Cosmology (ASTRO/PHYS 545)  
Spring 2019

A Solution for Problem Set 6

1. (30 pts) [Cosmology with massive neutrinos]  
(a) Neutrinos have temperature of $T_\nu = (4/11)^{1/3} T_\text{cmb} = 1.946 \, \text{K}$, and number density of

$$n_\nu = \frac{3}{4} \frac{g_{*e}}{g_{*\nu}} \left( \frac{T_\text{cmb}}{T_\nu} \right)^3 = \frac{3}{4} \frac{g_{*e}}{g_{*\nu}} \left( \frac{4}{11} \right) \times 411 \, \text{cm}^{-3} \approx 336.3 \, \text{cm}^{-3},$$  

which gives the neutrinos energy density of (they are massive, non-relativistic particles) $\rho_\nu = m_\nu n_\nu = 16.81, 33.63, 62.25, 134.5 \, \text{eV/cm}^3$ for $m_\nu = 0.05 \sim 0.4 \, \text{eV}$. 

(b) The neutrinos follow Fermi-Dirac distribution, given neutrino masses.

$$f_\nu(p) = \frac{1}{e^{p/T_\nu} + 1},$$  

and the median of the distribution $p_{1/2}$ can be calculated from

$$\int_0^{p_{1/2}} dp e^{p/T_\nu} = \frac{1}{2} \int_0^{\infty} dp e^{p/T_\nu} = \frac{3}{4} \zeta(3) T_\nu^2,$$

Rescaling the momentum as $x = p/T_\nu$, the equation becomes

$$\int_0^{x_{1/2}} x^2 dx = 0.901543,$$

whose solution is $x_{1/2} = 2.839$, or $p_{1/2} = 2.839 T_\nu$. The transition redshift can be calculated from equating $p_{1/2}(z) = 2.839 T_\nu (1 + z_m)$, which are $z_m = 104.0, 209.1, 419.1, 839.3$ for the four given neutrino masses.

(c) The energy density of the massive neutrinos are

$$\rho_\nu(z) = g_\nu \int_0^{\infty} x^2 \frac{e^{x/T_\nu} - 1}{e^{x/T_\nu} + 1} \, dx = g_\nu \frac{T_\nu^4}{2 \pi^2} \int_0^{\infty} x^2 \sqrt{x^2 + x^2_\nu} \, dx = 6.055 \times 10^{-6} (1 + z)^3 \rho_{\text{crit}}$$

Then, from the Friedmann equation, we find

$$H(z) = H_0 \sqrt{(1 - f_\nu)\Omega_m (1 + z)^3 + \Omega_\Lambda (1 + z)^4 + \Omega_\nu + \frac{\rho_\nu(z)}{\rho_{\text{crit}}}},$$

then the integral gives $d_\lambda(z = 1100) \approx 14.007, 14.010, 14.014, 14.017 \text{Gpc}$ for the four neutrino masses given here.

We show the result in Fig. 1. (d) Because $z_m \ll z_\text{re} / z_\text{eq}$, neutrinos can be counted as relativistic, and matter radiation equality time can be calculated from equating $\rho_{\text{crit}}(z) + \rho_\nu(z) = \rho_\Lambda(z) + \rho_\nu(z)$, that happens at $z_{\text{re}} = 3257, 3216, 3130, 2938$, respectively. For a large neutrino mass, we reduce the energy density of CDM and baryon that reduce the matter-radiation equality.

(e) For the flat Universe, the comoving angular diameter distance is

$$d_\lambda(z) = \int_0^{1100} \frac{dz}{H(z)} = \int_1^{1100/11} \frac{da}{a^2 H(a)}$$

Using Friedmann equation, we have

$$H(z) = H_0 \sqrt{(1 - f_\nu)\Omega_m (1 + z)^3 + \Omega_\Lambda (1 + z)^4 + \Omega_\nu + \frac{\rho_\nu(z)}{\rho_{\text{crit}}}},$$

We use $\Omega_\Lambda = 5.05 \times 10^{-5}$. For a flat Universe, we calculate the density parameters at redshift $z$ as

$$\Omega_i(z) = \frac{\rho_i(z)}{\sum_i \rho_i(z)}.$$

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Should $d_\lambda(z)$ increase or decrease as increasing the neutrino mass? To answer this question, we calculate

$$\frac{d d_\lambda(z)}{d m_\nu} = \int_0^{1100} \frac{dz}{H(z)}$$

Then, from the Friedmann equation, we find

$$2H \frac{d H(z)}{d m_\nu} = 8 \pi G \frac{d \rho_\nu(z)}{d m_\nu} \geq 0$$

2. (20 pts) [WIMP mass and cross-section]  
Using the equations in the lecture note, we can plot lines corresponding to (a) - (e) as shown in Fig. 2
Figure 1: Evolution of density parameters including massive neutrinos for $m_\nu = 0.05, 0.1, 0.2, 0.4$ eV from top to bottom.

Figure 2: A summary plot for thermal WIMP: Red lines show $\Omega_{cdm} h^2 = 0.12$, and blue lines show $\Omega_{cdm} h^2 = 0.49$ for $s$-wave annihilation (solid lines) and $p$-wave annihilation (dashed lines). Green and Cyan lines both show weak interaction mediated by, respectively, standard model weak gauge bosons ($W^\pm, Z^0$) and a new gauge boson of mass $M_{new} = 1$ TeV. Shaded region at the top right corner is excluded by the Unitarity bound that the coupling constant must be smaller than $\theta(1)$. 

Solution for hw6