phi_i$ integration.

$$I \equiv \int_{-\infty}^\infty d\phi_1 \exp \left[ -i\alpha_1 \phi_1 + \sum_{n=2}^{\infty} \frac{i^n}{n!} \phi_1^n \xi^{(n)}_{R} \right].$$  \hspace{1cm} (K.18)

We first calculate the quadratic term

$$I_0 = \int_{-\infty}^\infty d\phi_1 \exp \left[ -i\alpha_1 \phi_1 - \frac{1}{2} \phi_1^2 \sigma_R^2 \right] = \sqrt{\frac{2\pi}{\sigma_R^2}} \exp \left[ -\frac{\alpha_1^2}{2\sigma_R^2} \right].$$  \hspace{1cm} (K.19)

then, we calculate the other terms in $I$, which include $\phi_1^n (n > 3)$, by taking $n$-th derivative of $\alpha_1$ on $I_0$ as following.

$$I = \exp \left[ \sum_{n=3}^{\infty} \frac{(-1)^n}{n!} \xi^{(n)}_{R} \frac{\partial^n}{\partial \alpha_1^n} \right] \int_{-\infty}^\infty d\phi_1 \exp \left[ -\phi_1^2 \sigma_R^2 \right]$$

$$I = \frac{\sqrt{2\pi}}{\sigma_R} \exp \left[ \sum_{n=3}^{\infty} \frac{(-1)^n}{n!} \xi^{(n)}_{R} \frac{\partial^n}{\partial \alpha_1^n} \right] \exp \left[ -\frac{\alpha_1^2}{2\sigma_R^2} \right].$$  \hspace{1cm} (K.20)

Now, $P_1$ becomes

$$P_1(x_1) = \frac{1}{\sqrt{2\pi} \sigma_R} \int_0^{\infty} d\alpha_1 \exp \left[ \sum_{n=3}^{\infty} \frac{(-1)^n}{n!} \xi^{(n)}_{R} \frac{\partial^n}{\partial \alpha_1^n} \right] \exp \left[ -\frac{\alpha_1^2}{2\sigma_R^2} \right]$$

$$P_1(x_1) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} d\alpha_1 \exp \left[ \sum_{n=3}^{\infty} \frac{(-1)^n}{n!} \xi^{(n)}_{R} \frac{\partial^n}{\partial \alpha_1^n} \right] \exp \left[ -\frac{\alpha_1^2}{2\sigma_R^2} \right].$$  \hspace{1cm} (K.21)

where in the second equality, we change the variable $\alpha'_1 = \alpha_1/\sigma_R$. We use the Hermite polynomial, $H_n(x)$ to simplify the equation further. Especially, following two properties of Hermite polynomial is useful for our purpose.

$$H_n(x) = (-1)^n e^{-x^2} \frac{d^n}{dx^n} e^{-x^2}$$  \hspace{1cm} (K.22)

$$\lim_{x \to \infty} H_n(x) = 2^n x^n$$  \hspace{1cm} (K.23)

From equation (K.22), we find

$$(-1)^n \frac{d^n}{d\alpha'_1^n} e^{-\alpha'_1^2/2} = 2^{-n/2} \alpha'_1^{n/2} H_n \left( \frac{\alpha'_1}{\sqrt{2}} \right).$$
and when the argument is large, \((\alpha')^n \gg 1\) we approximate the derivative as

\[
(-1)^n \frac{d^n}{d\alpha_1^n} e^{-\alpha_1^2/2} \to e^{-\alpha_1^2/2} (\alpha')^n.
\]

Therefore, by using the asymptotic formula of Gaussian integration

\[
\int_a^\infty f(x)e^{-x^2/2} \approx \frac{f(a)}{\alpha_1} e^{-\alpha_1^2/2} + O\left(\frac{1}{\alpha_1^2}\right),
\]

we further simplify \(P_1\) as

\[
P_1(x) \sim \frac{1}{\sqrt{2\pi \nu}} \exp \left[ \sum_{n=3}^\infty \frac{\nu^n \xi^{(n)}}{n!} \frac{R \sigma_R}{\sigma_R} \right] e^{-\nu^2/2}.
\]

for the high peak limit \((\nu \gg 1)\).

**K.1.2 Calculation of \(P_2\)**

In this section, we focus only on \(N = 2\) case. The generating functional becomes

\[
\ln Z[J] = \sum_{n=2}^\infty \int d^3y_1 \cdots \int d^3y_n \xi^{(n)}(y_1, \cdots, y_n) \\
\times \prod_{r=1}^n \left[ \phi_1 W_R(x_r - y_r) + \phi_2 W_R(x_r - y_r) \right] \\
= \sum_{n=2}^\infty \frac{1}{n!} \int d^3y_1 \cdots \int d^3y_n \xi^{(n)}(y_1, \cdots, y_n) \sum_{m=0}^n \binom{n}{m} \phi_1^m \phi_2^{n-m} \\
\times \prod_{r_1=1}^m W_R(x_1 - y_{r_1}) \prod_{r_2=m+1}^n W_R(x_2 - y_{r_2}),
\]

where, in the second line, we use the binomial expansion and the symmetry of \(\xi^{(n)}\), namely, the correlation function does not depend on the order of argument. We can replace the integration over \(y_r\) as a smoothed correlation function.

\[
\int d^3y_1 \cdots \int d^3y_n \xi^{(n)}(y_1, \cdots, y_n) \prod_{r_1=1}^m W_R(x_1 - y_{r_1}) \prod_{r_2=m+1}^n W_R(x_2 - y_{r_2}) \\
= \xi^{(n)}_R \left( x_1, \cdots, x_1, \ x_2, \cdots, x_2 \right)_{m \text{ times } n-m \text{ times}}.
\]

For notational simplicity we denote such a smoothed correlation function as \(\xi^{(n)}_{R,m}\). Then, the generating functional becomes

\[
\ln Z[J] = \sum_{n=2}^\infty \frac{1}{n!} \sum_{m=0}^n \frac{\phi_1^m \phi_2^{n-m}}{m!(n-m)!} \xi^{(n)}_{R,m}.
\]
We calculate $P_2$ from equation (K.12)
\[
P_2(x_1, x_2) = \int_{-\infty}^{\infty} d\phi_1 \int_{-\infty}^{\infty} d\phi_2 \int_{-\infty}^{\infty} d\phi_1 \int_{-\infty}^{\infty} d\phi_2 \frac{e^{-i(\alpha_1 + \alpha_2)}}{(2\pi)^2} Z[J].
\] (K.29)

Let’s first consider the $\phi_i$ integration.
\[
I = \int_{-\infty}^{\infty} d\phi_1 \int_{-\infty}^{\infty} d\phi_2 \exp \left[ -i \sum_{r=1}^{2} \alpha_r \phi_r + \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{\phi_m}{m!(n-m)!} \xi^{(n)}_{R,m} \right]
\] (K.30)

As we have done in Section K.1.1, we first calculate the quadratic integration,
\[
I_0 = \int_{-\infty}^{\infty} d\phi_1 \int_{-\infty}^{\infty} d\phi_2 \exp \left[ -i \sum_{r=1}^{2} \alpha_r \phi_r - \left( \frac{1}{2} \phi_1^2 \sigma_{R}^2 + \frac{1}{2} \phi_2^2 \sigma_{R}^2 \right) \right]
\] 
\[
= \frac{2\pi}{\sigma_{R}^2} \exp \left[ -\frac{1}{2} \sum_{r=1}^{2} \frac{\alpha_r^2}{\sigma_{R}^2} \right].
\] (K.31)

Then, $\phi_i$ integration simplifies to be a sum of successive derivatives on $I_0$ as below.
\[
I = \frac{2\pi}{\sigma_{R}^2} \exp \left[ (-1)^2 \xi^{(2)}_{R}(x_{12}) - \frac{\partial^2}{\partial\alpha_1 \partial\alpha_2} + \sum_{n=3}^{\infty} (-1)^n \sum_{m=0}^{n} \frac{\xi^{(n)}_{R,m}}{m!(n-m)!} \frac{\partial^n}{\partial\alpha_1^m \partial\alpha_2^{n-m}} \right]
\]
\[
\times \exp \left[ -\frac{1}{2} \sum_{r=1}^{2} \frac{\alpha_r^2}{\sigma_{R}^2} \right].
\] (K.32)

Now, we have to calculate the $\alpha_i$ integration. For the notational simplicity, we define $\alpha'_i \equiv \alpha_i / \sigma_R$, and $w_m^{(n)}$ as following.
\[
\begin{cases}
  w_m^{(2)} = \xi^{(2)}_{R}(x_{12}) / \sigma_{R}^2 & (m = 1) \\
  w_m^{(2)} = 0 & (m = 0 \text{ or } 2) \\
  w_m^{(n)} = \xi^{(n)}_{R,m} / \sigma_{R}^n & (n > 2)
\end{cases}
\] (K.33)

Now the two-point probability $P_2(x_1, x_2)$ becomes
\[
P_2(x_1, x_2) = \frac{1}{2\pi} \int_{\nu} d\alpha'_1 \int_{\nu} d\alpha'_2 \exp \left[ \sum_{n=2}^{\infty} (-1)^n \sum_{m=0}^{n} \frac{w_m^{(n)}}{m!(n-m)!} \frac{\partial^n}{\partial\alpha'_1^m \partial\alpha'_2^{n-m}} \right]
\]
\[
\times \exp \left[ -\frac{1}{2} \sum_{r=1}^{2} \frac{\alpha'_r^2}{\sigma_R^2} \right].
\] (K.34)

Again, we use the Hermite polynomial, and take the high peak limit of $\nu \gg 1$
\[
P_2(x_1, x_2) \simeq \frac{1}{2\pi \nu^2} \exp \left[ \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{w_m^{(n)}}{m!(n-m)!} \nu^n \right] e^{-\nu^2}
\]

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K.1.3 Calculation of $P_3$

Let’s calculate for $P_3(x_1, x_2, x_3)$:

$$P_3(x_1, x_2, x_3) = \frac{1}{(2\pi)^3} \prod_{i=1}^{3} \int_{\nu \sigma_R} \int_{-\infty}^{\infty} d\alpha_i \int_{-\infty}^{\infty} d\phi_i e^{-i \sum_{r=1}^{3} \alpha_r \phi_r} Z[J]. \quad (K.35)$$

We first calculate the partition functional

$$\ln Z[J] = \sum_{n=2}^{\infty} \frac{i^n}{n!} \left( \prod_{i=1}^{n} \int d^3 y_i \right) \xi(y_1, \cdots, y_n) \prod_{r=1}^{n} \left[ \sum_{j=1}^{3} \phi_j W_R(x_j - y_r) \right]. \quad (K.36)$$

By using a multinomial expansion theorem,

$$\prod_{r=1}^{n} \left[ \sum_{j=1}^{3} \phi_j W_R(x_j - y_r) \right] = \sum_{m_1=0}^{n} \sum_{m_2=0}^{n-m_1} \frac{n!}{m_1!m_2!(n-m_1-m_2)!} \phi_1^{m_1} \phi_2^{m_2} \phi_3^{n-m_1-m_2} \prod_{r_1 \in [m_1]} W_R(x_1 - y_{r_1}) \prod_{r_2 \in [m_2]} W_R(x_2 - y_{r_2}) \prod_{r_3 \in [m_3]} W_R(x_3 - y_{r_3}) \quad (K.37)$$

where, $m_3 \equiv n - m_1 - m_2$, and $[m_i]$ stands for the set of indexes

$$[m_i] = \{ a_k | 1 \leq a_k \leq n, k = 1, \cdots, m_i \},$$

and empty when $m_i = 0$. Note that, $[m_i]$ are mutually exclusive, that is $[m_i] \cap [m_j] = \emptyset$, and $[m_1] \cup [m_2] \cup [m_3] = \{1, 2, \cdots, n\}$. For example, when $n=2$, it becomes

$$\prod_{r=1}^{2} \left[ \sum_{j=1}^{3} \phi_j W_R(x_j - y_r) \right] = \phi_1^2 W_R(x_1 - y_1) W_R(x_1 - y_2) + \phi_2^2 W_R(x_2 - y_1) W_R(x_2 - y_2)$$

$$+ \phi_3^2 W_R(x_3 - y_1) W_R(x_3 - y_2) + \phi_1 \phi_2 W_R(x_1 - y_1) W_R(x_2 - y_2)$$

$$+ \phi_1 \phi_3 W_R(x_1 - y_2) W_R(x_3 - y_1) + \phi_2 \phi_3 W_R(x_2 - y_1) W_R(x_3 - y_2)$$

$$+ \phi_3 \phi_1 W_R(x_3 - y_2) W_R(x_1 - y_1).$$

(K.38)
and we can write \( n = 2 \) component of \( \ln Z[J] \) as

\[
- \frac{1}{2} \int d^3 y_1 \int d^3 y_2 \xi^{(2)} (y_1, y_2) \times \text{eq.}[K.38]
\]

\[
= - \frac{1}{2} \left[ (\phi_1^2 + \phi_2^2 + \phi_3^2) \sigma_R^2 \right.
\]

\[
+ 2 \phi_1 \phi_2 \xi_R^{(2)} (x_1, x_2) + 2 \phi_2 \phi_3 \xi_R^{(2)} (x_2, x_3) + 2 \phi_3 \phi_1 \xi_R^{(2)} (x_1, x_3) \right].
\] (K.39)

Therefore, we rewrite \( Z[J] \) as

\[
\ln Z[J] = \sum_{n=2}^{\infty} \frac{i^n}{n!} \sum_{m_1=0}^{n-m_1} \sum_{m_2=0}^{n-m_2} n! \phi_1^{m_1} \phi_2^{m_2} \phi_3^{m_3} \xi_R^{(n)}_{m_1 m_2 m_3},
\] (K.40)

where

\[
\xi_R^{(n)}_{m_1 m_2 m_3} = \xi_R \left( \begin{array}{ccc} x_1, \ldots, x_1, & x_2, \ldots, x_2, & x_1, \ldots, x_3, \end{array} \right). 
\]

For example, for \( n = 2 \) case we write

\[
\frac{i^n}{n!} \sum_{m_1=0}^{n-m_1} \sum_{m_2=0}^{n-m_2} n! \phi_1^{m_1} \phi_2^{m_2} \phi_3^{m_3} \xi_R^{(n)}_{m_1 m_2 m_3} \bigg|_{n=2}
\]

\[
= - \frac{1}{2} \left[ \phi_1^2 \xi_R^{(2)}_{200} + \phi_2^2 \xi_R^{(2)}_{200} + \phi_3^2 \xi_R^{(2)}_{200} + 2 \phi_1 \phi_2 \xi_R^{(2)}_{110} + 2 \phi_2 \phi_3 \xi_R^{(2)}_{110} + 2 \phi_3 \phi_1 \xi_R^{(2)}_{110} \right].
\]

We follow the same procedure as we have calculated \( P_1 \) and \( P_2 \) in the previous section.

First, we calculate \( \phi_i \) integration by using a quadratic integration of

\[
I_0 = \int_{-\infty}^{\infty} d\phi_1 \int_{-\infty}^{\infty} d\phi_2 \int_{-\infty}^{\infty} d\phi_3 \exp \left[ -i \sum_{r=1}^{3} \alpha_r \phi_r - \frac{1}{2} \sum_{r=1}^{3} \phi_r^2 \sigma_R^2 \right]
\]

\[
= \frac{(2\pi)^{3/2}}{\sigma_R^3} \exp \left[ -\frac{1}{2} \sum_{r=1}^{3} \alpha_r^2 \sigma_R^2 \right].
\] (K.41)

Then, we can replace the \( \phi_r \)-integration by applying successive differentiations of \( (-i \alpha_r) \).

That is,

\[
I = \exp \left[ \sum_{i \neq j} \xi_R^{(2)} (x_i, x_j) \frac{\partial}{\partial \alpha_i} \frac{\partial}{\partial \alpha_j} \right.
\]

\[
+ \sum_{n=3}^{\infty} \sum_{m_1=0}^{n-m_1} \sum_{m_2=0}^{n-m_2} (-1)^n \xi_R^{(n)}_{m_1 m_2 m_3} \frac{\partial^n}{\partial \alpha_1^{m_1} \partial \alpha_2^{m_2} \partial \alpha_3^{m_3}} \bigg] I_0.
\] (K.42)
As we do not apply the derivative operator to the terms appears in $I_0$, we define $w_{m_1,m_2,m_3}^{(n)}$ as following.

$$
\begin{cases}
  w_{m_1,m_2,m_3}^{(2)} = \frac{\xi_R^{(2)}(x_{ij})}{\sigma_R^2} & (m_1 \neq 2, m_2 \neq 2, m_3 \neq 2) \\
  w_{m_1,m_2,m_3}^{(2)} = 0 & \text{otherwise} \\
  w_{m_1,m_2,m_3}^{(n)} = \frac{\xi_R^{(n)}(n > 2)}{\sigma_R^n}
\end{cases}
$$

By using the new notation, and changing the variable to $\alpha_i' \equiv \alpha_i / \sigma_R$, $P_3$ becomes

$$
P_3(x_1, x_2, x_3) = \frac{1}{(2\pi)^{3/2}} \prod_{r=1}^3 \int_0^\infty d\alpha_r'
\times \exp \left[ \sum_{n=2}^\infty \sum_{m_1,m_2} (-1)^n w_{m_1,m_2,m_3}^{(n)} \frac{\partial^n}{\partial \alpha_{m_1} \partial \alpha_{m_2} \partial \alpha_{m_3}} \right] \exp \left[ -\frac{1}{2} \sum_{r=1}^3 \alpha_r'^2 \right].
$$

Using the Hermite polynomial, and, finally, we imposing the high peak condition of $\nu \gg 1$, we find

$$
P_3(x_1, x_2, x_3) \approx \frac{1}{(2\pi \nu^2)^{3/2}} \exp \left[ \sum_{n=2}^\infty \sum_{m_1,m_2,m_3} \frac{w_{m_1,m_2,m_3}^{(n)} \nu^n}{m_1! m_2! m_3!} \right] e^{-2\nu^2}.
$$

### K.2 The two point correlation function of peaks

We calculate the two point correlation function of peaks by substituting $P_1$ from equation (K.25) and $P_2$ from equation (K.35) into equation (K.3):

$$
\xi_p(x_1, x_2) = \frac{P_2(x_1, x_2)}{P_1^2} - 1
\quad = \exp \left[ \sum_{n=2}^\infty \sum_{m=0}^n w_{m}^{(n)} \nu^n - 2 \sum_{n=3}^\infty \frac{\nu^n \xi_R^{(n)}}{n! \sigma_R^n} \right] - 1,
$$

where $w_{m}^{(n)}$ and $\xi_R^{(n)}$ are defined in equation (K.33) and equation (K.15), respectively. We further simplify the notation by using $w_{0}^{(n)} = w_{n}^{(n)} = \xi_R^{(n)} / \sigma_R^n$ for $n \geq 3$, and $w_{0}^{(2)} = w_{2}^{(2)} = 0$. 

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\[ \xi_p(x_1, x_2) = \exp \left[ \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{w_m^{(n)} \nu^n}{m! (n-m)!} - \sum_{n=3}^{\infty} \frac{\nu^n w_0^{(n)}}{n!} - \sum_{n=3}^{\infty} \frac{\nu^n w_0^{(n)}}{n!} \right] - 1 \]
\[ = \exp \left[ \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \frac{w_m^{(n)} \nu^n}{m! (n-m)!} \right] - 1 \]
\[ = \exp \left[ \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \left( n \frac{\nu^n \xi_{R,m}^{(n)}}{n! \sigma_R^m} \right) \right] - 1 \]  (K.46)

This result was first derived in Grinstein & Wise (1986) with a following ansatz
\[ n_p(x) = C \exp \left[ T \int d^3 y (y) W(x - y) \right]. \]  (K.47)

It was only after the full calculation of Matarrese et al. (1986) that \( T = \nu/\sigma_R \) is identified.

Note that for Gaussian case, where all the higher order correlation function vanishes \((\xi^{(n)} = 0 \text{ for } n > 3)\) equation (K.46) coincides with the result in Politzer & Wise (1984):
\[ \xi_p^G (x_1, x_2) = \exp \left[ \frac{\xi_{R}^{(2)} (x_{12})}{\sigma_R^2} \right] - 1 \approx \frac{\nu^2}{\sigma_R^2} \xi_{R}^{(2)} (x_{12}) \]  (K.48)

and, in the second equality, we can also reproduce the ‘Lagrangian bias’, \( b_L = \nu/\sigma_R \), by taking the large scale limit \((\xi_R \ll 1)\).

Recently, in the light of the scale dependent bias induced by primordial non-Gaussianity (see, Section I.3), Matarrese & Verde (2008) study the large scale limit of equation (K.46). For the large separation, we can expand the exponential in equation (K.46), and the two-point correlation function including the leading order correction due to the non-Gaussianity becomes
\[ \xi_p(|x_1 - x_2|) \approx \frac{\nu^2}{\sigma_R^2} \xi_{R}^{(2)} (x_1, x_2) + \frac{\nu^3}{\sigma_R^3} \xi_{R}^{(3)} (x_1, x_1, x_2). \]  (K.49)

Fourier transform of the non-Gaussian correction term is related to the matter bispectrum as
\[ \xi_{R}^{(3)} (x_1, x_1, x_2) \]
\[ = \int \frac{d^3 q_1}{(2\pi)^3} \int \frac{d^3 q_2}{(2\pi)^3} \int \frac{d^3 q_3}{(2\pi)^3} \delta_R (q_1) \delta_R (q_2) \xi_{R}^{(3)} (q_3) e^{i q_1 \cdot x_1} e^{i q_2 \cdot x_2} \]
\[ = \int \frac{d^3 q_1}{(2\pi)^3} \int \frac{d^3 q_2}{(2\pi)^3} B_R (q_1, q_2, -(q_1 + q_2)) e^{i (q_1 + q_2) \cdot (x_1 - x_2)}. \]  (K.50)
Therefore, by taking Fourier transformation of equation (K.49), we find the power spectrum of peaks as

\[
P_p(k) = \int d^3 r \xi_p(r) e^{-i k \cdot r}
= \frac{\nu^2}{\sigma_R^2} \int d^3 r \xi_R^{(2)}(r) e^{-i k \cdot r}
+ \frac{\nu^3}{\sigma_R^3} \int d^3 r \int \frac{d^3 q_1}{(2\pi)^3} \int \frac{d^3 q_2}{(2\pi)^3} B_R(q_1, q_2, -(q_1 + q_2)) e^{i(q_1 + q_2 - k) \cdot r}
= \frac{\nu^2}{\sigma_R^2} P_R(k) + \frac{\nu^3}{\sigma_R^3} \int \frac{d^3 q}{(2\pi)^3} B_R(q, -k, k - q)
\]

(K.51)

where \( r = x_1 - x_2 \).

**K.3 The three point correlation function of peaks**

We calculate the three point function from the relation between \( P_3 \) and correlation functions in equation (K.4). By using the calculation of \( P_1 \) in equation (K.25) and \( P_3 \) in equation (K.44), we calculate \( P_3/P_1^3 \) (in the high peak limit \( \nu \gg 1 \), and on large scales,
\( \xi^{(n)}_R(x_1, \cdots, x_n) \ll 1 \) up to four-point function order:

\[
P_3(x_1, x_2, x_3) \quad \quad \frac{P_3}{P_1^3} \\
= \exp \left[ \sum_{n=2}^{\infty} \sum_{m_1=0}^{n-m_1} \sum_{m_2=0}^{n-m_1-m_2} \frac{\nu^n w_{m_1 m_2 m_3}^{(n)}}{m_1! m_2!(n-m_1-m_2)!} \right] - 3 \sum_{n=3}^{\infty} \frac{\nu^n}{n!} \frac{\xi^{(n)}_R}{\sigma_R^n} \\
\approx 1 + \sum_{n=2}^{\infty} \sum_{m_1=0}^{n-m_1} \sum_{m_2=0}^{n-m_1-m_2} \frac{\nu^n w_{m_1 m_2 m_3}^{(n)}}{m_1! m_2!(n-m_1-m_2)!} - 3 \sum_{n=3}^{\infty} \frac{\nu^n}{n!} \frac{\xi^{(n)}_R}{\sigma_R^n} \\
\approx 1 + \frac{\nu^2}{\sigma_R^2} \left[ \xi^{(2)}_R(x_{12}) + \xi^{(2)}_R(x_{23}) + \xi^{(2)}_R(x_{31}) \right] \\
+ \frac{\nu^3}{\sigma_R^3} \left[ \frac{\xi^{(3)}_R}{6} + \frac{\xi^{(3)}_R}{2} + \frac{\xi^{(3)}_R}{6} + \frac{\xi^{(3)}_R}{120} \right] - 3 \frac{\nu^3}{\sigma_R^3} \frac{\xi^{(3)}_R}{6} \\
+ \frac{\nu^4}{\sigma_R^4} \left[ \frac{\xi^{(4)}_R}{24} + \frac{\xi^{(4)}_R}{6} + \frac{\xi^{(4)}_R}{2} + \frac{\xi^{(4)}_R}{24} + \frac{\xi^{(4)}_R}{130} \right] - 3 \frac{\nu^4}{\sigma_R^4} \frac{\xi^{(4)}_R}{24} \\
= 1 + \frac{\nu^2}{\sigma_R^2} \left[ \xi^{(2)}_R(x_{12}) + \xi^{(2)}_R(x_{23}) + \xi^{(2)}_R(x_{31}) \right] \\
+ \frac{\nu^3}{\sigma_R^3} \left[ \xi^{(3)}_R(x_{12}, x_2, x_3) + \xi^{(3)}_R(x_{23}, x_2, x_3) + \xi^{(3)}_R(x_{13}, x_1, x_2) + \xi^{(3)}_R(x_{13}, x_1, x_2) \right] \\
+ \frac{\nu^4}{\sigma_R^4} \left[ \frac{1}{2} \xi^{(4)}_R(x_{12}, x_2, x_2, x_3) + \frac{1}{2} \xi^{(4)}_R(x_{12}, x_2, x_3, x_3) + \frac{1}{2} \xi^{(4)}_R(x_{12}, x_2, x_2, x_3) \right. \\
\left. + \frac{1}{3} \xi^{(4)}_R(x_{23}, x_2, x_3, x_3) + \frac{1}{3} \xi^{(4)}_R(x_{23}, x_2, x_3, x_3) + \frac{1}{3} \xi^{(4)}_R(x_{23}, x_2, x_3, x_3) \right] \\
+ \frac{1}{3} \xi^{(4)}_R(x_{13}, x_1, x_2) + \frac{1}{3} \xi^{(4)}_R(x_{13}, x_1, x_2) + \frac{1}{3} \xi^{(4)}_R(x_{13}, x_1, x_2). \tag{K.52} \]

In order to calculate the three-point correlation function of peaks, we need to subtract the two-point correlation functions from equation (K.52). As we expand \( P_3/P_1^3 \) up to four-point
function order, we also expand equation (K.46) in the same order as

\[ \xi_p(x_1, x_2) \simeq \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{\nu^n w_m^{(n)}}{m!(n-m)!} - 2 \sum_{n=3}^{\infty} \frac{\nu^n \xi_m^{(n)}}{\sigma_R^n} \]

\[ = \frac{\nu^2}{\sigma_R} \xi_R^{(2)}(x_{12}) + \frac{\nu^3}{\sigma_R} \left[ \frac{\xi_R^{(3)}}{2} + \frac{\xi_R^{(3)}}{2} \right] + \frac{\nu^4}{\sigma_R} \left[ \frac{\xi_R^{(4)}}{6} + \frac{\xi_R^{(4)}}{4} + \frac{\xi_R^{(4)}}{6} \right] + \ldots \]

\[ = \frac{\nu^2}{\sigma_R} \xi_R^{(2)}(x_{12}) + \frac{\nu^3}{\sigma_R} \xi_R^{(3)}(x_1, x_1, x_2) \]

\[ + \frac{\nu^4}{\sigma_R} \left[ \frac{1}{3} \xi_R^{(4)}(x_1, x_1, x_2) + \frac{1}{4} \xi_R^{(4)}(x_1, x_2, x_2) \right] + \ldots \]

Subtracting the two-point correlation function, we find that the three-point correlation function is

\[ \zeta_p(x_1, x_2, x_3) = \frac{\nu^3}{\sigma_R} \xi_R^{(3)}(x_1, x_2, x_3) + \frac{\nu^4}{\sigma_R} \left[ \xi_R^{(2)}(x_{12}) \xi_R^{(2)}(x_{23}) + \text{(cyclic)} \right] \]

\[ + \frac{1}{2} \frac{\nu^4}{\sigma_R} \left[ \xi_R^{(4)}(x_1, x_1, x_2, x_2) + \text{(cyclic)} \right]. \]

Bispectrum is the Fourier transform of the three-point correlation function. As universe is statistically isotropic, the three-point correlation function only depends on the shape and size of triangles constructed by three points \((x_1, x_2, x_3)\), which can be fully specified by using two vectors, \(r \equiv x_1 - x_3\) and \(s \equiv x_2 - x_3\). That is,

\[ \zeta(r, s) = \zeta(r, s, |r-s|). \quad \text{(K.55)} \]

In order to see this, let’s Fourier transform the three point correlation function.

\[ \zeta(x_1, x_2, x_3) \]

\[ = \int \frac{d^3q_1}{(2\pi)^3} \int \frac{d^3q_2}{(2\pi)^3} \int \frac{d^3q_3}{(2\pi)^3} \delta(q_1)\delta(q_2)\delta(q_3)e^{i\mathbf{q}_1\cdot\mathbf{x}_1}e^{i\mathbf{q}_2\cdot\mathbf{x}_2}e^{i\mathbf{q}_3\cdot\mathbf{x}_3} \]

\[ = \int \frac{d^3q_1}{(2\pi)^3} \int \frac{d^3q_2}{(2\pi)^3} \int \frac{d^3q_3}{(2\pi)^3} B(q_1, q_2, q_3)\delta^D(q_{123})e^{i\mathbf{q}_1\cdot\mathbf{x}_1}e^{i\mathbf{q}_2\cdot\mathbf{x}_2}e^{i\mathbf{q}_3\cdot\mathbf{x}_3} \]

\[ = \int \frac{d^3q_1}{(2\pi)^3} \int \frac{d^3q_2}{(2\pi)^3} B(q_1, q_2, |q_1 + q_2|)e^{i\mathbf{q}_1\cdot(x_3-x_1)}e^{i\mathbf{q}_2\cdot(x_2-x_3)} \]

\[ = \int \frac{d^3q}{(2\pi)^3} B(q, q')e^{i\mathbf{q}\cdot\mathbf{r}} e^{i\mathbf{q}'\cdot\mathbf{s}} = \zeta(r, s) \]

Therefore, we calculate the bispectrum by inverse-Fourier transform of the three-point correlation function:

\[ B(q, q') = \int d^3r \int d^3s \zeta(r, s)e^{-i\mathbf{q}\cdot\mathbf{r}}e^{-i\mathbf{q}'\cdot\mathbf{s}} \]

\[ \text{(K.57)} \]
Finally, let’s think about the Fourier transformation of connected four point function,

\[ \xi^{(4)}_{R}(x_1, x_2, x_3, x_4) \]

\[ = \int \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3} \frac{d^3 q_4}{(2\pi)^3} \langle \delta_R(q_1) \delta_R(q_2) \delta_R(q_3) \delta_R(q_4) \rangle e^{i q_1 \cdot (x_1 - x_2)} e^{i q_2 \cdot x_1} e^{i q_3 \cdot x_2} e^{i q_4 \cdot x_3} \]

\[ = \int \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3} \frac{d^3 q_4}{(2\pi)^3} \langle \delta_R(q_1, q_2, q_3, q_4) \delta^D(q_{1234}) \rangle \]

\[ = \int \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3} \frac{d^3 q_4}{(2\pi)^3} \langle \delta_R(q_1, q_2, q_3, q_4) \delta^D(q_{1234}) \rangle e^{i q_1 \cdot (x_1 - x_2)} e^{i q_2 \cdot (x_1 - x_3)} e^{i q_3 \cdot (x_2 - x_3)} e^{i q_4 \cdot (x_1 - x_4)} \]

\[ = \int \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3} \frac{d^3 q_4}{(2\pi)^3} \langle \delta_R(q_1, q_2, q_3, q_4) \delta^D(q_{1234}) \rangle e^{i q_1 \cdot (q_1 + q_2)} e^{i q_2 \cdot q_3} \]

(K.58)

Similarly,

\[ \xi^{(4)}_{R}(x_1, x_2, x_3, x_4) \]

\[ = \int \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3} \frac{d^3 q_4}{(2\pi)^3} \langle \delta_R(q_1, q_2, q_3, q_4) e^{i q_1 \cdot r} e^{i (q_2 + q_3) \cdot r} \rangle \]

(K.59)

\[ \xi^{(4)}_{R}(x_1, x_2, x_3, x_4) \]

\[ = \int \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3} \frac{d^3 q_4}{(2\pi)^3} \langle \delta_R(q_1, q_2, q_3, q_4) e^{i q_1 \cdot (r - s)} e^{i (q_2 + q_3) \cdot s} \rangle \]

(K.60)

By using the Fourier relations above, we can easily calculate the Fourier transfor-
mation of equation (K.54).

\[ B_p(k_1, k_2) = \int d^3r \int d^3s \rho_p(r, s) e^{-i k_1 \cdot r} e^{-i k_2 \cdot s} \]

\[ = \frac{\nu^3}{\sigma_R^3} B_R(k_1, k_2) + \frac{\nu^4}{\sigma_R^4} \left[ P_R(k_1)P_R(k_2) + P_R(k_2)P_R(k_3) + P_R(k_3)P_R(k_1) \right] \]

\[ + \frac{\nu^4}{2\sigma_R^4} \int d^3r \int d^3s \int \frac{d^3q_1}{(2\pi)^3} \int \frac{d^3q_2}{(2\pi)^3} \int \frac{d^3q_3}{(2\pi)^3} \]

\[ \times \left[ T_R(q_1, q_2, q_3) e^{i q_1 \cdot r} e^{i q_2 \cdot s} e^{-i k_1 \cdot r} e^{-i k_2 \cdot s} \right. \]

\[ + T_R(q_1, q_2, q_3) e^{-i q_1 \cdot r} e^{-i q_2 \cdot s} e^{-i k_1 \cdot r} e^{-i k_2 \cdot s} \]

\[ + T_R(q_1, q_2, q_3) e^{i q_1 \cdot (r-s)} e^{-i q_2 \cdot s} e^{-i k_1 \cdot r} e^{-i k_2 \cdot s} \]

\[ = \frac{\nu^3}{\sigma_R^3} B_R(k_1, k_2) + \frac{\nu^4}{\sigma_R^4} \left[ P_R(k_1)P_R(k_2) + (2 \text{ cyclic}) \right] \]

\[ + \frac{\nu^4}{2\sigma_R^4} \int \frac{d^3q}{(2\pi)^3} \left[ T_R(q, k_1, k_2, k_3) + +(2 \text{ cyclic}) \right], \quad (K.61) \]

where \( k_3 = -k_2 - k_3 \).

Note that, even in the absence of the primordial non-Gaussianity, matter bispectrum \( B_R(k_1, k_2, k_3) \) is non-zero due to the gravitational instability, and that is given by

\[ B_R(k_1, k_2, k_3) = 2 F_2^s(k_1, k_2) P_R(k_1) P_R(k_2) + (\text{cyclic}). \quad (K.62) \]

Therefore, the galaxy bispectrum is given by

\[ B_p(k_1, k_2, k_3) = \frac{\nu^3}{\sigma_R^3} \left[ 2 F_2^s(k_1, k_2) P_R(k_1) P_R(k_2) + (\text{cyclic}) \right] \]

\[ + \frac{\nu^4}{\sigma_R^4} \left[ P_R(k_1)P_R(k_2) + (\text{cyclic}) \right]. \quad (K.63) \]

in the large scale where we expect that halo/galaxy bias is linear.