Solving problem sets is one of the most efficient ways of learning the subject. You are encouraged to collaborate with fellow students and/or to consult senior students, local postdocs and me. But, please write the solution by yourself.

Homework is due at the end of the class on 22 February 2019. No homework will be accepted after I post the solution on the course webpage (usually right after the class following the due date).

1. (20 pts) [Details of $e^- - e^+$ annihilation] In the class, we have seen what happens before and after the $e^- - e^+$ annihilation. This problem is designed to fill the gap, and you will figure out the time evolution of the photon temperature and the electron density during the process. Use following cosmological parameters: $\Omega_b h^2 = 0.023$, $Y_p = 0.24$, and $T_{\text{cmb}} = 2.726$ K.

For plotting, you may use your favorite programming language (C/C++, Fortran, python, IDL, Mathematica, MATLAB, etc), but the figure must start before $T = 0.511$ MeV and end after $T$ reaches $1.4 T_\nu$.

(a) The total comoving entropy density of thermal bath including photons and electrons is conserved throughout the process. From this, calculate the photon temperature as a function of the neutrino temperature. Plot $y = T/(1+z)$ as a function of $x = T_\nu$ in the unit of MeV.

(b) Plot the photon temperature $y = T/(1+z)$ (in K) as a function of cosmic time $x = t$ (in second). You can use the approximation that $g_\star$ varies slowly.

(c) How long did it take from $T = 0.511$ MeV to $T = 1.40 T_\nu$? Write your answer both in second and in the Hubble time ($t_H = H^{-1}$) at $T = 0.511$ MeV.

2. (30 pts) [Chemical potential in the early Universe] In this problem, we shall estimate the chemical potential of electron. The chemical potential of a particle species can be calculated from the number density difference between electron and positron. In the local thermal equilibrium state, the number density difference between electron and positron is given by

$$n_e - n_\bar{e} = g_e \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{1}{e^{[\epsilon(p)+\mu_e]/T} - 1} - \frac{1}{e^{[\epsilon(p)+\mu_\bar{e}]/T} - 1} \right].$$

(1)

Use following cosmological parameters: $\Omega_b h^2 = 0.023$, $Y_p = 0.24$, and $T_{\text{cmb}} = 2.726$ K.

(a) Explain how $n_e - n_\bar{e}$ at early times are related to the electron number density we observe now, and calculate $(n_e - n_\bar{e})/s_\gamma$, where $s_\gamma$ is the total entropy density $s_\gamma = (\rho_{\text{total}} + P_{\text{total}})/T$ of electron, positron and photon.

(b) Plot the evolution of chemical potential $\mu_e$ (in keV) from $T = 10$ MeV to $T = 1$ keV, in the log-log scale.

(c) In the same temperature range as problem 2-(b), plot $n_e/s_\gamma$ and $n_\bar{e}/s_\gamma$, in the log-log-scale.