Solving problem sets is one of the most efficient ways of learning the subject. You are encouraged to collaborate with fellow students and/or to consult senior students, local postdocs and me. But, please write the solution by yourself.

Homework is due at the end of the class on 1 March 2019. No homework will be accepted after I post the solution on the course webpage (usually right after the class following the due date).

1. (30 pts) [Cosmology with massive neutrinos] For this problem, let us assume that $m_{\nu_e} = m_{\nu_u} = m_{\nu_\tau} \equiv m_{\nu}$, which is approximately true when total mass of neutrinos are much larger than the observed mass gaps from the neutrino oscillation experiments. Massive neutrinos were relativistic particles at early times $T_{\nu} \gtrsim m_{\nu}$ and contributed $\Omega_R$, but transited to non-relativistic then contributed to $\Omega_M$, after $T_{\nu}$ had dropped below $m_{\nu}$. For a flat $\Lambda$CDM Universe with following cosmological parameters,

$$\Omega_M = 0.28, \Omega_\Lambda = 0.72, \Omega_b h^2 = 0.023, h = 0.7, T_{\text{cmb}} = 2.726 \text{K},$$

consider the cases that neutrino masses are $m_{\nu} = 0.05 \text{ eV, 0.1 eV, 0.2 eV and 0.4 eV}$. For all four cases with different neutrino masses

(a) Calculate the energy fraction of neutrinos to the total mass $f_{\nu} = \Omega_{\nu}/\Omega_M$ at present time.

(b) Calculate the transition redshift $z_{nr}$ at which the half of neutrinos become non-relativistic ($|p| < m_{\nu}$).

(c) Plot the redshift evolution of the density parameters $\Omega_{\text{cdm}}(z)$, $\Omega_b(z)$, $\Omega_{\nu}(z)$, and $\Omega_\Lambda(z)$ from redshift $z = 10^9$ to now. Your plot must show redshift in the x-axis, and $\Omega$ in the y axis in the logarithmic scale from $10^{-4}$ to 1. You should normalize $\Omega_{\text{cdm}}(z)$ as $\Omega_{\text{cdm}} + \Omega_b + \Omega_{\nu} = \Omega_M$ at present time.

(d) Calculate the redshift $z_{er}$ corresponding to the matter-radiation equality.

(e) Calculate the comoving angular diameter distance $d_A(z) = (1 + z)D_A(z)$ to $z = 1100$.

2. (20 pts) [WIMP mass and cross-section] In this problem, you will summarize the properties of the cold relic WIMP particles in the mass($M_{\text{cdm}}$) - cross section($\langle \sigma v \rangle$) plane. In the log-scale plane of $x$ (the mass of cold dark matter from $1 \text{ GeV}$ to $10^5 \text{ GeV}$) and $y$ (thermal average of $\langle \sigma v \rangle$ from $10^{-38} \text{ cm}^2$ to $10^{-34} \text{ cm}^2$), plot regions satisfying following condition ($h = 0.7$):

(a) $\Omega_{\text{cdm}} = 0.24$, $s$—wave annihilation, $\Omega_{\text{cdm}} = 1$, $s$—wave annihilation,

(b) $\Omega_{\text{cdm}} = 0.24$, $p$—wave annihilation, $\Omega_{\text{cdm}} = 1$, $p$—wave annihilation,

(c) Weak interaction via $W^+/Z^0$ bosons: $\langle \sigma v \rangle = G_F^2 m^2$

(d) Weak interaction via heavier bosons with mass $M_{\text{new}} = 1 \text{TeV}$: $\langle \sigma v \rangle = G_F^2 m^2(M_{\text{W}}/M_{\text{new}})$

(e) Unitarity bound: $\langle \sigma v \rangle = m^{-2}$. What is the critical mass of the dark matter above which the dark matter energy density would over-close the universe (that is, $\Omega_m > 1$)?