Cosmology (ASTRO/PYHS 545)
Spring 2019

A Solution for Problem Set 12

1. (30 pts) [Evolution of matter and radiation]
   (a) We show the result of the calculation for \( Q = 0.001 \) in Fig. 1, \( Q = 0.01 \) and \( Q = 0.1 \) in Fig. 2, \( Q = 1.0, Q = 10.0 \), and \( Q = 100.0 \) in Fig. 3. For all figures, we show the density contrast of matter (Red), the density contrast of radiation (Blue), and the Bardeen's potential (Green).

   Figure 1: The evolution of linear perturbations for the large-scale mode that stays outside horizon at all time.

   

(b) Combining the solutions between \( Q = 0.01 \) and \( Q = 100 \), we find the transfer function shown in Fig. 4.

(c) Computing the \( \sigma_s \), we find that

\[
\sigma_s(\Lambda CDM) = 0.607 \\
\sigma_s(o CDM) = 0.356 \\
\sigma_s(s CDM) = 0.882
\]

2. (20 pts) [Effect of massive neutrino on matter evolution]
   (a) From the distribution function of the neutrinos,

\[
f_s(p) = \frac{1}{e^{p/T} - 1},
\]

we compute the average momentum as

\[
\langle p_v \rangle = \int \frac{d^3p}{(2\pi)^3} f_s(p) = \int \frac{d^3p}{(2\pi)^3} p f_s(p)
\]

\[
= \frac{7\pi^4}{180\zeta(3)} T_v \approx 3.15 T_v.
\]

The neutrino temperature is

\[
T_v = 0.16767 / a \text{ meV},
\]

and we calculate the neutrino velocity from

\[
v_v = \frac{\langle p_v \rangle}{m_v} = 5.28 \times 10^{-4} \frac{a}{m_{\text{neV}}}. \tag{7}
\]

(b) We estimate the free-streaming length as

\[
\lambda_s(a) = \frac{v_v}{aH} = \frac{5.28 \times 10^{-4}}{a^2 m_{\text{neV}} H_0 \sqrt{\Omega_m/a^3 + \Omega_\nu}} \\
= 1.583 \text{ Mpc}/h \frac{m_{\text{neV}}}{\sqrt{a^3 \Omega_m + a^4 \Omega_\nu}}, \tag{8}
\]

with \( H_0 = 2997.96 [\text{Mpc}/h] \). In homework 6, we show that the transition happens at

\[
a_{nr} = \frac{2.839 T_{eq}}{m_\nu} = 0.00952, 0.00476, 0.00238, 0.00119,
\]

for the four masses, that brings

\[
\lambda_s(a_{nr}) = 1357, 1917, 2690, 3581, \tag{9}
\]

and

\[
\lambda_s(a_t) = 12.9, 9.13, 6.40, 4.26. \tag{11}
\]

Note that the neutrino free-streaming scales decrease in time.

(c) The neutrinos do not contribute the gravitational collapse, therefore, the Poisson's equation becomes

\[
k^2 \Phi = -\frac{3}{2} \mathcal{H}^2 (1 - f_s) \delta_m = -\frac{6}{\eta^2} (1 - f_s) \delta_m, \tag{12}
\]
here we use that $a(\eta) = \eta^2$ during the matter dominated universe, and the density field satisfies

$$\delta''_m + \frac{2}{\eta} \delta'_m - \frac{6}{\eta^2} (1 - f_\nu) \delta_m = 0.$$  \hfill (13)

The growing mode solution is

$$\delta_m \propto \eta^{-\frac{1}{2} + \frac{1}{4} \sqrt{25 - 24 f_\nu}} = a^{\frac{1}{2} \left( \sqrt{25 - 24 f_\nu} - 1 \right)}.$$ \hfill (14)

(d) The scales smaller than the free-streaming scale at present time suffer the most. Basically, from the $a_{nr}$ to present time ($a = 1$). In the presence of neutrinos:

$$\frac{d \ln \delta_m}{d \ln a} = \frac{1}{4} \left( \sqrt{25 - 24 f_\nu} - 1 \right) \approx 1 - \frac{3}{5} f_\nu.$$ \hfill (15)

The change in growth factor is comes with powers of $a^{-0.6 f_\nu}$. Using

$$f_\nu = \frac{336.3 m_\nu}{1445.95} = 0.233 m_\nu,$$ \hfill (16)

we can write $a_{nr} = 0.000111 / f_\nu$, then find that

$$\frac{\Delta \delta_m}{\delta_m} = 0.6 f_\nu \ln(a_{nr}) = 0.6 f_\nu \ln(0.000111 / f_\nu) \approx -5.46 f_\nu.$$ \hfill (17)

Therefore, it suppress the power spectrum by about

$$\frac{\Delta P}{P} \approx -10 f_\nu.$$ \hfill (18)

on scales smaller than the free-streaming scales.

Figure 3: The evolution of linear perturbations for the small-scale modes that enter the horizon during the radiation era.
Figure 4: The transfer function at $y = 100$, in comparison with BBKS transfer function.

Figure 5: The linear matter power spectrum from the fixed primordial power spectrum $\Delta^2_{R} = 2.15 \times 10^{-9}$, for three cosmological models: $\Lambda$CDM ($\Omega_\Lambda = 0.7$, $\Omega_m = 0.3$) oCDM ($\Omega_\Lambda = 0.0$, $\Omega_m = 0.3$) sCDM ($\Omega_\Lambda = 0.0$, $\Omega_m = 1.0$).