Solving problem sets is one of the most efficient ways of learning the subject. You are encouraged to collaborate with fellow students and/or to consult senior students, local postdocs and me. But, please write the solution by yourself.

Homework is due at the end of the class on 25 March 2019. No homework will be accepted after I post the solution on the course webpage (usually right after the class following the due date).

Preview: This should be an interesting problem set; in it, you will work out several possibilities, many relevant to current research, for inflation, dark energy, and dark matter. The problems themselves are technically pretty simple, but together they cover quite a bit of territory. Problem 1 is a straightforward exercise in which you will show that a rolling scalar field (i.e., a scalar field in a kinetic-energy-dominated phase) acts like matter with pressure \( P = \rho \). Problem 2 is a pretty involved problem in which you will work out pretty fully the phenomenological consequences of a particular model for inflation. Problem 3 works through a particularly intriguing quintessence model (i.e., a scalar-field model for negative-pressure dark energy in the Universe today) in which the dark-energy density tracks that of the dominant component (e.g., radiation or matter) of the cosmological energy density. Problem 4 is an order-of-magnitude calculation that shows that magnetic monopoles produced at a GUT phase transition should overwhelm the density of the Universe today (if there were no inflation); this reproduces a calculation that John Preskill was the first to do around 1980. Problem 5 shows that oscillations in an anharmonic scalar-field potential can give rise to exotic equations of state.

1. (5 pts) [A \( w = 1 \) equation of state from a rolling scalar field.] Consider a massless scalar field; i.e., a scalar field \( \phi(t,x) \) whose potential-energy density is \( V(\phi) = 0 \). Now suppose that this scalar field is initially rolling, so \( \dot{\phi} \neq 0 \), and that the kinetic-energy density associated with this rolling dominates the energy density of the Universe. Show from the stress-energy tensor \( P = \rho \) for this type of matter. Show that this implies that \( \rho \propto a^{-6} \), where \( a \) is the scale factor, in two ways: (1) by recalling how the energy density of matter with an equation of state \( P = \rho \) scales with \( a \); and (2) by solving the equation of motion for \( \phi \) in an expanding universe. (This should be a very simple problem.)

2. (15 pts) [Phenomenology of \( \lambda \phi^4 \) inflation.] Consider \( V(\phi) = \lambda \phi^4 \), where \( \lambda \) is the self-coupling constant. Assume that the field rolls toward \( \phi = 0 \) from the positive side. Calculate the value of \( \phi \) where each of the slow-roll conditions (i.e., \( \epsilon \ll 1 \) and \( \eta \ll 1 \)) first break down. Do they break down at the same place? Assuming that inflation ends when \( \epsilon = 1 \), calculate the number of e-foldings of inflation that occur for an initial value \( \phi_i \). Demonstrate that the slow-roll solutions
with $\phi = \phi_i$ and $t = t_i$ are ($M_{pl}$ is the reduced Planck mass)

$$
\phi = \phi_i \exp \left[ -\sqrt{\frac{16 \lambda M_{pl}^2}{3}} (t - t_i) \right],
$$

(1)

$$
a = a_i \exp \left( \frac{\phi_i^2}{8 M_{pl}^2} \left\{ 1 - \exp \left[ -\sqrt{\frac{64 \lambda M_{pl}^2}{3}} (t - t_i) \right] \right\} \right).
$$

(2)

Use the solution for $\phi$ to calculate the time that inflation ends. Demonstrate that the number of e-foldings calculated using the solution for $a$ is the same as that which you calculated above.

Expand the solution for $a$ at small $t - t_i$ to demonstrate that the inflation is approximately exponential in the initial stage. Calculate the time constant $\kappa$ [from $a \sim \exp(\kappa t)$] and demonstrate that it equals the (slow-roll) Hubble parameter during inflation.

3. (10 pts) [Tracker field.] Consider a scalar field that rolls down a potential-energy density $V(\phi) = V_0 e^{-\phi/\phi_i}$. Now suppose that the energy density of the Universe is dominated by ordinary non-relativistic matter (so $a \propto t^{2/3}$), and that the energy density of the rolling scalar field is negligible compared with the non-relativistic matter. Show that there is a solution to the scalar-field equation of motion such that the energy density $\rho_\phi = (1/2)\dot{\phi}^2 + V(\phi)$ of the scalar field scales as $\rho_\phi \propto a^{-3}$, the same as the ordinary matter. Does the same thing happen if the energy density of the Universe is dominated by relativistic matter? This is the basis for the “tracker-field” solutions that have been discussed in the literature recently.

4. (10 pts) [The monopole problem.] Calculate the relic density of magnetic monopoles (a dragon), assuming that there is one GUT-mass ($\sim 10^{15}$ GeV) monopole produced per Hubble volume at the GUT phase transition ($T \sim 10^{15}$ GeV). You should get an unreasonably large number. There is a bound $\Omega_{\text{monopole}} < 10^{-6}$ (the Parker bound) to the relic density of magnetic monopoles in the Universe today. Calculate the number of e-folds of inflation after the GUT transition required to solve the monopole problem. Monopole is a topological defect and the energy density scales as $\rho_{\text{monopole}} \propto a^{-3}$. Assumes that the inflation happens at the GUT scale, and radiation domination starts at the same scale.

5. (10 pts) [Anharmonic scalar-field oscillations.] In class we argued that if we have a real scalar field $\phi$ with a quadratic potential $V(\phi) = (1/2)m^2 \phi^2$, and if $m \gg H$ (implying that the oscillation frequency is large than the expansion rate), then coherent oscillations of the scalar field imply that the pressure $P = 0$ when averaged over an oscillation cycle and thus that the energy density $\rho \propto a^{-3}$. Now consider oscillations in a potential $V(\phi) = c|\phi|^n$, where $c$ is a constant. Show that coherent oscillations in such a potential give rise to an energy density that decays as $\rho \propto a^{-\alpha}$, and determine $\alpha$. Of course, you should recover $\alpha = 3$ for $n = 2$. What value of $n$ is required to produce $\alpha = 4$ (i.e., radiation)? Can you think of a physical argument that justifies your result? Likewise, is there a value of $n$ that produces $\alpha = 0$? Can you explain this result in physical terms?