Problem Set 9

1. (30 pts) [Some fun with gauge]
(a) Suppose there are two coordinate systems $x^\mu$ and $y^\mu$ related by
\[ y^\mu = x^\mu + \xi^\mu, \tag{1} \]
and both satisfy the gauge conditions $A = B = 0$. From the gauge transformation equations, $\xi$ must satisfy
\[ \xi^0 = 0, \quad \frac{1}{a} \xi^0 = a \xi^\mu. \tag{2} \]
Therefore, as long as they are related by two purely spatial functions $A(x)$ and $B(x)$,
\[ \xi^0 = A(x), \quad \xi^\mu = A(x) \int \frac{dt}{a^2} + B(x) \tag{3} \]
the two coordinates can satisfy the synchronous gauge conditions at the same time. That is, the synchronous gauge conditions do not uniquely specify the coordinate system.
(b) The general vector field is transformed as
\[ V^\mu(y) = \frac{\partial y^\mu}{\partial x^\nu} V^\nu(x) = (\delta^\mu_\nu + \xi^\mu_\nu) V^\nu(x). \tag{4} \]
Now decomposing the background and perturbation, we have
\[ \delta V^\mu(y) + \delta V^\mu(y) = \left( \delta^\mu_\nu + \xi^\mu_\nu \right) \left( \delta V^\nu(x) + \delta V^\nu(x) \right) \]
\[ \simeq \delta V^\mu(x) + \xi^\mu_\nu \delta V^\nu(x) + \delta V^\mu(x). \tag{5} \]
From which we find the final gauge transformation equation as
\[ \delta V^\mu(y) = \delta V^\mu(x) - \delta V^\mu_\nu \delta V^\nu(x). \tag{6} \]
In FRW universe, the background vector field must be $V^\nu = (1, 0, 0, 0)$, therefore the gauge transformation becomes
\[ \delta V^\mu(y) = \delta V^\mu(x) + \xi^\mu_\nu \]
and we can decompose it as the scalar and vector modes as
\[ \delta V^\mu(y) = \delta V^\mu_0(x) + \xi^\mu_0 \tag{8} \]
\[ V(x) \rightarrow V(x) + \xi \tag{9} \]
\[ V'_\perp(x) \rightarrow V'_\perp(x) + \xi. \tag{10} \]

2. (20 pts) [Curvature perturbation during inflation] On super-horizon scales, the field fluctuation is constant, and we can approximate
\[ \delta \rho = \dot{\phi} \delta \phi + V_\phi \delta \phi \simeq -3H \dot{\phi} \delta \phi, \tag{16} \]
so we can write
\[ \zeta = \psi - H \frac{\delta \rho}{\dot{\rho}} = -\phi + \frac{3H^2 \dot{\phi} \delta \phi}{-3H(\dot{\rho} + \ddot{\rho})} \]
\[ = -\dot{\phi} - \frac{3H^2}{3(\dot{\rho} + \ddot{\rho})} \frac{H^{-1} \phi + \dot{\phi}}{4\pi G}, \tag{17} \]
with Eq. (7) in the homework. We then use the Friedmann equation $3H^2 = 8\pi G \rho$ and $\dot{\rho} + \ddot{\rho} = (1 + w)\rho$ to arrive at the desired expression:
\[ \zeta = -\dot{\phi} - \frac{2}{3} \frac{H^{-1} \phi + \dot{\phi}}{1 + w}. \tag{18} \]
First, subtracting Eq. (6) from Eq. (8) cancels $\delta \varphi$ term:
\[
\dot{\phi} + 7H\dot{\phi} + 2(3H^2 + H)\phi = -8\pi G V'(\hat{\varphi})\delta \varphi. \tag{19}
\]
Here, we set $\nabla^2 \varphi = 0$ as we are considering super-horizon scales. We then use Eq. (7) to re-write the RHS as
\[
-8\pi G V'(\hat{\varphi})\delta \varphi = -2\frac{V'(\hat{\varphi})}{\hat{\varphi}} (\dot{\phi} + H\phi)
= 2\frac{\ddot{\varphi}}{2} + 3H\frac{\dot{\varphi}}{\hat{\varphi}} (\dot{\phi} + H\phi). \tag{20}
\]
We then find the differential equation of
\[
\dot{\phi} + 7H\dot{\phi} + 2(3H^2 + H)\phi - \left( \frac{2\dot{\varphi}}{\hat{\varphi}} + 6H \right) (\dot{\phi} + H\phi) = 0,
\tag{21}
\]
which simplifies further as
\[
\dot{\phi} + \left( H - \frac{2\dot{\varphi}}{\hat{\varphi}} \right) \phi + 2 \left( H - H \frac{\dot{\varphi}}{\hat{\varphi}} \right) \dot{\phi} = 0. \tag{22}
\]
To calculate the time derivative of $\zeta$, we need to calculate first
\[
\dot{w} = \frac{\dot{p}}{\rho} - p \frac{\dot{\rho}}{\rho^2} = \frac{\dot{p}}{\rho} + 3HP \frac{\rho + P}{\rho^2} = \frac{\dot{p}}{\rho} + 3H\omega(1 + w). \tag{23}
\]
By using
\[
\rho_\varphi = \frac{1}{2} \dot{\varphi}^2 + V, \quad P_\varphi = \frac{1}{2} \dot{\varphi}^2 - V, \tag{24}
\]
we can also obtain
\[
\dot{\rho} = \ddot{\varphi} - V'(\hat{\varphi}) \dot{\varphi} = \ddot{\varphi} + \left( \frac{\dot{\varphi}}{\hat{\varphi}} + 3H \dot{\varphi} \right) \dot{\varphi} = 2\ddot{\varphi} + 3H\dot{\varphi}^2. \tag{25}
\]
Combining the two,
\[
\dot{w} = \left[ \frac{2\ddot{\varphi} + 3H\dot{\varphi}^2}{\rho + P} + 3H\omega \right] (1 + w)
= \left[ 2\ddot{\varphi} + 3H(1 + w) \right] (1 + w). \tag{26}
\]
Finally, using $(\rho + P) = \dot{\varphi}^2 = -\dot{H}/4\pi G$, we find
\[
H(1 + w) = H \frac{\rho + P}{\rho} = H \frac{-\dot{H}/4\pi G}{3H^2/8\pi G} = -\frac{2\dot{H}}{3H} \tag{27}
\]
from which
\[
\frac{\dot{w}}{1 + w} = 2\frac{\ddot{\varphi}}{\hat{\varphi}} + 3H(1 + w) = 2 \left( \frac{\ddot{\varphi}}{\hat{\varphi}} - \frac{\dot{H}}{\dot{H}} \right) \tag{28}
\]
Then,
\[
-\frac{3}{2} \zeta H(1 + w) = \frac{3}{2} H(1 + w) + \left[ -H^{-1} \dot{H} \phi + \dot{\phi} + H\dot{\phi} \right] - (\dot{\phi} + H\phi) \frac{\dot{w}}{1 + w}
= -\frac{\dot{H}}{H} \phi + \left[ -\frac{\dot{H}}{H} \phi + \dot{H} \phi + H\dot{\phi} \right] - 2(\dot{\phi} + H\phi) \left( \frac{\ddot{\varphi}}{\hat{\varphi}} - \frac{\dot{H}}{\dot{H}} \right)
= \dot{\phi} + \left( H - 2\frac{\dot{\varphi}}{\hat{\varphi}} \right) \dot{\phi} - 2H \left( \frac{\ddot{\varphi}}{\hat{\varphi}} - \frac{\dot{H}}{\dot{H}} \right) \Phi = 0 \tag{29}
\]