V. Thermal History of the Universe (Cont'd)

- SM particles

**Leptons**
\[
\begin{align*}
\ell & \quad e \text{ 511 keV} & \mu \text{ 105 MeV} & \tau \text{ 1.8 GeV} \\
\nu & \quad \nu_e < 2 eV & \nu_\mu < 2 eV & \nu_\tau < 2 eV
\end{align*}
\]

**Quarks**
\[
\begin{align*}
quarks & \quad u \text{ 2.3 MeV} & c \text{ 1.3 GeV} & t \text{ 173 GeV} \\
d & \text{ 4.8 MeV} & s \text{ 95 MeV} & b \text{ 4.18 GeV}
\end{align*}
\]

**Bosons**
\[
\begin{align*}
\text{Bosons} & \quad W, \ g_i \ (i=1, \ldots, 8) \\
W^\pm & \text{ 80.4 GeV} \\
Z & \text{ 91.2 GeV} \\
H & \text{ 125.7 GeV}
\end{align*}
\]

- Phase transitions

- EW phase transition \( (T_c = m_H/g \approx 196.7 \text{ GeV}) \)
- QGP (QCD) phase transition \( (T_c \approx 150 \text{ MeV}) \)
\[ n_i = g_i \int \frac{d^3p}{(2\pi)^3} f(p) = \left\{ \begin{array}{ll} \frac{5}{3} \frac{\sigma_i}{T_i} & (\text{B}) \\ \frac{3}{4} \frac{5}{3} \frac{\sigma_i}{T_i} & (\text{F}) \end{array} \right. \]

\[ \rho_i = g_i \int \frac{d^3p}{(2\pi)^3} p f(p) = \left\{ \begin{array}{ll} \frac{\pi^2}{30} \frac{\sigma_i}{T_i} & (\text{B}) \\ \frac{7}{8} \frac{\pi^2}{30} \frac{\sigma_i}{T_i} & (\text{F}) \end{array} \right. \]

\[ P_i = \rho_i/3, \quad s_i = \frac{\rho_i + P_i}{T_i} = \frac{4\rho_i}{3T_i} \]

* Relativistic particles

* Non-relativistic particles ; \( m \gg T \); \( E \approx m + \frac{p^2}{2m} \gg T \\

\[ n_i = g_i \left( \frac{m_i T_i}{2\pi} \right)^{3/2} e^{-(m_i - \mu_i)/T_i} \]

\[ \rho_i = g_i \int \frac{d^3p}{(2\pi)^3} \left( m + \frac{p^2}{2m} \right) f(p) = n_i \left( m_i + \frac{3}{2} T_i \right) \]

\[ P_i = g_i \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3m} f(p) = n_i T_i \]

\[ s_i = \frac{\rho_i + P_i - m_i n_i}{T_i} = \left( \frac{m_i - \mu_i}{T_i} + \frac{5}{2} \right) n_i \]

Note: \( \frac{n_i^{nr}}{n_i^{rel}} \sim \left( \frac{m_i}{T} \right)^{3/2} e^{-m_i/T} \ll 1 \) when \( T \ll m \) \( \implies s_i \ll n_i \)

\( \implies \text{most of entropy carried by relativistic particles!} \)
\[ \frac{n_x}{n_y} \sim \left( \frac{m}{T} \right)^{3/4} e^{-m/T} \quad \text{It is harder to pair-produce } X \& X \text{ at low } T. \]

\[ X + \overline{X} \rightarrow 2Y \]

Q: When can we ignore $\overline{X}$ completely?

\[ \begin{align*}
\beta &= \frac{n_x - n_{\overline{x}}}{S} \approx 10^{-9} \\
\frac{n_x n_{\overline{x}}}{S^2} &\sim \left( \frac{m}{T} \right)^3 e^{-2m/T}
\end{align*} \]

\[
\therefore \frac{n_x}{S}, -\frac{n_{\overline{x}}}{S} \text{ are solutions of } t^2 + \beta t - \left( \frac{m}{T} \right)^3 e^{-2m/T} = 0
\]

\[ \Rightarrow \frac{\overline{n}}{S} = \sqrt{\left( \frac{\beta}{2} \right)^2 + \left( \frac{m}{T} \right)^3 e^{-2m/T}} - \frac{\beta}{2} \]

\[ \Rightarrow \overline{X} \text{ negligible when } \left( \frac{\beta}{2} \right)^2 \gg \left( \frac{m}{T} \right)^3 e^{-2m/T} \]

\[ \Rightarrow \ln \left( \frac{\beta}{2} \right) > \frac{3}{2} \ln \left( \frac{m}{T} \right) - \frac{m}{T} \]

or, \[ T < \frac{m}{\ln(2/\beta)} \sim \frac{m}{21} \]

For example, Wien tail photons can create $e^+e^-$ at \( T > \frac{511}{21} \sim 25 \text{ keV} \).
Thermal history of the early Universe

**Keys:**
1. Comoving entropy conservation: \( \frac{d}{dE} (S a^2) = 0 \)
2. Entropy is carried by relativistic particles
3. Various energy scales (photon mass, symmetry breaking scales)

(i) Effective relativistic degrees of freedom

\[
\begin{align*}
S(T) &= \frac{\pi^2}{30} \vartheta_x(T) T^4 \\
S(T) &= \frac{2\pi^2}{45} \vartheta_x(T) T^3 \\
\eta(T) &= \frac{\zeta(3)}{16} \vartheta_x(T) T^3
\end{align*}
\]

* for relativistic bosons in LTE with \( \gamma \): \( \Delta \vartheta_x = \Delta \vartheta_x = \Delta \vartheta_x = 0 \)

* for relativistic fermions in LTE with \( \gamma \): \( \Delta \vartheta_x = \Delta \vartheta_x = \frac{7}{8} g_i, \Delta \vartheta_x = \frac{3}{4} g_i \)

- if NOT in LTE with \( \gamma \): \( \Delta \vartheta_x \rightarrow \Delta \vartheta_x \left( \frac{T'}{T} \right)^{3} \) or 4
  (like temperature \( T' \))

* for non-relativistic pts: Need to integrate \( f(p) \).
(ii) Evolution of temperature: $T(a)$

$$g_{\*s} T^3 a^3 = \text{const.} \quad \Rightarrow \quad T \propto g_{\*s}^{-\frac{1}{3}} a^{-1}$$

(iii) Friedmann equation during the radiation-dominated epoch

$$3H^2 = 8\pi G \rho_R = 8\pi G \left( \frac{\pi^2}{30} \frac{g_{\*s}}{T^4} \right) = \frac{3}{4t^2}$$

$$\Rightarrow \quad t = \sqrt{\frac{45}{16 \pi^3 g_{\*s}(T)}} \frac{1}{T^2} \approx \frac{2.420}{\sqrt{g_{\*s}}} T_{\text{MeV}}^{-2} \text{ sec}$$
\( \mathcal{g}_x \) and \( \mathcal{g}_{xs} \)

- at very high \( T \) (\( \approx 200 \) GeV, after EW phase transition)

; all SM pts are in LTE & relativistic!

(a) Bosons \( \sqrt{S} \to 0 \)

\[ W^\pm, Z, \gamma, H, \phi, \phi_i (i=1, \ldots, g) \] are massive, \( S=1 \)

\[ \Rightarrow \Delta \mathcal{g}_B = 3(\text{W}^\pm, \text{Z}) \times 3 + 2(\gamma) + 8(\phi_i) \times 2 + 1(H) \]

\[ = 28 \]

(b) Fermions

\[ \Delta \mathcal{g}_F = 3(\text{e}, \mu, \tau) \times 2 (S=\frac{1}{2}) \times 2 (\bar{e}, \bar{\mu}, \bar{\tau}) \]

\[ + 6(\text{quarks}) \times 2 (S=\frac{1}{2}) \times 2 (\bar{\text{q}}) \times 3 \text{colors}(r, g, b) \]

\[ + 3(\nu_e, \nu_\mu, \nu_\tau) \times 2 (\text{\bar{\nu}e, \bar{\nu}\mu, \bar{\nu}\tau}) \]

\[ = 12 + 72 + 6 = 90 \]

\[ \Rightarrow \Delta \mathcal{g}_x = \Delta \mathcal{g}_{xs} = \Delta \mathcal{g}_B + \frac{7}{8} \Delta \mathcal{g}_F = 28 + \frac{7}{8 \cdot 90} = \frac{10675}{90} \approx 118 \]

\[ \Delta \mathcal{g}_n = \Delta \mathcal{g}_B + \frac{3}{4 \cdot 90} \Delta \mathcal{g}_F = 28 \times \frac{1}{4} \cdot 90 = 95.75 \]
A story of neutrinos

\[ \nu \rightarrow e^+, \mu^+, \tau^+, \ldots \quad \sigma_w \propto G_F^2 T^2 \]

\[ G_F \approx 1.166 \times 10^{-5} \text{ GeV}^{-2} \approx (300 \text{ GeV})^{-2} \approx (\theta/\mu) \]

\[ \Gamma_{\nu} \sim n_\nu \sigma_w \sim T^3 \cdot T^2 = G_F^2 T^5 \]

\[ H \sim \sqrt{\frac{8\pi G}{3} T^4} \sim \sqrt{\frac{8\pi G}{3}} T^2 \]

\[ T_{\nu-de} \leq T \]

\[ G_F^2 T^5 = \sqrt{\frac{8\pi G}{3}} T^2 \quad \Rightarrow \quad T_{\nu-de} = \left( \frac{8\pi G}{3G_F^4} \right)^{1/4} \approx 1.2 \text{ MeV} \]

\[ \begin{cases} T > T_{\nu-de} & : \nu \text{ coupled to } e^+, \mu^+, \tau^+ \text{ & LTE}! \\ T < T_{\nu-de} & : \nu \text{ de-coupled from the cosmic thermal bath} \end{cases} \]

Key: \( m_e \approx 0.5 \text{ MeV} \Rightarrow e^+e^- \text{ pair annihilation after } T_{\nu-de}. \]

\[ \Rightarrow \text{ heat } \gamma, \text{ but NOT } \nu. \]

\[ \begin{cases} \text{after } T_{\nu-de} : \quad T_\gamma(a) = T_{\nu-de} \left( \frac{\sigma_\nu}{\sigma_\gamma} \right) ; \quad T_e = T_\gamma = T_{\nu-de} \left( \frac{\sigma_\nu}{\sigma_e} \right) \\ \text{after } T_{ee} : \quad \text{---} ; \quad T_\gamma > T_e \quad \text{By how much?} \end{cases} \]

\[ S a^3 = (S_e + S_\gamma + S_\tau) a^3 \equiv \text{constant}. \]

\[ \Rightarrow \frac{7}{8} \cdot 4 + 2 \quad T_\gamma^3 (a_i) a_i^3 = 2 T_\gamma (a_f) a_f^3 \]

\[ \Rightarrow T_\gamma (a_f) = \left( \frac{11}{4} \right)^{1/3} T_\gamma (a_i) \left( \frac{a_i}{a_f} \right) = \left( \frac{11}{4} \right)^{1/3} T_\gamma (a_i) \times \left[ \frac{T_\gamma (a_f)}{T_\gamma (a_i)} \right] \]

\[ \therefore \quad \frac{T_\gamma}{T_\nu} = \left( \frac{11}{4} \right)^{1/3} \]
9 After $ee$ pair annihilation

$$\theta_* = \theta_Y + \theta_{\nu} = 2 + \frac{7}{8} N_{\nu} \times 2 \times \left(\frac{4}{11}\right)^{4/5} = 2 \left(1 + 0.2271 N_{\nu}\right)$$

$$N_{\nu} = 3 + \alpha$$ \quad ; \quad \alpha = 0.046 = 0.035 + 0.011 \quad \text{(in SM)}$$

$e, \mu, \tau$

$\frac{T}{e_0}$

plasma effect (finite-$T$ effect)

entropy transfer from $ee \rightarrow \nu\bar{\nu}$

$$\therefore \quad t = 2.420 \sqrt{\theta_*} \frac{T_{\text{MeV}}^{-1}}{T_{\text{MeV}}} \text{ sec} \sim 1.32 \frac{T_{\text{MeV}}^{-2}}{T_{\text{MeV}}} \text{ sec}$$

$$\left\{ \begin{array}{l}
\Omega_r = \frac{\pi^2}{30} \theta_* T_Y^4 = \left(1 + 0.2271 N_{\nu}\right) \frac{\pi^2}{15} T_Y^4 \\
\approx 2.0036 \times 10^{-15} \left(1 + 0.2271 N_{\nu}\right) \left(\frac{T_Y}{2.726 K}\right)^4 \text{ eV}^4 \\
\end{array} \right.$$  \quad \left( T_Y \sim 2.726 K \sim 0.2349 \text{ meV} \right)

$$\frac{\Omega_{\text{crito}}}{\theta_*} = \frac{3\hbar^2}{8\pi^2 G} = 8.096 \times 10^{-11} \text{ eV}^4$$

$$\therefore \quad \Omega_r = \frac{\Omega_r}{\Omega_{\text{crito}}} \sim 4.186 \times 10^{-5} \hbar^{-2} \sim 8.544 \times 10^{-5} \hbar_{0.7}^{-2}$$

Note: \quad $\theta_{\star s} = \theta_Y + \frac{7}{8} N_{\nu} \times 2 \times \left(\frac{4}{11}\right) = 2 \left(1 + 0.3182 N_{\nu}\right)$
VI. Out of Equilibrium

* During RD, majority (radiations) of pets were in LTE (last section).

\[ 3H^2 = 8\pi G \rho_r \] is enough to describe act.

* If all pets have been in equilibrium at all time \( \Rightarrow \) boring \( T = 2.726 \text{ K} \).

\( \Rightarrow \) out of equilibrium processes

\[ \begin{align*}
1 & \text{ Baryogenesis / Leptogenesis} \\
2 & \text{ Weakly Interacting Massive Particles (DM, sterile neutrinos, ...)} \\
3 & \text{ Neutrino decoupling} \\
4 & \text{ BBN} \\
5 & \text{ Recombination of CMB}
\end{align*} \]

* Main theme : \( \begin{align*} 
\Gamma > H & \Rightarrow \text{LTE} \\
\Gamma < H & \Rightarrow \text{freeze-out} \Rightarrow \text{Simply diluted as} \frac{1}{a^3}
\end{align*} \)

(a) \( \frac{S}{\sqrt{\rho_r}} \] [massive pets freeze-out] \( \sim m_X / a^3 \sim m_X T^3 \)

\( \rho_r \propto T^4 \)

\( \Rightarrow \rho_m > \rho_r \) at lower \( T \) (beginning of MD-epoch)

(b) \( \nu \)-decoupling, BBN, CMB

\( \Rightarrow \) cosmological probes! (\( \because \) if stayed in LTE forever, \( T = 2.726 \text{ K} \) would tell all about everything!)
transition from $\Gamma > H$ to $\Gamma < H$?

Let $\langle \sigma v \rangle \sim T^n \implies \Gamma = n \langle \sigma v \rangle \sim T^{n+3}$

$$H = \sqrt{\frac{\kappa}{4\pi}} \frac{T^2}{m_{pl}}$$

$\therefore$ if $n+3 > 2 \implies n > 1 \iff \Gamma$ drops faster than $H$.

e.g. Weak Interaction: $\sigma_w = \left( \frac{g_w}{m_{w}^2} \right)^2 T^2 + v \sim C \implies \langle \sigma v \rangle \sim T^2 \checkmark$

EM Interaction: $\sqrt{\Gamma} = \text{const.}$

$\sigma_{\text{kn}} \sim T^{-2}! \implies n \sigma_{\text{kn}} v \sim T < H \sim T^2$ at high-$T$!

(\therefore \text{interactions can freeze-out at high $T$.})
(1) Kinetic theory

Boltzmann equation: \[ \frac{df}{d\lambda} = C[f] \]

\[ \Downarrow \quad \text{Liouville operator} \]

\[ f = f(\mathbf{x}, \mathbf{p}; t) \]

\[ \Rightarrow \quad \frac{df}{d\lambda} + \frac{dx}{d\lambda} \frac{df}{dx} + \frac{dp}{d\lambda} \frac{df}{dp} = C[f] \]

(2) Boltzmann equation in the FRW universe

\[ f = f(p; t) \]

\[ \Rightarrow \quad \frac{df}{d\lambda} + \frac{dp}{d\lambda} \frac{df}{dp} = C[f] \]

i) LHS

\[ \text{geodesic eq.} \quad \frac{dp}{d\lambda} = -\nabla p \quad \Rightarrow \quad \frac{df}{d\lambda} = \frac{dt}{d\lambda} \frac{df}{dt} = \epsilon \left[ \frac{df}{dt} - \mathbf{H} \frac{df}{dp} \right] \]

ii) RHS for the binary interaction \((12 \leftrightarrow 34)\)

\[ C[f] = (\text{ptl added}) - (\text{ptl removed}) \]

\[ \int \frac{df}{d\lambda} = \frac{\theta_2 \theta_3 \theta_4}{2 \mathcal{E}_1} \int \frac{1}{2 \mathcal{E}_2} \int \frac{1}{2 \mathcal{E}_3} \int \frac{1}{2 \mathcal{E}_4} (2\pi)^4 \delta^{(4)}(\mathbf{p}_1 + \mathbf{p}_3 - \mathbf{p}_1 - \mathbf{p}_2) |M_{12}|^2 \]

\[ \times \left[ f_3(p_3) f_4(p_4) \left( (\pm f_1(p_1)) (\pm f_1(p_1)) \right) \right. \]

\[ - \left. f_1(p_1) f_2(p_2) (1 \pm f_3(p_3)) (1 \pm f_4(p_4)) \right] \]
Equivalently, using \( n_i = g_i \int_{\pi_i} f_i(p_i) \):

\[
\Rightarrow \quad g_i \int_{\pi_i} e \frac{df_i}{d\lambda} = g_i \int_{\pi_i} \left( \frac{df_i}{dt} - H(p) \frac{df_i}{dp} \right)
\]

\[
= \frac{d}{dt} n_i - H \left\{ g_i \frac{1}{2\pi^2} \int_{\pi_i} \int_{\pi_i} \int_{\pi_i} \int_{\pi_i} \frac{\partial f_i}{\partial p_i} dp_i \right\}
\]

\[
= \frac{d}{dt} n_i + 3H n_i = \frac{1}{\alpha^3} \frac{d}{dt} (n_i \alpha^3)
\]

\[
\frac{1}{\alpha^3} \frac{d}{dt} (n_i \alpha^3) = \int_{\pi_i} \frac{\partial}{\partial \lambda_i} \left[ \frac{\partial}{\partial \lambda_i} \right] + \int_{\pi_i} \frac{\partial}{\partial \lambda_i} \left[ \frac{\partial}{\partial \lambda_i} \right] \left\{ \frac{\partial}{\partial \lambda_i} \right\} \left\{ \frac{\partial}{\partial \lambda_i} \right\}
\]

\[
\times \left[ f_3(p_3) f_4(p_4) (1 \pm f_1(p_1)) (1 \pm f_2(p_2))
\right.
\]

\[
- f_1(p_1) f_2(p_2) (1 \pm f_3(p_3)) (1 \pm f_4(p_4)) \right]
\]
After decoupling at $T = T_0$ & $a = a_0$ (e.g. $T_0 \approx 1$ MeV & $\nu$) 

$\Rightarrow C[f] = 0$

$\Rightarrow \frac{\partial f}{\partial t} - H_p \frac{\partial f}{\partial p} = 0 \Rightarrow a^3 n = \text{constant}$

* What about $f$?

let $f(p; t) = \left[ e^{(E - \mu(t))/T(t)} - 1 \right]^{-1}$

$\frac{\partial f}{\partial t} = -f^2 e^{(E - \mu)/T} \left\{ -\frac{1}{T} \dot{\epsilon} + \frac{\dot{\mu} - \dot{\mu}T}{T} \right\}$

$\frac{\partial f}{\partial p} = -f^2 e^{(E - \mu)/T} \left\{ \frac{1}{T} \frac{P^2}{E} \right\}$

$\Rightarrow \frac{\partial f}{\partial t} - H_p \frac{\partial f}{\partial p} = -f^2 e^{(E - \mu)/T} \left\{ -\frac{1}{T} \dot{\epsilon} + \frac{\dot{\mu} - \dot{\mu}T}{T} - H_p \frac{P^2}{E} \right\} = 0$

$= \mu \left( \frac{\mu}{T} - \frac{\dot{\mu}}{T} \right) - E \left( \frac{\dot{P}}{T} + H_p \frac{P^2}{E^2} \right) = 0$

i) Particles relativistic at decoupling time

$f(p) = \frac{1}{e^{(p - \mu)/T} - 1}$ at freeze-out, $E = p$

$\Rightarrow \frac{\mu}{T} = \frac{\dot{\mu}}{\dot{T}} = -\frac{\dot{a}}{a} \Rightarrow T \alpha = \text{const.} \quad H/T = \text{const}$

$\Rightarrow T \propto \frac{1}{a} \quad ; \quad \mu \propto T \propto \frac{1}{a}$

$\Rightarrow S a^3 \equiv \text{const.}$

ii) Particles non-relativistic at decoupling time

$E = m + p^2/2m$

$\Rightarrow \mu - \mu \frac{\dot{T}}{T} + (m + \frac{p^2}{2m}) \frac{\dot{T}}{T} + \frac{p^2}{E} H = (m - \mu) \left( \frac{\dot{T}}{m - \mu} + \frac{\dot{T}}{T} \right) + p^2 \left( \frac{\dot{T}}{T} + 2 \frac{\dot{a}}{a} \right) = 0$

$\Rightarrow T \propto \frac{1}{a^2} \quad ; \quad (\mu - m) \propto T \propto \frac{1}{a^2}$

$\Rightarrow S a^3 \equiv \text{const.}$