Cosmology (ASTRO/PHYS 545)
Spring 2019

Extra credit problem set

Due 6 May 2019

Solving problem sets is one of the most efficient ways of learning the subject. You are encouraged to collaborate with fellow students and/or to consult senior students, local postdocs and me. But, please write the solution by yourself.

This is an extra-credit homework due by the end of the semester (3 May 2019). No homework will be accepted after I post the solution on the course webpage.

1. (20 pts) [Christoffel symbols from geodesic equation] One can calculate the Christoffel symbols by matching the Euler-Lagrange equation

\[
\frac{d}{d\lambda} \left( \frac{\partial L}{\partial \dot{x}^\mu} \right) = \frac{\partial L}{\partial x^\mu}
\]

of the Lagrangian of free particle

\[
L = g_{\mu\nu}(t, x) \dot{x}^\mu \dot{x}^\nu,
\]

to the geodesic equation:

\[
\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0.
\]

Here, \(\dot{x}^\mu \equiv d x^\mu / d \lambda = P^\mu\) is the derivative with respect to the affine parameter \(\lambda = \tau / m\), and Greek letters stand for the space-time indices.

With the the metric perturbations in Poisson gauge

\[
ds^2 = -(1 + 2\Phi(t, x))dt^2 + a^2(t) \left[ (1 + 2\Psi(t, x))\delta_{ij} + h_{ij}(t, x) \right] dx^i dx^j,
\]

with traceless, transverse \((h^i_i = h_{ij,i} = 0)\) tensor perturbation \(h_{ij}\), show that the Christoffel symbols are given as

\[
\Gamma^0_{00} = \Phi, \quad \Gamma^0_{0i} = \Phi,_{,i}, \quad \Gamma^0_{ij} = a^2 \left\{ \left[ H(1 - 2\Phi + 2\Psi) + \Psi \right] \delta_{ij} + \left[ Hh_{ij} + \frac{1}{2} \dot{h}_{ij} \right] \right\},
\]

\[
\Gamma^i_{00} = \frac{1}{a^2} \Phi,^i, \quad \Gamma^i_{0j} = H + \Psi, \quad \Gamma^i_{j0} = H, \quad \Gamma^i_{ij} = \Psi,_{,i} \delta^j_+ + \Psi,_{,j} \delta^i_+ - \Psi,_{,ij} \delta^0_+ + \frac{1}{2} \left\{ h_{ij} + h^i_j + h^j_i - h^i_j \right\}.
\]

2. (30 pts) [Einstein tensor] From the Christoffel symbols, calculate the perturbed Ricci tensor

\[
\delta R_{00} = \frac{1}{a^2} \nabla^2 \Phi - 3\dot{\Psi} + 3H\dot{\Phi} - 6H\dot{\Psi},
\]

\[
\delta R_{0i} = -2 \left[ \Psi - H\Phi \right],
\]

\[
\delta R_{ij} = a^2 \left[ \ddot{\Psi} + H(-\Phi + 6\Psi) + 2(H + 3H^2)(-\Phi + \Psi) - \frac{1}{a^2} \nabla^2 \Psi \right] \delta_{ij} - \left[ \Phi + \Psi \right]_{ij}
\]

\[
+ \frac{a^2}{2} \left[ \dot{h}_{ij} + 3H\dot{h}_{ij} + 2(H + 3H^2)h_{ij} - \frac{1}{a^2} \nabla^2 h_{ij} \right],
\]

(7)
the perturbed Ricci scalar

$$\delta R = 6\dot{\Psi} + 6H(4\Psi - \Phi) - 12(2H^2 + \dot{H})\Phi - \frac{2}{a^2}\nabla^2[\Phi + 2\Psi], \quad (8)$$

and the perturbed Einstein tensor

$$\delta G_{00} = 6H\dot{\Psi} - \frac{2}{a^2}\nabla^2\Psi \quad (9)$$

$$\delta G_{0i} = -2[\dot{\Psi} - H\Phi]_i \quad (10)$$

$$\delta G_{ij} = a^2\left[-2\ddot{\Psi} + 2H(\dot{\Phi} - 3\dot{\Psi}) + 2(2H + 3H^2)(\Phi - \Psi) + \frac{1}{a^2}\nabla^2(\Phi + \Psi)\right]\delta_{ij} - [\Phi + \Psi]_{ij}$$

$$+ \frac{a^2}{2}\left[\ddot{h}_{ij} + 3\dot{h}_{ij} - 2(2\dot{H} + 3H^2)h_{ij} - \frac{1}{a^2}\nabla^2h_{ij}\right]. \quad (11)$$