Chapter 4

Thermodynamics in the expanding universe

We have shown the expansion of the Universe reduces the momentum of indivisual particle proportional to the scale factor $p(t) \propto 1/a(t)$. That means, at earlier times in the Universe, each particle moves with higher and higher momentum. On top of that, the number density of particles increases as the cubic of the scale factor $a^3(t)$ to make the early universe a hot and dense state. In this chpater, we will study the thermal history of the Universe: what happens at those hot, dense states in the early Universe?

As the Universe is an expanding and everything in the Universe changes in time, the usual notion of equilibrium thermodynanics cannot be directly applied here. Luckly, however, for many cases, we do not have to deal with the full kinetic theory that describes the time evolution non-equilibrium state. That is, for the cases where particles interact so frequently that the interaction time scale $t_{\rm i} = (n\sigma v)^{-1}$ is much smaller than the expansion time scale $t_{\rm H} = H^{-1}$:

$$t_{\rm i} \ll t_{\rm H}, \tag{4.1}$$

these particles are in a *local equilibrium* state. If this is the case, we can describes the thermodynamical states of particles by maximizing their entropy (this is exactly what we mean by the local equilibrium state).

Eventurally, however, the interaction time scale typically had dropped more quickly than the Hubble time scale, and the interaction went out of equilibrium. Then, we need to invoke the kinetic theory to set up the Boltzmann equation describing the out-of-equilibrium process. That will be the subject of the next chapter.

4.1 Brief thermal history

We start our discussion from brief thermal history of the Universe. Before beginning, however, we have to clarify one thing: what do we mean by the **temperature** of the Universe? As discovered by Pensiaz and Wilson in 1965, and determined much more precisely in the early 1990s by NASA's COBE (Cosmic Background Explorer) FIRAS (Far Infrared Absolute Spectrometer), the Unierse is filled with a gas of microwave photons with a *blackbody spectrum*:

$$I_\nu = \frac{8\pi\nu^3}{\exp(\nu/T_0) - 1}, \tag{4.2}$$

with $T_0 = 2.726\,\mathrm{K}$. This gas of photons is called Cosmic Microwave Background (CMB). It is, however, important to clarify that the CMB photons are not in thermal equilibrium, although the energy distribution has the form of that for the massless particle in thermal equilibrium. Thermal equilibrium implies the frequent energy-momentum exchange via collision between particles. But, the photon gas that we observe in the microwave only rarely interact with electrons (recall the optical depth of $\tau \simeq 0.06$ from $z = 1089$ to now), and its mean-free-path is comparable to the size of the observable universe!

The extremely rare interaction, on the other hand, preserves the form of the spectral energy distribution. The only change is the frequency of photons due to cosmological redshift: a CMB photon with frequency ν_0 now had $\nu = (1+z)\nu_0$ at redshift z. That is, to keep the form of the spectral energy distribution invariant, one has to increases the *temperature* with the same factor $T(z) = (1+z)T_0$. As we have stressed before, this does *NOT* mean that the CMB photons are in thermal equilibrium, but it means that the form of the spectral energy distribution is the same as a massless boson gas in the thermal equilibrium with temperature $T(z)$.

The fact that the spectral energy distribution of the CMB is extremely close to the Planck curve suggests that the CMB photons indeed were in thermal equilibrium at early history of the Universe. This happens at redshift higher than the CMB recombination time $z \simeq 1100$, before which the Universe is opaque for the CMB photons and the CMB photons were in thermal equilibritum. The temperature of the Universe was about 0.2 eV and the Universe was about 400,000 years after the Big-Bang. Before the recombination, the temperature of the Universe is the thermal temperature of photon.

Following is the list of important events that happens prior to the CMB recombination:

- $T \sim 3\,\mathrm{eV}$, $t \sim 10^{4\sim5}$ yrs: matter-radiation equality; before this time, the energy density of the Universe is dominated by radiation.

- $T \sim$ keV, $t \sim 10^5$ sec: photons fall out of chemical equilibrium; at earlier times, free-free and double compton scattering can change the photon number rapidly compared with the expansion rate; no CMB photons are created or destroyed after this time.

- $T \sim$ MeV, $t \sim 10^{0\sim2}$ sec: Big-bang nucleosynthesis (BBN) happens; this is the time when neutrons combine with protons to form Deuterium, Tritium, Helium, Lithium, Beryllium nuclei. This is probably the earliest epoch that we test our theoretical prediction with direct observation. The standard calculation of BBN matches impressively well with observed light element abundance.

- $T \sim 150\,\mathrm{MeV}$, $t \sim 10^{-5}$ sec: The quantum-chromodynamics (QCD) phase transition. This is when quarks and gluons first become bound into neutrons and protons. Before this time, u-quarks, d-quarks and gluons are in the plasma state called QGP (quark-gluon plasma).

- $T \sim 100\,\mathrm{GeV}$, $t \sim 10^{-11}$ sec: Electro-weak symmetry breaking. Around this energy, Higgs mechanism takes place to give mass to weak bosons: $\mathrm{W}^{\pm}$ and Z. Before this time, they were massless.

- $T \sim 10^{12}\,\mathrm{GeV}$, $t \sim 10^{-30}$ sec: The Peccei-Quinn phase transition, if the Peccei-Quinn mechanism is the correct explanation for the strong-CP problem. This is highly speculative.

- $T \sim 10^{16}\,\mathrm{GeV}$, $t \sim 10^{-38}$ sec: The GUT phase transition, before which the stron and electroweak interactions are indistinguishable. This is highly speculative.

- $T \sim 10^{19}\,\mathrm{GeV}$, $t \sim 10^{-43}$ sec: The Planck scale. Fundamental strings? look quantum gravity? quantum birth of the Universe? This is all highly speculative. We do not yet know the physics at this scale, a quantum theory of gravity.

4.2 Maximum entropy states

When the scattering is frequent enough, the local equilibrium state in the expanding Universe maximizes the entropy:

$$S = \ln \Gamma, \tag{4.3}$$

where Γ is the number of possible microscopic state for a given macrostate. In this section, we calculate the maximum entropy states for bosons (with integer spin, no exclusion) and fermions (with half-integer spin, exclusion).

4.2.1 Boson-Einstein distribution function

Let us consider an ideal gas of N-bosons with total energy E in a box of volume V. We denote the number of bosons in energy state between ε and $\varepsilon + \delta\varepsilon$ as ΔN_ε and the number of possible microscopic states that a particle can occupy in the one particle phase space (degeneracy) as Δg_ε.

Then, we calculate the total number of possible microscopic states that the ΔN_ε particles can have as

$$\Delta G_\varepsilon = \begin{pmatrix} \Delta N_\varepsilon + \Delta g_\varepsilon - 1 \\ \Delta N_\varepsilon \end{pmatrix} = \frac{(\Delta N_\varepsilon + \Delta g_\varepsilon - 1)!}{(\Delta N_\varepsilon)!(\Delta g_\varepsilon - 1)!} \tag{4.4}$$

The total number of microstates for the whole system is then

$$\Gamma(\{\Delta N_\varepsilon\}) = \prod_\varepsilon \Delta G_\varepsilon, \tag{4.5}$$

from which we calculate the entropy as

$$\begin{aligned} S = \ln \Gamma &= \sum_\varepsilon \ln[(\Delta N_\varepsilon + \Delta g_\varepsilon - 1)!] - \ln[(\Delta N_\varepsilon)!] - \ln[(\Delta g_\varepsilon - 1)!] \\ &= \sum_\varepsilon \ln[(\Delta N_\varepsilon + \Delta g_\varepsilon)!] - \ln(\Delta N_\varepsilon + \Delta g_\varepsilon) - \ln[(\Delta N_\varepsilon)!] - \ln[(\Delta g_\varepsilon)!] + \ln \Delta g_\varepsilon. \end{aligned} \tag{4.6}$$

Using Stirling's formula $\ln N! \simeq N \ln N - N$, we approximate the entropy as

$$S \simeq \sum_\varepsilon (\Delta N_\varepsilon + \Delta g_\varepsilon) \ln(\Delta N_\varepsilon + \Delta g_\varepsilon) - \ln(\Delta N_\varepsilon + \Delta g_\varepsilon) - (\Delta N_\varepsilon) \ln(\Delta N_\varepsilon) - (\Delta g_\varepsilon) \ln(\Delta g_\varepsilon) + \ln \Delta g_\varepsilon. \tag{4.7}$$

Now we define the mean occupation number n_ε as

$$\Delta N_\varepsilon = n_\varepsilon \Delta g_\varepsilon, \tag{4.8}$$

to have

$$\begin{aligned} S &\simeq \sum_\varepsilon \Delta g_\varepsilon (1 + n_\varepsilon) \ln[\Delta g_\varepsilon (1 + n_\varepsilon)] \\ &\quad - \ln[\Delta g_\varepsilon (1 + n_\varepsilon)] - (n_\varepsilon \Delta g_\varepsilon) \ln(n_\varepsilon \Delta g_\varepsilon) - (\Delta g_\varepsilon) \ln(\Delta g_\varepsilon) + \ln \Delta g_\varepsilon \\ &\simeq \sum_\varepsilon \Delta g_\varepsilon [(1 + n_\varepsilon) \ln(1 + n_\varepsilon) - n_\varepsilon \ln n_\varepsilon], \end{aligned} \tag{4.9}$$

where we approximate $\Delta g_\varepsilon \gg 1$ in the last line. We want to maximize Eq. (4.9) with the constraint that the total energy and number of particles in the system are

$$E = \sum_\varepsilon \Delta N_\varepsilon \varepsilon = \sum_\varepsilon \varepsilon n_\varepsilon \Delta g_\varepsilon, \; N = \sum_\varepsilon \Delta N_\varepsilon = \sum_\varepsilon n_\varepsilon \Delta g_\varepsilon. \tag{4.10}$$

This can be done by using Lagrangian multiplier method with Lagrangian

$$\begin{aligned} L =& S + \lambda_1 \left(\sum_\varepsilon \varepsilon n_\varepsilon \Delta g_\varepsilon - E \right) + \lambda_2 \left(\sum_\varepsilon n_\varepsilon \Delta g_\varepsilon - N \right) \\ =& \sum_\varepsilon \Delta g_\varepsilon \left[(1+n_\varepsilon)\ln(1+n_\varepsilon) - n_\varepsilon \ln n_\varepsilon + \lambda_1 \varepsilon n_\varepsilon + \lambda_2 n_\varepsilon \right] - \lambda_1 E - \lambda_2 N. \end{aligned} \tag{4.11}$$

Usign Euler-Lagrange equation, we have

$$\begin{aligned} \frac{\partial L}{\partial n_\varepsilon} &= \sum_\varepsilon \Delta g_\varepsilon \left[\ln(1+n_\varepsilon) - 1 - \ln n_\varepsilon + 1 + \lambda_1 \varepsilon + \lambda_2 \right] \\ &= \sum_\varepsilon \Delta g_\varepsilon \left[\ln(1+n_\varepsilon) - \ln n_\varepsilon + \lambda_1 \varepsilon + \lambda_2 \right] = 0, \end{aligned} \tag{4.12}$$

or

$$n_\varepsilon = \left[e^{-\lambda_1 \varepsilon - \lambda_2} - 1 \right]^{-1}. \tag{4.13}$$

To match with the usual definition in thermodynamics[1], we identify

$$\lambda_1 = -\left(\frac{\partial S}{\partial E} \right)_N \equiv -\frac{1}{T} \tag{4.17}$$

$$\lambda_2 = -\left(\frac{\partial S}{\partial N} \right)_E \equiv \frac{\mu}{T}, \tag{4.18}$$

then the occupation density in the local equilibrium becomes

$$n_\varepsilon = \frac{1}{e^{(\varepsilon-\mu)/T} - 1}. \tag{4.19}$$

Here, T and μ are, respectively, temperature and the chemical potential, and this function is called Bose-Einstein distribution function.

[1]This comes from the thermodynamic identity:

$$dS = \frac{1}{T} dE + \frac{P}{T} dV - \frac{\mu}{T} dN, \tag{4.14}$$

or

$$dE = T dS - P dV + \mu dN. \tag{4.15}$$

The interpretation of chemical potential follows from Eq. (4.15) as

$$\mu = \left(\frac{\partial E}{\partial N} \right)_{S,V} \tag{4.16}$$

the energy required to change the number of particle when fixing entropy and volume. The minus sign in the definition Eq. (4.14) comes about to define the poticle flow from higher to lower chemical potential.

4.2.2 Fermi-Dirac distribution function

Due to the Pauli exclusion principle, no two Fermi particles occupy the same quantum states. Therefore, the maximum number of particles occupying Δg_ε internal states is Δg_ε, and the total number of possible microscopic states with ΔN_ϵ particles and Δg_ε degenerate states is

$$\Delta G_\varepsilon = \begin{pmatrix} \Delta g_\varepsilon \\ \Delta N_\varepsilon \end{pmatrix} = \frac{(\Delta g_\varepsilon)!}{(\Delta N_\varepsilon)!(\Delta g_\varepsilon - \Delta N_\varepsilon)!}. \tag{4.20}$$

We repeat the same calculation as before. The entropy is

$$\begin{aligned} S = \ln\Gamma &= \sum_\varepsilon \ln[(\Delta g_\varepsilon)!] - \ln[(\Delta N_\varepsilon)!] - \ln[(\Delta g_\varepsilon - \Delta N_\varepsilon)!] \\ &\simeq \sum_\varepsilon \Delta g_\varepsilon \ln\Delta g_\varepsilon - \Delta N_\varepsilon \ln\Delta N_\varepsilon - (\Delta g_\varepsilon - \Delta N_\varepsilon)\ln(\Delta g_\varepsilon - \Delta N_\varepsilon) \\ &= \sum_\varepsilon \Delta g_\varepsilon \ln\Delta g_\varepsilon - n_\varepsilon \Delta g_\varepsilon \ln n_\varepsilon - n_\varepsilon \Delta g_\varepsilon \ln\Delta g_\varepsilon - (1-n_\varepsilon)\Delta g_\varepsilon \ln(1-n_\varepsilon) - (1-n_\varepsilon)\Delta g_\varepsilon \ln\Delta g_\varepsilon \\ &= -\sum_\varepsilon [n_\varepsilon \ln n_\varepsilon + (1-n_\varepsilon)\ln(1-n_\varepsilon)]\Delta g_\varepsilon, \end{aligned} \tag{4.21}$$

then we maximize the Lagrangian:

$$L = S = -\sum_\varepsilon [n_\varepsilon \ln n_\varepsilon + (1-n_\varepsilon)\ln(1-n_\varepsilon) - \lambda_1 \varepsilon n_\varepsilon - \lambda_2 n_\varepsilon]\Delta g_\varepsilon - \lambda_1 E - \lambda_2 N. \tag{4.22}$$

The Euler-Lagrangian equation is

$$\frac{\partial L}{\partial n_\varepsilon} = -\sum_\varepsilon \{\ln n_\varepsilon - \ln(1-n_\varepsilon) - \lambda_1\varepsilon - \lambda_2\}\Delta g_\varepsilon = 0, \tag{4.23}$$

solving that reads

$$n_\varepsilon = e^{\lambda_1\varepsilon+\lambda_2}\left[1 + e^{\lambda_1\varepsilon+\lambda_2}\right]^{-1} = \left[e^{-\lambda_1\varepsilon-\lambda_2} + 1\right]^{-1}, \tag{4.24}$$

and using the same identification, we find the occupation number of

$$n_\varepsilon = \frac{1}{e^{(\varepsilon-\mu)/T} + 1}. \tag{4.25}$$

This is called Fermi-Dirac distribution function.

4.2.3 A word for chemical potential

Let us consider the following chemical reaction:

$$\nu_1 X_1 + \nu_2 X_2 + \cdots \rightleftharpoons \nu'_1 Y_1 + \nu'_2 Y_2 \cdots, \tag{4.26}$$

in the chemical equilibrium. One can also re-define the variables $\nu'_i = -\nu_j$ and $Y_i = X_j$ to rewrite above chemical reaction as

$$\sum_i^K \nu_i X_i = 0. \tag{4.27}$$

The conservation of number density indicates $\forall i \in (1, \cdots, K)$,

$$\frac{\delta N_{X_i}}{\nu_i} = \text{const.} \equiv \delta N. \tag{4.28}$$

To get the equilibrium condition let us assume that the reaction happens at constant temperature T and volume V. In that case, we have three variables T, V, N, and the Helmholtz free energy $A = E - TS$ is a good function to use:

$$dA = dE - TdS - SdT = -SdT - PdV + \mu dN \tag{4.29}$$

from Eq. (4.15). In the equilibrium state, $\delta A = 0$, which implies

$$0 = \delta A = \sum_{i=1}^{K} \mu_i dN_{X_i} = \sum_{i=1}^{K} \mu_i \nu_i \delta N. \tag{4.30}$$

Since δN is an arbitrary constant, we find a condition for chemical equilibrium:

$$\sum_{i=1}^{K} \mu_i \nu_i = 0. \tag{4.31}$$

For example, from the pair creation-annihilation process of particle X and anti-particle $\bar{X}$:

$$X + \bar{X} \rightleftharpoons 2\gamma, \tag{4.32}$$

and $\mu_\gamma = 0$, we have $\mu_X = -\mu_{\bar{X}}$ in the chemical equilibrium.

4.3 Phase space in an expanding Universe

The maximum entropy states that we calculated in the previous section is the same as the equilibrium state phase space distribution function in the usual statistical mechanics. When applying this to the expanding Universe, however, one have to carefully define the *phase space* first. As a phase space is a six dimensional space of space coordinate **x** and the momentum coordinate **p**, constructing the phase space requires to fix the time coordinate. Also, in order to enjoy the compressless property in the phase sapce thanks to Liouville theorem, momentum must be chosen to be the conjugate momentum of the space coordinate. Of course, the coordinate can be naturally chosen in the FRW Universe that we are dealing with here, and we find the conjugate momentum from the classical action (that we choose time coordinate as a variable):

$$S = \frac{1}{2}\int d\lambda \left[g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right] = \frac{1}{2}\int dt \left(\frac{dt}{d\lambda}\right) g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} \tag{4.33}$$

with constraint

$$g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\left(\frac{dt}{d\lambda}\right)^2 = -m^2. \tag{4.34}$$

The conjugate momentum is

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = \left(\frac{dt}{d\lambda}\right) g_{i\nu}\frac{dx^\nu}{dt} = g_{i\nu}\frac{dx^\nu}{d\lambda}. \tag{4.35}$$

That is, the conjugate momentum is simply a three spatial component of the four momentum $P_\mu = dx_\mu/d\lambda$. Corresponding Hamiltonian is

$$H = p_i\dot{x}^i - L, \tag{4.36}$$

and the position x^i and conjugate momentum p_i satisfy Hamilton's equation of motion:

$$\dot{x}_i = \frac{\partial H}{\partial p^i},\ \dot{p}_i = -\frac{\partial H}{\partial x^i}. \tag{4.37}$$

Therefore, by using Liouville's theorem[2], we can define a single-particle phase space with x^i and p_i. Note that the phase space volume element $d^3xd^3p = dx^1dx^2dx^3dp_1dp_2dp_3$ is invariant under the coordinate transformation $(t,\mathbf{x}) \to (\tilde{t},\tilde{\mathbf{x}}) = (\tilde{t}(t,\mathbf{x}),\tilde{\mathbf{x}}(t,\mathbf{x}))$ [3].

We define the distribution function as the number of particles found in the volume element d^3xd^3p:

$$dN = f(\mathbf{x},\mathbf{p},t)d^3pd^3x. \tag{4.43}$$

Liouville's theorem says in the absence of the particle sink or source, the phase space density $f(\mathbf{x},\mathbf{p},t)$ stays the same along the path of particles in the phase space. In addition, the phase space volume is invariant under the coordinate choice, which makes $f(\mathbf{x},\mathbf{p},t)$ a scalar.

4.3.1 Cosmological dimming, revisited

Applying this to cosmic radiation reads cosmological dimming, or the *brightness theorem*. The specific surface brightness I_ν of a beam of radiation is flow of energy δu per time interval δt, normally through

[2] Recap on Liouville's theorem: without source or sink of particles, the phase space density $\rho(\mathbf{x},\mathbf{p},t)$ must satisfy the continuity equation in $6-d$, denoted by $\mathbf{q} = (\mathbf{x},\mathbf{p})$:

$$\frac{\partial\rho}{\partial t} + \nabla^i_q(\rho\dot{q}^i) = \frac{\partial\rho}{\partial t} + \frac{\partial(\rho\dot{x}^i)}{\partial x^i} + \frac{\partial(\rho\dot{p}^i)}{\partial p^i} = 0 \tag{4.38}$$

Then, convective time derivative along the trajectory is given by

$$\begin{aligned}\frac{d\rho}{dt} &= \frac{\partial\rho}{\partial t} + \dot{x}^i\frac{\partial\rho}{\partial x^i} + \dot{p}^i\frac{\partial\rho}{\partial p^i}\\ &= -\frac{\partial(\rho\dot{x}^i)}{\partial x^i} - \frac{\partial(\rho\dot{p}^i)}{\partial p^i} + \dot{x}^i\frac{\partial\rho}{\partial x^i} + \dot{p}^i\frac{\partial\rho}{\partial p^i}\\ &= -\rho\left[\frac{\partial\dot{x}^i}{\partial x^i} + \frac{\partial\dot{p}^i}{\partial p^i}\right] = -\rho\left[\frac{\partial}{\partial x^i}\left(\frac{\partial H}{\partial p^i}\right) + \frac{\partial}{\partial p^i}\left(-\frac{\partial H}{\partial q^i}\right)\right] = 0.\end{aligned} \tag{4.39}$$

Here, we use the Hamilton's equation in the last line. The Liouville's theorem says that the phase space density is conserved along the trajectories if there is no source or sink of particles.

[3] The phase space volume calculated at the constant-$\tilde{t}$-hypersurface is

$$d^3\tilde{x}d^3\tilde{p} = \frac{\partial(\tilde{x}^1,\tilde{x}^2,\tilde{x}^3,\tilde{p}_1,\tilde{p}_2,\tilde{p}_3)}{\partial(x^1,x^2,x^3,p_1,p_2,p_3)}d^3xd^3p, \tag{4.40}$$

where $\partial(\cdots)/\partial(\cdots)$ is the Jacobian of the coordinate transform

$$\tilde{x}^i = \tilde{x}^i(\tilde{t},x^j),\ \tilde{p}_i = \left(\frac{\partial x^\mu}{\partial\tilde{x}^i}\right)_{\tilde{x}}p_\mu. \tag{4.41}$$

Caution: here, I am emphasizing agagin that we calculate the phase space volume calculated at constant-$\tilde{t}$-hypersurface! The Jacobian is,

$$J = \det\left(\left.\frac{\partial\tilde{x}^i}{\partial x^j}\right|_{\tilde{t}=\text{const.}}\right)\det\left(\left.\frac{\partial\tilde{p}^j}{\partial p^k}\right|_{\tilde{t}=\text{const.}}\right) = \det\left(\left.\frac{\partial\tilde{x}^i}{\partial x^j}\right|_{\tilde{t}=\text{const.}}\right)\det\left(\left.\frac{\partial x^j}{\partial\tilde{x}^k}\right|_{\tilde{t}=\text{const.}}\right) = 1. \tag{4.42}$$

the area δA, into the solid angle $\delta\Omega$, and unit bandwidth $\delta\nu$:

$$I_\nu = \frac{\delta u}{\delta t \delta A \delta\Omega \delta\nu}. \quad (4.44)$$

From Eq. (4.43), the photon number flux density per unit interval of p and solid angle is $p^2 f$. The energy of photon is, $E = p$, which makes $I_\nu = p^3 f$. Let p_e the photon's momentum observed at the galaxy that emits the photon. When it get observed, the photon momentum is redshifted to $p_o = p_e/(1+z)$, and because f is unchanged, the observed specific surface brightness is

$$I_{\nu_o} = \frac{I_{\nu_e}}{(1+z)^3}, \quad (4.45)$$

then the total surface brightness is

$$I_o = \int I_{\nu_o} d\nu_o = \int \frac{I_{\nu_e}}{(1+z)^3} \frac{d\nu_e}{(1+z)} = \frac{I_e}{(1+z)^4}. \quad (4.46)$$

This is the cosmological dimming factor that we have calculated in the previous section! Note that it only depends on the redshift.

4.4 Phase space density (distribution function) in the FRW universe

In the FRW universe, the phase space distribution function must be independent on the spatial coordinate $\mathbf{x}$ (spatial homogeneity), and on the direction of the momentum $\mathbf{p}$ (isotropy): $f(\mathbf{x}, \mathbf{p}, t) = f(p, t)$. For the local equilibrium state, the distribution function should have the form that we find in the previous section, and

$$f(p) = \frac{1}{e^{(\varepsilon-\mu)/T} \pm 1}, \quad (4.47)$$

where (+) sign correspond to the fermions, and (−) sign corresponds to the bosons.

The number of particles in the phase space volume element $d^3x d^3p$ is now written as

$$dN(\mathbf{x}, \mathbf{p}, t) = g f(\mathbf{x}, \mathbf{p}, t) \frac{d^3x d^3p}{(2\pi)^3} = g f(p, t) \frac{d^3x d^3p}{(2\pi)^3}. \quad (4.48)$$

Here, g is the internal degeneracy factor, and we divide the phase space volume by the uncertainty cell of one dimensional size of $\Delta p \Delta x \equiv (2\pi)$. The degeneracy Δg_ε that we introduced in the previous section is then

$$\Delta g_\varepsilon = \frac{g}{(2\pi)^3} \int_V d^3x \int_{E=\varepsilon} d^3p = \frac{gV}{(2\pi)^3} \int_{E=\varepsilon} d^3p \quad (4.49)$$

4.4.1 Energy density, number density, pressure

From a generic distribution function $f(\mathbf{x}, \mathbf{p}, t)$, we calculate the number density, energy density and pressure as following.

- **number density**
  The number density of particle is simply from Eq. (4.48)

$$n = \frac{\delta N}{\delta V} = g \int \frac{d^3p}{(2\pi)^3} f(\mathbf{x}, \mathbf{p}, t). \quad (4.50)$$

- **energy density**
  From the relativistic expression of the energy $\varepsilon = \sqrt{p^2+m^2}$, for a particle of mass m and momentum, $p^2 = a^2\tilde{g}_{ij}p^i p^j$, we find the energy density

$$\rho = \sum \varepsilon \frac{\delta N}{\delta V} = g \int \frac{d^3p}{(2\pi)^3} f(\mathbf{x},\mathbf{p},t)\sqrt{m^2+p^2}. \tag{4.51}$$

- **pressure**
  To get the expression for the pressure, let us consider a small area element $\Delta\sigma$ normal to a direction $\hat{\mathbf{n}}$. Particles within the spherical shell of radius $R = |\mathbf{v}|t$ and $|\mathbf{v}|(t+\Delta t)$ now will hit the element between time t and $t+\Delta t$, if the velocity vector falls within the solid angle $\Delta\sigma/R^2(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})$. Total number of such particles within the solid angle $\Delta\Omega$ is then given by:

$$\Delta N_\sigma = \left[\frac{\Delta\sigma/R^2(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})}{4\pi}\right] \times \bar{n}(v)R^2|\mathbf{v}|\Delta t\Delta\Omega = \bar{n}(v)(\hat{\mathbf{n}}\cdot\mathbf{v})\Delta\sigma\Delta t\frac{\Delta\Omega}{4\pi}. \tag{4.52}$$

  Here, the curly braket is the fraction of particles targetting the area element $\Delta\sigma\hat{\mathbf{n}}$ assuming the statistical isotropy. If the particles are reflected elastically, each transfers the momentum $2(\mathbf{p}\cdot\hat{\mathbf{n}})$ to the target. Therefore, the contribution of the particle with velocity v to the pressure is

$$\Delta p = \sum_{\Delta\Omega}\frac{2(\mathbf{p}\cdot\hat{\mathbf{n}})\Delta N_\sigma}{\Delta\sigma\Delta t} = \sum_{\Delta\Omega} 2(\mathbf{p}\cdot\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{v})\bar{n}(v)\frac{\Delta\Omega}{4\pi} = 2\frac{\bar{n}(v)}{\varepsilon}\int(\mathbf{p}\cdot\hat{\mathbf{n}})^2\frac{d\Omega}{4\pi}, \tag{4.53}$$

  where we use $\mathbf{p} = \varepsilon\mathbf{v}$. Without loss of generality, we can set $\hat{\mathbf{n}} \parallel \hat{\mathbf{z}}$ to do the integral:

$$\Delta p = 2\frac{\bar{n}(v)|\mathbf{p}|^2}{\varepsilon}\int_0^1 \mu^2\frac{d\mu}{2}\int_0^{2\pi}\frac{d\varphi}{2\pi} = \frac{1}{3}\frac{\bar{n}(v)|\mathbf{p}|^2}{\varepsilon}. \tag{4.54}$$

  Here, we integrated over the hemisphere ($\mu = 0...1$), where particles are facing tothe surface toward $-\hat{\mathbf{n}}$ direction. Using the phase space density function, we find the pressure as

$$P = g\int\frac{e^3p}{(2\pi)^3}f(\mathbf{x},\mathbf{p},t)\frac{|\mathbf{p}|^2}{3\sqrt{m^2+p^2}}. \tag{4.55}$$

Using the distribution function that we found in the previous section, the number density n_i, energy density ρ_i and pressure P_i for a given particle i are given by

$$n_i = g_i\int\frac{d^3p}{(2\pi)^3}\frac{1}{e^{(\varepsilon-\mu_i)/T_i}\pm 1}, \tag{4.56}$$

$$\rho_i = g_i\int\frac{d^3p}{(2\pi)^3}\frac{\varepsilon}{e^{(\varepsilon-\mu_i)/T_i}\pm 1}, \tag{4.57}$$

$$P_i = g_i\int\frac{d^3p}{(2\pi)^3}\frac{p^2}{3\varepsilon}\frac{1}{e^{(\varepsilon-\mu)/T}\pm 1}, \tag{4.58}$$

where $-$ sign is for Bosons and $+$ sign is for Fermions, and $\varepsilon = \sqrt{m_i^2+p^2}$ is the energy of a particle.

4.4.2 Entropy density

Let us come back to the entropy and consider it as a function of number density, energy denstity and pressure. In terms of the phase space distribution function f_ε, the entropy for the boson is (Eq. (4.9))

$$S = \sum_\varepsilon \Delta g_\varepsilon \left[(1+f_\varepsilon)\ln(1+f_\varepsilon) - f_\varepsilon \ln f_\varepsilon\right] = \frac{gV}{(2\pi)^3}\int d^3p \left[(1+f_\varepsilon)\ln(1+f_\varepsilon) - f_\varepsilon \ln f_\varepsilon\right] \tag{4.59}$$

and for the fermion is (Eq. (4.21))

$$S = -\sum_\varepsilon \Delta g_\varepsilon \left[(1-f_\varepsilon)\ln(1-f_\varepsilon) + f_\varepsilon \ln f_\varepsilon\right] = -\frac{gV}{(2\pi)^3}\int d^3p \left[(1-f_\varepsilon)\ln(1-f_\varepsilon) + f_\varepsilon \ln f_\varepsilon\right] \tag{4.60}$$

Note that I simply replaced n_ε to f_ε.

Let us substitute the equilibrium distibution function

$$f_\varepsilon = \frac{1}{e^{(\varepsilon-\mu)/T} \mp 1}, \tag{4.61}$$

to the entropy equations above. First, we note the identity

$$\ln(1 \pm f_\varepsilon) = \frac{\varepsilon - \mu}{T} + \ln f_\varepsilon, \tag{4.62}$$

from which we calculate the derivative

$$\frac{d\ln(1\pm f_\varepsilon)}{d\varepsilon} = \frac{1}{T} + \frac{d\ln f_\varepsilon}{d\varepsilon} = \frac{1}{T}\left[1 - f_\varepsilon e^{(\varepsilon-\mu)/T}\right] = \mp\frac{f_\varepsilon}{T}. \tag{4.63}$$

Then the expression for the entropy becomes

$$\begin{aligned} S &= \pm\frac{gV}{2\pi^2}\int_0^\infty p^2dp\left[(1\pm f_\varepsilon)\ln(1\pm f_\varepsilon) \mp f_\varepsilon \ln f_\varepsilon\right] \\ &= \pm\frac{gV}{2\pi^2}\int_0^\infty p^2dp\left[\ln(1\pm f_\varepsilon) \pm f_\varepsilon\left(\ln(1\pm f_\varepsilon) - \ln f_\varepsilon\right)\right] \\ &= \pm\frac{gV}{2\pi^2}\int_0^\infty p^2dp\left[\ln(1\pm f_\varepsilon) \pm f_\varepsilon\left(\frac{\varepsilon-\mu}{T}\right)\right] \end{aligned} \tag{4.64}$$

and using Eq. (4.63), the first term in the integral can be rewritten by means of integration by part as

$$\begin{aligned} \int_0^\infty p^2dp\ln(1\pm f_\varepsilon) &= \frac{1}{3}p^3\ln(1\pm f_\varepsilon)\Big|_{p=0}^{p=\infty} - \frac{1}{3}\int_0^\infty dp p^3\frac{d\ln(1\pm f_\varepsilon)}{dp} \\ &= \pm\frac{1}{3T}\int_0^\infty dp p^3\frac{d\varepsilon}{dp}f_\varepsilon = \pm\frac{1}{3T}\int_0^\infty dp p^3\frac{p}{\varepsilon}f_\varepsilon. \end{aligned} \tag{4.65}$$

Combining all, we have the expression for the entropy

$$S = \frac{gV}{2\pi^2}\frac{1}{T}\int_0^\infty p^2dpf_\varepsilon\left[\frac{p^2}{3\varepsilon} + \varepsilon - \mu\right] = V\left(\frac{\rho + P - \mu n}{T}\right). \tag{4.66}$$

Note that the equation above is consistnet with the standard definition of the temperature, presure and chemical potential. That is, the temperature, pressure, chemical potential are the quantities equalized between two systems when, respectively, energy ($E = \rho V$), volume, and the particle number ($N = nV$) reach the equilibrium (maximum entropy) state:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N}, \ \frac{P}{T} = \left(\frac{\partial S}{\partial V}\right)_{E,N}, \ \frac{\mu}{T} = -\left(\frac{\partial S}{\partial N}\right)_{E,V}. \tag{4.67}$$

As we have discussed earlier, in an expanding Universe, the equilibrium state exist only locally, and we define the entropy density as

$$s = \frac{S}{V} = \frac{\rho + P - \mu n}{T}. \tag{4.68}$$

One important aspect of the entropy density in the expanding Universe is the time evolution of the comoving entropy density (sa^3):

$$\frac{d(sa^3)}{dt} = -\frac{\mu}{T}\frac{d(na^3)}{dt}, \tag{4.69}$$

which is the outcome of the energy-momentum conservation $\dot{\rho}+3H(\rho+P) = 0$, and following identities:

$$\left.\frac{\partial P}{\partial T}\right|_{\mu} = s, \ \left.\frac{\partial P}{\partial \mu}\right|_{T} = n. \tag{4.70}$$

You will show this in the homework.

Eq. (4.69) implies that **the comoving entropy** sa^3 **of a particle species is separately conserved** when comoving particle number density is conserved, $d(na^3)/dt = 0$, or chemical potential is smaller than the temperature ($\mu \ll T$, non-degenerate). This is the case for all the particle species that we are interested in—although we do not know precise lepton number (thus, chemical potential of the nentrinos), there is a good reason to believe that it is of the same order as the baryon number in the standard model of particle physics (namely, $B - L$ is conserved, and takes a value of order unity).

When it comes to the entropy of the whole Universe, **the comoving entropy of the homogeneous and isotropic universe must be conserved as there is no external souces of entropy**. Again, you will explicitly show this in the homework by combining the entropies of all particle species. The total entropy density of universe may be written as

$$s_{\rm tot} = \frac{\rho_{\rm tot} + P_{\rm tot}}{T}, \tag{4.71}$$

and $d(a^3 s_{\rm tot})/dt = 0$. Therefore, the FRW universe is, in some sense, such a boring system. Every part of the universe is evolving exactly the same way (homogeneity!), without building up complexity. This may be similar to the stage that Boltzmann referred as 'thermal death of the Universe', where every part of the Universe has reached the maximum entropy state.

The story completely changes with tiny small initial fluatuations, which might be originated from quantum fluctuations during inflation. However small it is, the gravitational instability will enhance the structures and evolves into the complex large-scale structures, galaxies, stars, planetary system, and eventually the life. We will talk more about it later in the class.

By the way, there is an alternative—perhaps easier—way of reaching at the same conclusion. But, I don't follow the path because I don't find it easy to accept the idea of pdV work done in the expanding Universe. For your information, I summarize you how to do that here.

Starting from the second law of thermodynamics, $TdS = dU + PdV$ and replacing $U = \rho V$, we find

$TdS = (\rho + P)dV + Vd\rho = d[(\rho + P)V] - Vdp$. Then, use the identity (only true when $\mu = 0$. see, Eq. (4.70)) $dP/dT = (\rho + P)/T$, we have

$$dS = \frac{d[(\rho + P)V]}{T} - (\rho + P)V\frac{dT}{T^2} = d\left[\frac{(\rho + P)V}{T} + \text{const}\right]. \tag{4.72}$$

From the first law of thermodynamics (energy conservation), $d[(\rho + P)V] = Vdp$, then dS = 0. That conserves the entropy per comoving volume as $S \propto sa^3$. This is, for example, how Kolb & Turner's textbook derives the conservation of comoving entorpy.

4.5 Thermodynamical quantities

In the previous sections, we have calculated equilibrium phase space density in the FRW universe. Once the phase space density is given, we can calculate the thermodynamical quantities from Eqs. (4.56)–(4.58). But, these integrals cannot be done analytically except for some extreme cases and must be done numerically. In this section, we study the evolution of thermodynamical quantities as temperature of the Universe drops.

4.5.1 Ultra-relativistic particles

When particles are ultra-relativistic ($T \gg m$), we ignore the mass of the particles, and Eqs. (4.56)–(4.57) reduce to

$$n_i = \frac{g_i}{2\pi^2}\int_0^\infty dp\frac{p^2}{e^{(p-\mu_i)/T_i} \pm 1} \tag{4.73}$$

$$\rho_i = \frac{g_i}{2\pi^2}\int_0^\infty dp\frac{p^3}{e^{(p-\mu_i)/T_i} \pm 1}, \tag{4.74}$$

and $P_i = \rho_i/3$, which is the equation of state for the relativisitic particles.

Let us first consider chemical potential of particles in the Universe. As we have discussed before, the chemical potential of the antiparticles ($\bar{X}$) is given by $\mu_{\bar{X}} = -\mu_X$. Using this, we calculate the particle number excess over antiparticles as

$$n_i - \bar{n}_i = \frac{g_i}{2\pi^2}\int_0^\infty dpp^2\left[\frac{1}{e^{(p-\mu_i)/T_i} \pm 1} - \frac{1}{e^{(p+\mu_i)/T_i} \pm 1}\right]. \tag{4.75}$$

For fermions, the integral reads exact answer:

$$n_i^{(F)} - \bar{n}_i^{(F)} = \frac{g_i T_i^3}{6}\left[\left(\frac{\mu_i}{T_i}\right) + \frac{1}{\pi}\left(\frac{\mu_i}{T_i}\right)^3\right]. \tag{4.76}$$

For bosons, we need to make an approximation, and we find

$$n_i^{(B)} - \bar{n}_i^{(B)} \simeq \frac{g_i T_i^3}{3}\left(\frac{\mu_i}{T_i}\right), \tag{4.77}$$

for the non-degenerate Bose gas ($\mu \ll T$). Therefore, the small baryon-to-photon ratio indicates that the chemical potential for baryons (mostly carried by protons and neutrons) is also small, of the same order as the baryon-to-photon ratio: $\mu_i/T \simeq (n-\bar{n})/s \simeq \eta_{10}$. Then, electric neutrality of the Universe dictates the chemical potential of the electron must be the same as that of proton (they share the same temperature in equilibrium state). You will estimate the chemical potential in the early Universe in the homework. Find out how small it is!

When chemical potential is much smaller compared to temperature, $\mu \ll T$, the number density and energy density become

$$n_i = \frac{g_i}{2\pi^2}\int_0^\infty dp \frac{p^2}{e^{p/T_i} \pm 1} = \frac{g_i}{2\pi^2} T^3 \int_0^\infty dx \frac{x^2}{e^x \pm 1}, \tag{4.78}$$

$$\rho_i = \frac{g_i}{2\pi^2}\int_0^\infty dp \frac{p^3}{e^{p/T_i} \pm 1} = \frac{g_i}{2\pi^2} T^4 \int_0^\infty dx \frac{x^3}{e^x \pm 1}, \tag{4.79}$$

where the integrals are, for bosons

$$\int_0^\infty dx \frac{x^2}{e^x - 1} = 2\zeta(3) \tag{4.80}$$

$$\int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}, \tag{4.81}$$

with $\zeta(x)$ being Riemann zeta function $\zeta(3) \simeq 1.20206$. The integrals for fermions can be written in terms of the integrals above because

$$\int_0^\infty dx \frac{x^n}{e^x - 1} - \int_0^\infty dx \frac{x^n}{e^x + 1} = \int_0^\infty dx \frac{2x^n}{e^{2x} - 1} = \frac{1}{2^n}\int_0^\infty dt \frac{t^n}{e^t - 1} \tag{4.82}$$

reads

$$\int_0^\infty dx \frac{x^n}{e^x + 1} = \left(1 - \frac{1}{2^n}\right)\int_0^\infty dx \frac{x^n}{e^x - 1}. \tag{4.83}$$

At temperature T (when $\mu \ll T$), the number density, energy density, and entropy density for bosons are

$$n_i^{(B)} = \frac{\zeta(3)}{\pi^2} g_i T_i^3,\ \rho_i^{(B)} = \frac{\pi^2}{30} g_i T_i^4,\ s_i^{(B)} = \frac{\rho_i^{(B)} + P_i^{(B)}}{T_i} = \frac{2\pi^2}{45} g_i T_i^3, \tag{4.84}$$

and for fermions are

$$n_i^{(F)} = \frac{3\zeta(3)}{4\pi^2} g_i T_i^3,\ \rho_i^{(F)} = \frac{7}{8}\frac{\pi^2}{30} g_i T_i^4,\ s_i^{(F)} = \frac{7}{8}\frac{2\pi^2}{45} g_i T_i^3. \tag{4.85}$$

Note the number density and entropy density both have T^3 dependence, which keeps the entropy per baryon and fermion constant:

$$\frac{s_i^{(F)}}{n_i^{(F)}} = \frac{7}{6}\frac{s_i^{(B)}}{n_i^{(B)}} = \frac{7}{6}\frac{2\pi^4}{45\zeta(3)} \simeq 4.202,\ \frac{s_i^{(B)}}{n_i^{(B)}} \simeq 3.602 \tag{4.86}$$

average energy per particles drops proportional to the reciproccal of temperature:

$$\frac{\rho_i^{(F)}}{n_i^{(F)}} = \frac{7}{6}\frac{\rho_i^{(B)}}{n_i^{(B)}} = \frac{7}{6}\frac{\pi^4}{30\zeta(3)} T_i \simeq 3.151 T_i,\ \frac{\rho_i^{(B)}}{n_i^{(B)}} \simeq 2.701 T_i, \tag{4.87}$$

which is consistent with the cosmological redshift.

For low-z universe, $T \lesssim 10\,\text{keV}$, when the photon (boson with $g_i = 2$ for two polarization states) and massless neutrinos (fermion with $g_i = 2$ for two helicity states: ν_L, $\bar{\nu}_R$) are the only relativistic particles, both photon temperature and neutrino temperature scales as $T_i \propto a^{-1}$, and the expressions above are consistent with the result from particle number conservation $n \propto a^{-3}$ and energy-momentum conservation $\rho_{\rm R} \propto a^{-4}$. For high-z universe, $T \gtrsim 1\,\text{MeV}$, this simple temperature scaling does not work anymore because of the events such as QCD phase transition, EW phase transition, etc, which change relativistic particle content. We will come back to this point in the next section.

4.5.2 Non-relativistic particles

For non-relativisitic particles, $m \gg T$, the exponential factors in the denominator of the distribution dominates over ± 1 and fermions and bosons follows essentially the same distribution function:

$$f(\varepsilon) \simeq e^{-(\varepsilon-\mu_i)/T_i} \simeq e^{-(m-\mu_i)/T_i} e^{-p^2/(2mT_i)}, \tag{4.88}$$

where we use $\varepsilon \simeq m + p^2/(2m) + \cdots$. We calculate equilibrium number density with the distibution function as[4]

$$\begin{aligned} n_i =& \frac{g_i}{2\pi^2} e^{-(m-\mu_i)/T_i} \int_0^\infty dp p^2 e^{-p^2/(2mT_i)} = \frac{g_i}{2\pi^2} e^{-(m-\mu_i)/T_i} \frac{1}{4}\sqrt{\pi(2mT_i)^3} \\ =& g_i \left(\frac{mT_i}{2\pi}\right)^{3/2} e^{-(m-\mu_i)/T_i}, \end{aligned} \tag{4.93}$$

which makes the particle abundance over antiparticle as

$$n_i - \bar{n}_i = 2g_i \left(\frac{mT_i}{2\pi}\right)^{3/2} e^{-m/T_i} \sinh\left(\frac{\mu_i}{T_i}\right). \tag{4.94}$$

That is, the antiparticle number density is suppressed compared to the particle number density by a factor of

$$\bar{n}_i = n_i e^{-2\mu_i/T_i}. \tag{4.95}$$

[4]Gaussian integral can be most conveniently done from

$$\left(\int_{-\infty}^\infty dx e^{-x^2}\right)^2 = \int_{-\infty}^\infty dx e^{-x^2} \int_{-\infty}^\infty dy e^{-y^2} = \int d^2 r e^{-r^2} = 2\pi \int_0^\infty dr r e^{-r^2} = \pi \int_0^\infty d(r^2) e^{-r^2} = \pi. \tag{4.89}$$

Equivalently,

$$\int_{-\infty}^\infty dx e^{-ax^2+bx} = e^{\frac{b^2}{4a}} \int_{-\infty}^\infty dx e^{-a\left(x-\frac{b}{2a}\right)^2} = e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}}. \tag{4.90}$$

From here, you have two choices to calculate the integral of form

$$\begin{aligned} \int_{-\infty}^\infty dx x^2 e^{-ax^2} =& -\frac{d}{da}\left(\int_{-\infty}^\infty dx e^{-ax^2+bx}\bigg|_{b=0}\right) = \frac{1}{2}\sqrt{\frac{\pi}{a^3}} \\ =& \frac{\partial^2}{\partial b^2}\int_{-\infty}^\infty dx e^{-ax^2+bx}\bigg|_{b=0} = \sqrt{\frac{\pi}{a}}\left(\frac{d}{db}\left[\frac{b}{2a}e^{\frac{b^2}{4a}}\right]\right)_{b=0} = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}. \end{aligned} \tag{4.91}$$

Of course, they agree. Choose one that you like. By the way, if you master this way of doing the Gaussian integral, then you are ready to solve quantum field theory! I use the former to get

$$\int_{-\infty}^\infty dx x^4 e{-}ax^2 = \frac{3}{4}\sqrt{\frac{\pi}{a^5}}. \tag{4.92}$$

Energy density is

$$\rho_i = \frac{g_i}{2\pi^2} e^{-(m-\mu_i)/T_i} \int_0^\infty dp p^2 \sqrt{m^2+p^2} e^{-p^2/(2mT_i)}$$
$$\simeq mn + \frac{g_i}{2\pi^2} e^{-(m-\mu_i)/T_i} \frac{1}{2m} \int_0^\infty dp p^4 e^{-p^2/(2mT_i)} = n\left(m + \frac{3}{2}T_i\right), \tag{4.96}$$

and pressure is

$$P_i = \frac{g_i}{2\pi^2} e^{-(m-\mu_i)/T_i} \int_0^\infty dp p^2 \frac{p^2}{3\sqrt{m^2+p^2}} e^{-p^2/(2mT_i)}$$
$$\simeq \frac{g_i}{2\pi^2} e^{-(m-\mu_i)/T_i} \frac{1}{3m} \int_0^\infty dp p^4 e^{-p^2/(2mT_i)} = nT_i, \tag{4.97}$$

that gives the entropy density as

$$s_i = \frac{\rho_i + P_i + \mu_i n}{T_i} = \left(\frac{m-\mu_i}{T_i} + \frac{5}{2}\right) n_i. \tag{4.98}$$

Comparing equilibrium number density of non-relativistic particles, Eq. (4.93), with that of relativistic particles in Eqs. (4.84)–(4.85), the particle number density of non-relativistic particles ($m \gg T$) is suppressed by a factor of

$$\frac{n_i^{(NR)}}{n_i^{(B)}} \sim \left(\frac{m}{T}\right)^{3/2} e^{-m/T}, \tag{4.99}$$

for particles with $\mu/T \ll 1$, and *the entropy density is also suppressed by the same exponential factor*. In the homework, you will calculate contribution of proton entropy to the total entropy at $T \simeq$ MeV (when protons are non-relativistic).

4.5.3 The end of cosmic-antiparticle-background

The exponential factor in the equilibrium density of the matter particles comes about because it is harder and harder to create the particle-antiparticle pairs from the thermal bath when average energy of photon is less than the rest-mass energy of the particle: $\langle E \rangle \propto T \lesssim m$. But, still, because we have so many photons for one baryon, the photons at the exponential tail of the Planck curve can produce some pairs, but not as much as when the particles were relativistic because number of high energy photons is exponentially suppressed in the Wein tail.

Eventually, when temperature of the Universe drops sufficiently below, $T \ll m$ particle-antiparticle pair production essentially stops so that we can completely ignore the antiparticle. Let us estimate when this happens. We do it as a function of

$$\beta = \frac{n - \bar{n}}{s} \simeq 10^{-9} \tag{4.100}$$

and from Eq. (4.93),

$$\frac{n\bar{n}}{s^2} \simeq \left(\frac{m}{T}\right)^3 e^{-2m/T}. \tag{4.101}$$

Then, n/s and $-\bar{n}/s$ are two solutions of the quadratic equation

$$x^2+\beta x-\left(\frac{m}{T}\right)^3 e^{-2m/T}=\left(x+\frac{\beta}{2}\right)^2-\frac{\beta^2}{4}-\left(\frac{m}{T}\right)^3 e^{-2m/T}=0. \tag{4.102}$$

Solving the equation, we find

$$\frac{n}{s}=\sqrt{\left(\frac{\beta}{2}\right)^2+\left(\frac{m}{T}\right)^3 e^{-2m/T}}+\frac{\beta}{2},\ \frac{\bar{n}}{s}=\sqrt{\left(\frac{\beta}{2}\right)^2+\left(\frac{m}{T}\right)^3 e^{-2m/T}}-\frac{\beta}{2}. \tag{4.103}$$

That means, the antiparticle number density is neglegibly small if $(\beta/2)^2 \gg (m/T)^3 \exp[-2m/T]$. Taking the log (and now $>$ may be enough to ensure $\gg$),

$$\ln\left(\frac{\beta}{2}\right)>\frac{3}{2}\log\left(\frac{m}{T}\right)-\frac{m}{T}. \tag{4.104}$$

We iteratively invert it as $\log(m/T)$ changes much slower compare to m/T:

$$\frac{m}{T}>\ln\left(\frac{2}{\beta}\right)+\frac{3}{2}\ln\left(\ln\left(\frac{2}{\beta}\right)+\cdots\right). \tag{4.105}$$

For the observed value of $\beta \simeq 10^{-9}$, we find

$$T \lesssim \frac{m}{\ln(2/\beta)} \simeq \frac{m}{21}. \tag{4.106}$$

That is, when the temperature of the Universe drops below 5 % of the mass of the particle, then particle-antiparticle pair production ceases and we can practically ignore the antiparticles. For protons and neutrons, this happens at $T \lesssim 50\,\mathrm{MeV}$, and for for electrons, this happens at $T \lesssim 20\,\mathrm{keV}$. Again, I emphasize one more time that it is because we have about a billion times more photons than baryons and electrons.

4.6 Thermal history of the early Universe

In the previous section, we study the energy density, pressure, number density and entropy for two extreme cases: ultra-relativistic and non-relativistic particles. Let us now turn to the thermal history of the entire Universe. The goal of this section is to calculate the time evolution of the temperature and scale factor in the early Universe.

The various events that we have discussed at the beginning of this chapter affect the thermal evolution of the Universe. As the Universe expand and temeprature drops, two main things happens: [1] when temeprature of the Universe drops below the mass energy of some particle, the particle undergoes relativistic-to-nonrelativistic transition, then particle-antiparticle pair production stops at $T \lesssim m/20$. [2] when temperature of the Universe drops below the symmetry breaking scale, some drastic change happens to the nature of particles involved in the symmetry: e.g. QCD phase transition, EW phase transition.

4.6.1 Effective relativistic degrees of freedom

As we have seen in the previous sections, once a particle speices transits from relativistic to nonrelativisitc, then the number density, energy density and entropy of the particles are significantly reduced. That is, the thermal evolution of the early Universe is controlled mainly by the relativistic particles.

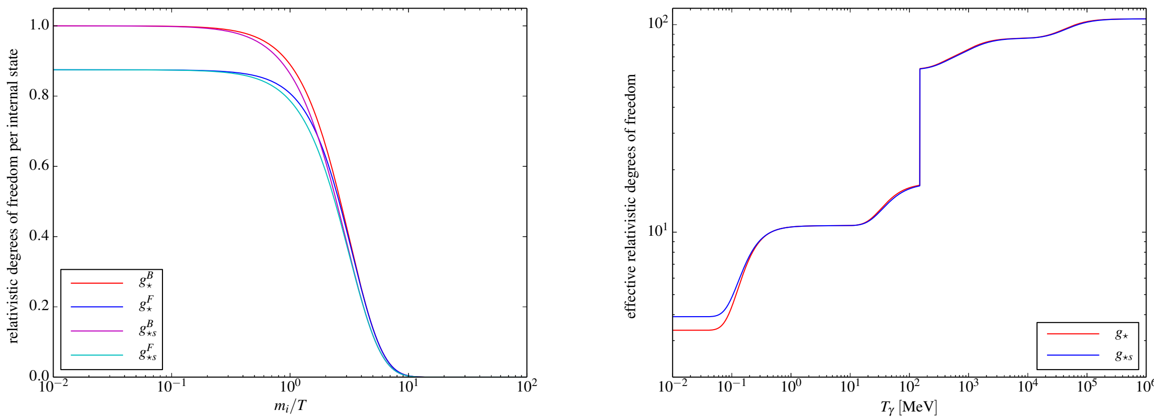

Figure 4.1: (*Left*) Efective relativistic degrees of freedom as a funciton of $x_i \equiv m_i/T$. As temperature drops, the particle species becomes non-relativistic and the effective degrees of freedom is reduced for $T < m_i$. (*Right*) Efective relativistic degrees of freedom as a funciton of the photon temperature T_γ in the early Universe. We include all the standard model particles and assuming that QCD phase transition happens sharply at $T_\gamma = 150$ MeV.

But, the relativistic-nonrelativistic transition does not happen instantaneously, as we have seen in the previous section, and one must take into account the smooth transition properly to calculate correct thermal history.

This motivates us to defin the effective relativistic degrees of freedom as the coefficient between temperature and the total energy density ($g_\star$) and the entropy ednsity ($g_{\star s}$):

$$\rho(T) = \frac{\pi^2}{30} g_\star(T) T^4 \tag{4.107}$$

$$s(T) = \frac{2\pi^2}{45} g_{\star s}(T) T^3. \tag{4.108}$$

$$n(T) = \frac{\zeta(3)}{\pi^2} g_{\star n}(T) T^3. \tag{4.109}$$

Compare these two equations to Eq. (4.84), we find that relativistic bosons contribute

$$\Delta g_\star = \Delta g_{\star s} = \Delta g_{\star n} = g_i, \tag{4.110}$$

and relativistic fermions contribute

$$\Delta g_\star = \Delta g_{\star s} = \frac{7}{8} g_i, \ \Delta g_{\star n} = \frac{3}{4} g_i. \tag{4.111}$$

to $g_\star$ and $g_{\star s}$, repectively, when the particles are in the equilibrium states with photon (T here is the temperature of photons). If some particles are thermally decoupled from photon bath and evolves independently (with temperature T', e.g. neutrinos at $T \lesssim 1.5\,$MeV), the contribution to the effective relativistic species is revised as

$$\Delta g_\star = g_i \left(\frac{T'}{T}\right)^4, \ \Delta g_{\star s} = g_i \left(\frac{T'}{T}\right)^3, \ \Delta g_{\star n} = g_i \left(\frac{T'}{T}\right)^3 \tag{4.112}$$

for Bosons, similarly, for Fermions with appropriate factors ($3/4$ for $g_{\star m}$, $7/8$ for $g_\star$ and $g_{\star n}$).

The effective relativistic d.o.f for massive particles changes drastically when temperature is around the mass of the paticle $T_i \sim m_i$. When ignoring the chemical potential (this is a good approximation as we see in the previous section), we find that

$$g_{\star n,i} = \frac{\pi^2}{\zeta(3)T^3} g_i \int_0^\infty \frac{p^2 dp}{2\pi^2} \frac{1}{e^{\sqrt{p^2+m_i^2}/T_i} \pm 1} = \frac{g_i}{2\zeta(3)} \left(\frac{T_i}{T}\right)^3 \int_{x_i}^\infty u du \frac{\sqrt{u^2 - x_i^2}}{e^u \pm 1} \tag{4.113}$$

$$g_{\star,i} = \frac{30}{\pi^2 T^4} g_i \int_0^\infty \frac{p^2 dp}{2\pi^2} \frac{\sqrt{m_i^2 + p^2}}{e^{\sqrt{p^2+m_i^2}/T_i} \pm 1} = \frac{15}{\pi^4} g_i \left(\frac{T_i}{T}\right)^4 \int_{x_i}^\infty u^2 du \frac{\sqrt{u^2 - x_i^2}}{e^u \pm 1} \tag{4.114}$$

$$g_{\star s,i} = \frac{45}{2\pi^2 T^3} \frac{g_i}{T_i} \int_0^\infty \frac{p^2 dp}{2\pi^2} \frac{m_i^2 + 4p^2/3}{\sqrt{m_i^2 + p^2}\left(e^{\sqrt{p^2+m_i^2}/T_i} \pm 1\right)} = \frac{15}{\pi^4} g_i \left(\frac{T_i}{T}\right)^3 \int_{x_i}^\infty u^2 du \frac{\sqrt{u^2 - x_i^2}}{(e^u \pm 1)} \left(1 - \frac{x_i^2}{4u^2}\right). \tag{4.115}$$

For the second equalities, we simplify the integral by setting $u^2 \equiv (m_i^2 + p^2)/T_i^2$, so that $udu = pdp/T_i^2$ and define $x_i \equiv m_i/T_i$: At $x_i \to 0$ limit, we reproduce the formula for the relativistic particles.

Occasionally, we need to evaluate the derivatives of $g_\star$'s with respect to the temperature T_i:

$$\frac{dg_{\star n,i}}{d\ln T_i} = -\frac{dg_{\star n,i}}{d\ln x_i} = \frac{g_i}{2\zeta(3)} \left(\frac{T_i}{T}\right)^3 \int_{x_i}^\infty u du \frac{1}{e^u \pm 1} \frac{x_i^2}{\sqrt{u^2 - x_i^2}}, \tag{4.116}$$

and similarly,

$$\frac{dg_{\star,i}}{d\ln T_i} = \frac{15}{\pi^4} g_i \left(\frac{T_i}{T}\right)^4 \int_{x_i}^\infty u^2 du \frac{1}{e^u \pm 1} \frac{x_i^2}{\sqrt{u^2 - x_i^2}}, \tag{4.117}$$

$$\frac{dg_{\star s,i}}{d\ln T_i} = -\frac{15}{\pi^4} g_i \left(\frac{T_i}{T}\right)^3 \int_{x_i}^\infty u^2 du \frac{1}{e^u \pm 1} \frac{3x_i^2(x_i^2 - 2u^2)}{4u^2\sqrt{u^2 - x_i^2}}. \tag{4.118}$$

4.6.2 Thermal history with standard model of particle physics

Now let us consider the effective relativistic degrees of freedom in our Universe with the standard model particles. This may be the correct description of the thermal history at temperature $T \lesssim$ TeV —except for some exceptional cases where beyond-standard-model effects show up at lower energy scales (e.g. with massive neutrinos). Let use first review the particle contents in the standard model.

The standard model of particle physics contains three families of fermion particles, which are doublet under the electro-weak interaction. In the lepton sector, they are (e, ν_e), (μ, ν_μ), (τ, ν_τ), and in the baryon sector, they are (u, d), (c, s), (t, b) quarks. On top of the electric charge, each quark has one of three color charges (r, g, b). These are all spin-1/2 fermions with $g_i = 2$ and all have correpsonding anti-particleis except for neutrinos. In the standard model, neutrinos care about the handedness (helicity, or chirality), so that all neutrinos in standard model are left-handed: so, we write them as $\nu_{\rm L}$. All anti-neutrinos, at the same time, are right-handed $(\bar{\nu}_{\rm R})$[5]. When all the standard model fermions are

[5]This picture may be changed with massive neutrino, which indicates the physics beyond the standard model—neutrinos are massless in the standard model. There, right handed neutrinos (with a large mass so that we cannot observe them in the lab experiments up until now) are possible.

relativistic, therefore, the relativistic degrees of freedome from them is

$$\begin{aligned} g_F =& 3(r,g,b)\times 2(\uparrow,\downarrow)\times 6(u,d,c,s,t,b)\times 2(\bar{q})+2(\uparrow,\downarrow)\times 3(e,\mu,\tau)\times 2(\bar{e},\bar{\mu},\bar{\tau})+3(\nu's)\times 2(\bar{\nu}'s) \\ =& 90. \end{aligned} \tag{4.119}$$

Bosons in the standard model either mediate three fundamental interactions—electromagnetic interaction (γ), weak interaction ($W^{\pm}$, Z^0), and strong interaction (g, 8 gluons)— or controls the electroweak symmetry breaking (H, the Higgs boson). The electroweak symmetry breaking, where the Higgs field moves away from the origin to its vacuum-expectation-value (VEV) $v \simeq 246\,\mathrm{GeV}$, happens around the critical temperature[6] $T_{\mathrm{EW}} \simeq 196.7\,\mathrm{GeV}$. The $W^{\pm}$ bosons and Z^0 bozon get their mass after this critical temperature. Therefore, around $T \simeq 200\,\mathrm{GeV}$, all the standard model bosons are massless, and their contribution to the relativistic degrees of freedom is

$$g_B = 2(\gamma)+3\times 3(W^{\pm},Z^0)+2\times 8(g)+1(H) = 28. \tag{4.120}$$

Here, I use that the photon and gluons are massless spin-1 particles with $g=2$, and W and Z bosons are massive spin-1 particles with $g=3$. The Higgs boson is a scalar ($s=0$) particle and therefore $g=1$. At around $T \simeq 200\,\mathrm{GeV}$, the relativistic degree of freedom is

$$g_\star = g_{\star s} = 28+90\times\frac{7}{8} = 106.75,\ \ g_{\star n} = 95.5, \tag{4.121}$$

which makes

$$\begin{aligned} \rho(T) \simeq& 35.12T^4 \simeq 8.149\times 10^{24}\left(\frac{T}{100\,\mathrm{GeV}}\right)^3 g/\mathrm{cm}^4 \\ s(T) \simeq& 61.98T^3 \simeq 8.056\times 10^{48}\left(\frac{T}{100\,\mathrm{GeV}}\right)^3 \mathrm{cm}^{-3} \\ n(T) \simeq& 11.6313T^3 \simeq 1.514\times 10^{48}\left(\frac{T}{100\,\mathrm{GeV}}\right)^3 \mathrm{cm}^{-3}, \end{aligned} \tag{4.122}$$

when the temperature of the Universe is

$$T = 1.160\times 10^{15}\left(\frac{T}{100\,\mathrm{GeV}}\right)\mathrm{K}. \tag{4.123}$$

Yes, the whole Universe was in a hot-dense state! Note that if there is no other physics beyond the standard model, then the estimation given above should work all the way to the Planck scale—although for scales above electroweak phase transition, we should change g_W and g_Z to 2 as they are massless for $T > T_{\mathrm{EW}}$.

After the electroweak phase transition, standard model particles become non-relativistic and drop out of the thermal bath in the descending order of mass: $t = 173$ GeV, $H = 125$ GeV, $Z^0 = 91$ GeV, $W^{\pm} = 80$ GeV, $b = 4$ GeV, $\tau = 1.777$ GeV, $c = 1$ GeV. At around $T \sim 150$ MeV QCD phase transition happens so that free quarks and gluons form baryons such as protons (p), and neutrons (n) as well as mesons (π^0, $\pi^{\pm}$). Then, pions $\pi \simeq 139$ MeV and $\mu = 106$ MeV go non-relaivistic. We show the evolution of $g_\star$ and $g_{\star s}$ during this period in the right panel of Fig. 4.1.

The Tablea 4.1 summarizes the particle content of the standard model of particle physics.

[6]The critical temperature of the electroweak symmetry breaking is $T_c = m_H/g$, where m_H is the mass of the Higgs boson and g is the weak coupling constant, $G_F = \sqrt{2}/8(g/m_W)^2 = 1.1663787(6)\times 10^{-5}\,\mathrm{GeV}^{-2}$. Now we know this number pretty well, because we have measured the mass of the Higgs boson from LHC ($m_H = 125.7\pm 0.4\,\mathrm{GeV}$).

symbol	mass	spin	anti-particle	g
u	2.3 MeV	1/2	$\bar{u}$	$2(u,\bar{u}) \times 2(\uparrow,\downarrow) \times 3(\text{color})$
d	4.8 MeV	1/2	$\bar{d}$	”
c	1.275 GeV	1/2	$\bar{c}$	”
s	95 MeV	1/2	$\bar{s}$	”
t	173.21 GeV	1/2	$\bar{t}$	”
b	4.18 GeV	1/2	$\bar{b}$	”
e	0.511 MeV	1/2	e^+	$2(e,e^+) \times 2(\uparrow,\downarrow)$
μ	105.7 MeV	1/2	μ^+	”
τ	1.777 GeV	1/2	τ^+	”
ν_e	<2 eV	1/2	$\bar{\nu}_{e,R}$	$2(\nu_{e,L}, \bar{\nu}_{e,R})$
ν_τ	<2 eV	1/2	$\bar{\nu}_{\tau,R}$	”
ν_μ	<2 eV	1/2	$\bar{\nu}_{\mu,R}$	”
γ	0	1		2
g	0	1		8(gluons) × 2
W^+	80.4 GeV	1		3
W^-	80.4 GeV	1		3
Z	91.2 GeV	1		3
H	125.7 GeV	0		1

Table 4.1: Particle content of the standard model of particle physics. Data from particle data group (http://pdg.lbl.gov/index.html).

4.6.3 Neutrino decoupling and e/e^+-annihilation

At high temperatures, neutrinos are in thermal equilibrium with cosmic plasma by electron-neutrino scattering with weak interaction cross section given by

$$\sigma_{\rm wk} \simeq G_F^2 T^2 \tag{4.124}$$

where $G_F \simeq 1.166 \times 10^{-5}\,\text{GeV}^{-2}$ is the Fermi constant for weak interaction coupling. When electrons are relativistic, the number density of electrons are

$$n_e \propto T^3, \tag{4.125}$$

that give rises to the collosion rate as

$$1/t_{\rm coll} = n_e \sigma_{\rm wk} \simeq \mathcal{O}(1) G_F^2 T^5. \tag{4.126}$$

Now we compare this to the Hubble time scale

$$1/t_{\rm H} = H \simeq \mathcal{O}(1)\sqrt{\frac{8\pi G}{3}} T^2. \tag{4.127}$$

The electron-neutrino scattering freezes out when

$$t_{\rm coll} > t_{\rm H} \rightarrow G_F^2 T^5 < \mathcal{O}(1)\sqrt{\frac{8\pi G}{3}} T^2, \tag{4.128}$$

or

$$T_{\nu-\text{decoupling}} < \mathcal{O}(1)\left(\frac{8\pi G}{3G_F^4}\right)^{1/6} \simeq \mathcal{O}(1) 0.0012\,\text{GeV}. \tag{4.129}$$

More accurate calculation show that the numerical coefficient is in fact very close to unity and $T_{\nu_e} \simeq 1.34\,\mathrm{MeV}$, $T_{\nu_\mu,\nu_\tau} \simeq 1.5\,\mathrm{MeV}$ for different neutrino species [**?**]. Therefore, after $T \lesssim 1.5\,\mathrm{MeV}$, neutrinos decouple from the cosmic plasma of photon, electron and positron; then, the neutrinos evolve independently by conserving their comoving entropy separately from the other particles.

Neutrino decouples from the thermal bath right before the e^-/e^+ annihilation period happens at around $T = m_e \simeq 0.511\,\mathrm{MeV}$. Before e^-/e^+ annihilation time, although decoupled, temperature of neutrinos is the same as that of photons during its free-streaming as for both cases $T = T_\nu \propto 1/a$ (we will show that in the next section with kinetic theory). But, the pair annihilation of electrons transfer entropy *only to the photons*, and not to neutrinos, and, as a result, the temperature of photon drops a bit slower than $1/a$ leading to the heating of photon relative to neutrinos.

The final temperature difference between photon and neutrinos can be calculated by using the conservation of comoving entropy of the cosmic plasma consisting of photon, electron and antielectron. Before e^-/e^+ annihilaiton, all three particles are relativistic, and we have

$$g_{\star s}^{\gamma\,(\text{before})} = 2 + \frac{7}{8} \times 2 \times 2 = \frac{11}{2}. \tag{4.130}$$

Here, $g_{\star s}^{\gamma}$ refers to the relativistic degrees of freedom counting only for the species coupled to photon. After the annihilation is complete ($T \lesssim m_e/21 = 24.3\,\mathrm{keV}$), only photon remains to be relativistic and

$$g_{\star s}^{\gamma\,(\text{after})} = 2 \tag{4.131}$$

The conservation of comoving entropy before and after the e-annilation reads

$$\left[a^{(\text{before})}\right]^3 g_{\star s}^{\gamma\,(\text{before})} \left[T^{(\text{before})}\right]^3 = \left[a^{(\text{after})}\right]^3 g_{\star s}^{\gamma\,(\text{after})} \left[T^{(\text{after})}\right]^3, \tag{4.132}$$

then

$$T^{(\text{after})} = \left(\frac{g_{\star s}^{\gamma\,(\text{before})}}{g_{\star s}^{\gamma\,(\text{after})}}\right)^{1/3} \frac{a^{(\text{before})}}{a^{(\text{after})}} T^{(\text{before})} = \left(\frac{g_{\star s}^{\gamma\,(\text{before})}}{g_{\star s}^{\gamma\,(\text{after})}}\right)^{1/3} T_\nu^{(\text{after})}. \tag{4.133}$$

Plugging in the numbers, we find

$$\frac{T}{T_\nu} = \left(\frac{11}{4}\right)^{1/3}, \tag{4.134}$$

at $T < T_\nu \simeq 1.5\,\mathrm{MeV}$. Using this, we calculate the neutrino temperature now is

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_{\text{CMB}} \simeq 1.946\,\mathrm{K}. \tag{4.135}$$

You will calculate the details of this process in the homework.

One final remark. Like any other processes in nature, the neutrino decoupling process does not happen instantaneously. That is, the $e + \nu_e \rightleftharpoons e + \nu_e$ scattering is not completely shut off at $T < T_\nu$, and some, albeit small, energy transfer from electron-positron pair to neutrino sector is allowed to heat the neutrino temperature and lead to the spectral distortion of neutrino. Result of this leackage is often parameterized as the effective number of neutrino species, N_{eff}, and a detailed study of this process shows that $N_{\text{eff}} = 3.035$ by using the standard model of particle physics [**?**]. There are additional $\Delta N_{\text{eff}} = 0.011$ from the plasma effects (QED correction at finite temperature) that changes the neutrino interaction rate [**?**], to make the total $N_{\text{eff}} = 3.046$ [**?**, **?**]. You can measure N_{eff} from the abundance of Helium and Deuterium as well as the temperature anisotropy of CMB. We will come back to this point later in the lecture. But, for now, let's ignore and stick on $N = 3$ families of neutrino species.

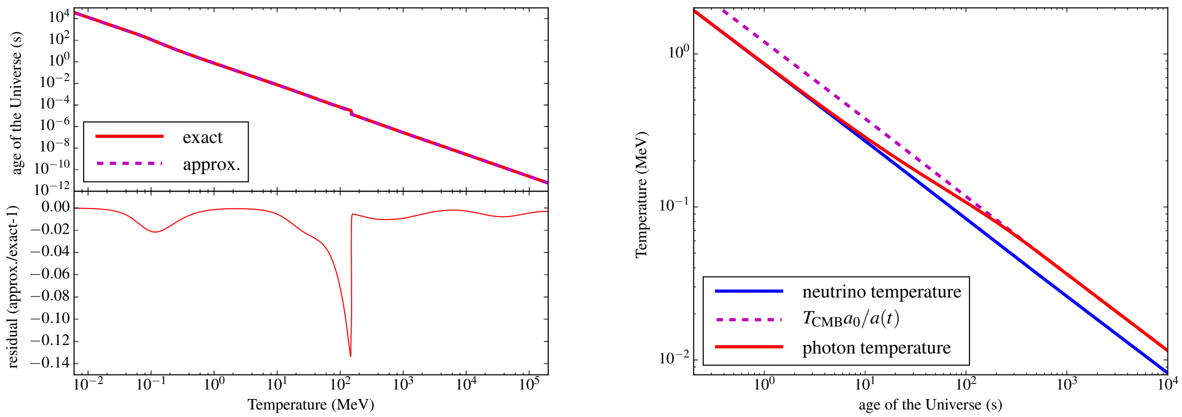

Figure 4.2: *Left:* (*Top*) Age of the Universe as a function of temperature from exact integration, Eq. (4.143), and constant-$g_\star$ approximation, Eq. (4.140), and its residual (*Bottom*). The approximation works pretty well except for the abrupt change at QCD phase transition. Even so, the residual is only 14% at maximum. *Right:* Time evolution of the photon temperature and neutrino temperature during e^-/e^+-annihilation.

4.6.4 Temperature and redshift

We have thus far written all quantities as a function of temperature of the Universe. How is the temperature of the Universe related to the scale factor? We use that the comoving entropy density sa^3 stays constant along the evolution. From Eq. (4.108), the comoving density is

$$g_{\star s} T^3 a^3 = \text{const}, \tag{4.136}$$

then we calculate the temperature-redshift relation, through $a(1+z) = a_0$, as

$$T(a) = \left(\frac{g_{\star s0}}{g_{\star s}}\right)^{1/3} \frac{a_0}{a} T_{\rm CMB}. \tag{4.137}$$

Here, $T_{\rm CMB} = 2.726\,\text{K}$ is the temperature of CMB at present, $g_{\star s0}$ is the relativistic degrees of freedom at present:

$$g_{\star s0} = 2 + \frac{7}{8} \times 2 \times N_{\rm eff} \times \left(\frac{T_{\nu 0}}{T_{\rm CMB}}\right)^3 \simeq 3.909, \tag{4.138}$$

where $N_{\rm eff} = 3$ is the effective number of neutrino species, and $T_{\nu 0} \simeq 1.946\,\text{K}$ is the present day neutrino temperature.

4.6.5 Redshift and time

The observational bound of the spatial curvature of the Universe at present is $|\Omega_{\rm k}| \lesssim 0.01$, that means we can completely ignore the curvature fraction at earlier times. Then, the Friedmann equation is

$$3H^2 = 8\pi G\rho = 8\pi G\left(\frac{\pi^2}{30} g_\star(T) T^4\right). \tag{4.139}$$

When $g_\star(T)$ is constant, $H^2 \propto t^{-2} \propto T^4$ yield $H = 1/(2t)$ and we have

$$\frac{3}{4t^2} = 8\pi G\left(\frac{\pi^2}{30}g_\star(T)\right)T^4 \longrightarrow t = \left(\frac{45}{16\pi^3 G g_\star(T)}\right)^{1/2}\frac{1}{T^2} = 2.420 g_\star^{-1/2} T_{\rm MeV}^{-2}\ {\rm s}. \tag{4.140}$$

For example, at $T = 1\,{\rm MeV}$, the relativistic degree of freedome is

$$g_\star(1\,{\rm MeV}) = 2 + \frac{7}{8}(2\times 2 + 2\times 3) = 10.75, \tag{4.141}$$

and the Universe is $t \simeq 2.420/\sqrt{10.75} = 0.738\,{\rm s}$ old, and at $T = 200\,{\rm GeV}$, the relativistic degree of freedom is $g_\star(200\,{\rm GeV}) = 106.75$ and the Universe is $t = 5.8556\times 10^{-12}\,{\rm s}$ old. Corresponding Hubble parameter is

$$H \simeq 1.360\times 10^{-25}\sqrt{g_\star}T_{\rm MeV}^2\,{\rm GeV} \simeq \left(4.831\,g_\star^{-1/2}T_{\rm MeV}^{-2}\,{\rm s}\right)^{-1} \simeq \left(2.081\,g_\star^{-1/2}T_{\rm MeV}^{-2}R_\odot\right)^{-1}. \tag{4.142}$$

That means, the physical size of the Universe at 1 MeV is about $0.6R_\odot$.

But, as you can see from the right panel of Fig. 4.1, $g_\star$ is not always a constant, and rather have a broad feature whenever there is a thermal event. In that case, we can form a differential equation for $a(t)$ by using Eq. (4.139) along with Eq. (4.137) as:

$$3H^2 = 3\left(\frac{1}{a}\frac{da}{dt}\right)^2 = 8\pi G\frac{\pi^2}{30}g_\star(T)\left(\frac{g_{\star sp}}{g_{\star s}(T)}\right)^{4/3}\left(\frac{a_p}{a}\right)^4 T_p^4, \tag{4.143}$$

where the subscript p refers to some pivot scale to which we normalize the temperature and scale factor. If we choose current CMB temperature as the pivot scale, then the equation becomes

$$3H^2 = 8\pi G\rho_{\rm rad}\left(\frac{g_\star(T)}{g_{\star 0}}\right)\left(\frac{g_{\star s0}}{g_{\star s}(T)}\right)^{4/3}\left(\frac{a_0}{a}\right)^4 = 3H_0^2\Omega_{\rm rad}\left(\frac{g_\star(T)}{g_{\star 0}}\right)\left(\frac{g_{\star s0}}{g_{\star s}(T)}\right)^{4/3}\left(\frac{a_0}{a}\right)^4, \tag{4.144}$$

with

$$g_{\star 0} = 2 + \frac{7}{8}\times 2\times N_{\rm eff}\times\left(\frac{T_{\nu 0}}{T_{\rm CMB}}\right)^{4/3} \simeq 3.363, \tag{4.145}$$

being the relativistic degrees of freedom $g_{\star 0}$ at present time.

To see the effect of time-varying relativistic degrees of freedom, let us assume that nothing has happened before $T_p = 200\,{\rm GeV}$, and the relativistic degrees of freedom is fixed to $g_\star = g_{\star s} = 106.75$ for $T \gtrsim 200\,{\rm GeV}$. Then, from the analysis above, $t_p = 5.8556\times 10^{-12}\,{\rm s}$ old, and the scale factor (and redshift) is

$$\frac{a_p}{a_0} = \left(\frac{g_{\star s,0}}{g_{\star s,p}}\right)^{1/3}\frac{T_{\rm CMB}}{T_p} = 3.90\times 10^{-16}, \tag{4.146}$$

and $z_p = 2.56\times 10^{15}$. We then solve the differential equation by integrating

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} =& t_p\sqrt{\frac{8\pi G}{3}\frac{\pi^2}{30}g_\star(y)\left(\frac{g_{\star sp}}{g_{\star s}(y)}\right)^{4/3}\frac{1}{y^4}T_p^4} = \sqrt{\frac{1}{4}\frac{g_\star(y)}{g_{\star p}}\left(\frac{g_{\star sp}}{g_{\star s}(y)}\right)^{4/3}\frac{1}{y^2}}\\ \simeq& 1.08901\frac{\sqrt{g_\star(y)[g_{\star s}(y)]^{-4/3}}}{y^2},\end{aligned} \tag{4.147}$$

where we rescale the variables as $y = a/a_p$ and $x = t/t_p$, and use Eq. (4.140) to eliminate constants. The solution for the differential equation is

$$\int_1^y dy\, y \sqrt{\frac{[g_{\star s}(y)]^{4/3}}{g_\star(y)}} = 1.08901(x-1). \tag{4.148}$$

We show the result in Fig. 4.2. It turns out that the constant-$g_\star$ approximation in Eq. (4.140) works pretty well!