Cosmology (ASTRO/PHYS 545)  
Spring 2019  

A Solution for Problem Set 7  

1. (20 pts) [Alpher, Herman, Gamow’s estimation of $T_{\text{amb}} \approx 5K$]  
   (a) Freeze-out condition is  
   \[ n_e \langle \sigma v \rangle = H = \sqrt{\frac{8 \pi G}{3} \rho_\gamma}, \]  
   \[ (1) \]  
   then  
   \[ n_b = \frac{1}{\langle \sigma v \rangle} \sqrt{\frac{8 \pi}{3M^3}} \pi^2 T_4^4 \approx 2.17 \times 10^{17} \text{cm}^{-3} \]  
   \[ (2) \]  
   (b) The redshift can be estimated as  
   \[ \frac{1 + z}{1 + \gamma} \approx \frac{\rho_b}{\rho_\gamma} \approx 3.867 \times 10^{23} \]  
   \[ (3) \]  
   that gives $z \approx 7.29 \times 10^{-7}$.  
   (c) The redshift above corresponds to $T = 10^9 K$, and the temperature of the Universe now is  
   \[ T_b = \frac{10^9}{1 + z} \approx 13.7 K. \]  
   \[ (4) \]  
   That is pretty good guess as of 1948!  
   (d) Had they included the neutrinos, the radiation density must be modified  
   \[ \rho_r = \rho_\gamma + \rho_\nu = \frac{\pi^2}{15} \left( 1 + \frac{7}{8} \times 3 \times \left( \frac{4}{11} \right)^{4/3} \right) T_4^4 \]  
   \[ \approx 1.68 \rho_\gamma. \]  
   \[ (5) \]  
   This would increase the estimate of $n_b$ by 30%, and the redshift estimate by 9%. Therefore, the temperature is estimated to drop by 9%, yielding $T_b = 12.5 K$.  

2. [Simplified Peebles equation.] (a) Rearranging the simplified Peebles equation, we find  
   \[ \int_{X_0}^{X(z)} \frac{dX_e}{X^2_e} = 1 - \frac{1}{X_0(z)} = - \int_0^{f(z)} dt \left( \frac{\Lambda_{2y}}{\Lambda_{2y} + 4\beta} \right) a_y n_{H,\text{tot}}, \]  
   or  
   \[ X_e(z) = \left[ 1 + \int_0^{f(z)} dt \left( \frac{\Lambda_{2y}}{\Lambda_{2y} + 4\beta} \right) a_y n_{H,\text{tot}} \right]^{-1}. \]  
   \[ (7) \]  
   To facilitate the integration, we change the variable to $y = a/a_{eq}$:  
   \[ \int_0^{f(z)} dt \left( \frac{\Lambda_{2y}}{\Lambda_{2y} + 4\beta} \right) a_y n_{H,\text{tot}} \]  
   \[ = \int_0^{y(z)} dy \frac{y}{yH(y)} \left( \frac{\Lambda_{2y}}{\Lambda_{2y} + 4\beta} \right) a_y n_{H,\text{tot}} \]  
   \[ (8) \]  
   Figure 1: The free-electron fraction calculated from the approximation given in the problem2 (solid line) along with the accurate solution (dashed line). The approximation works pretty well!  

with  
\[ yH(y) = 7.357 \times 10^{-14} \left( \frac{\Omega_m h^2}{0.143} \right)^2 \frac{1 + y}{2y^2} \text{s}^{-1}, \]  
\[ \Lambda_{2y} = 8.2206 \text{s}^{-1}, \]  
\[ 4\beta = \frac{1.0406 \times 10^9 T_{4}^{0.8834}}{1 + 0.67037 T_{4}^{0.53}} \exp \left( -\frac{3.9545}{T_{4}} \right) \text{s}^{-1}, \]  
\[ a_y n_{H,\text{tot}} = 4.178 \times 10^{-9} \left( \frac{\Omega_B h^2}{0.023} \right) T_{4}^{2.3834} \text{s}^{-1}. \]  
\[ (9) \]  
   To complete the relation, we need the temperature-$y$ relation:  
   \[ T_4 = \frac{0.93684}{y} \left( \frac{\Omega_m h^2}{0.143} \right), \]  
   \[ (10) \]  
   and the redshift-$y$ relation:  
   \[ 1 + z = 3436.72 \left( \frac{\Omega_m h^2}{0.143} \right). \]  
   \[ (11) \]  
   The resulting recombination history is shown in Fig. 1 along with a more accurate solution from HyRec.  
   (b) The freeze-out value for the ionization fraction we get from the above integral is  
   \[ A = 2.618 \times 10^{-4}. \]  
   \[ (12) \]  
   How would it depend on cosmology? As $T_4$ dependence is more complicated, let us rewrite the integral in terms of $T_4$. From  
   \[ dy = -\frac{dT_4}{T_4} \frac{0.93684}{y} \left( \frac{\Omega_m h^2}{0.143} \right), \]  
   \[ (13) \]
we find
\[
\int_0^{y(z)} \frac{d y}{y H(y)} \left( \frac{\Lambda_2 y}{\Lambda_2 y + 4 \beta} \right) a_B n_{H, \text{tot}} \\
\propto \int_0^\infty \frac{d T_4}{T_4} 0.93684 \left( \frac{\Omega_m h^2}{0.143} \right)
\times \left( \frac{\Omega_m h^2}{0.143} \right)^{-3/2} \left[ \frac{1}{y(T_4) H(T_4)} \right]_{y(T_4) = 0.143}
\times \left( \frac{\Lambda_2 y}{\Lambda_2 y + 4 \beta} \right) \left( \frac{\Omega_m h^2}{0.023} \right) a_B n_{H, \text{tot}} \left( \Omega_b h^2 = 0.023 \right)
\propto \left( \frac{\Omega_m h^2}{0.143} \right)^{-1/2} \left( \frac{\Omega_m h^2}{0.023} \right) \left( \frac{\Omega_m h^2}{0.143} \right)^{-1/2} \left( \frac{\Omega_m h^2}{0.023} \right)
\]
(14)

Here, we approximate \( \sqrt{(1 + y)/(2y^2)} \approx 1/\sqrt{2y} \) for freeze-out happening deeply at the matter domination \( (y \gg 1) \). Therefore,

\[
X_e \approx \int_0^{y(z)} \frac{d y}{y H(y)} \left( \frac{\Lambda_2 y}{\Lambda_2 y + 4 \beta} \right) a_B n_{H, \text{tot}} \left[ y(z) \right]^{-1}
\propto \Omega_m h^2 \left( \frac{\Omega_m h^2}{0.143} \right)^{-1/2} \left( \frac{\Omega_m h^2}{0.023} \right) \left( \frac{\Omega_m h^2}{0.143} \right)^{-1/2} \left( \frac{\Omega_m h^2}{0.023} \right)
\]
(15)

and we have completed the equation

\[
X_e^{\text{freeze-out}} = 2.618 \times 10^{-4} \left( \frac{\Omega_m h^2}{0.143} \right)^{1/2} \left( \frac{\Omega_m h^2}{0.023} \right)^{-1}.
\]
(16)

(c) Based on the redshift evolution of the free-electron fraction we calculate above, we plot the visibility function

\[
g(z) = \frac{d \tau}{dz} e^{-\tau(z)},
\]
(17)

where

\[
\tau(z) = \int n_e \sigma_T dt,
\]
(18)

is the optical depth of the CMB photon (without reionization), in Fig. 2. The curve peaks at \( z = 1055 \).

Figure 2: Photon's visibility function, probability distribution function for the last-scattering redshift of the CMB photons.