Solving problem sets is one of the most efficient ways of learning the subject. You are encouraged to collaborate with fellow students and/or to consult senior students, local postdocs and me. But, please write the solution by yourself.

Homework is due at the end of the class on 15 February 2019. No homework will be accepted after I post the solution on the course webpage (usually right after the class following the due date).

1. (30 pts) [Thermodynamics in an expanding universe] In the first problem set, you have shown that general relativistic energy momentum conservation \( T_{\nu;\mu} = 0 \) leads to

\[
\dot{\rho} + 3H(\rho + P) = 0, \tag{1}
\]

where \( \dot{\cdot} \) denotes the time derivative, and \( H = \dot{a}/a \) is the Hubble parameter. Using Eq. (1) and the results from the local thermal equilibrium calculations we have done in the class, you will find out the time evolution of the entropy density

\[
s = \frac{\rho + P - \mu n}{T} \tag{2}
\]

in this problem. Here, \( \mu \) and \( T \) are, respectively, and the chemical potential and the temperature, and \( n, \rho, P \) may be written as

\[
n = g \int \frac{d^3p}{(2\pi)^3} f_\mp, \quad \rho = g \int \frac{d^3p}{(2\pi)^3} \epsilon f_\mp, \quad P = g \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3} f_\mp, \tag{3}
\]

with the fermionic \((f_+)\) or bosonic \((f_-)\) phase space distribution functions:

\[
f_\pm = \frac{1}{e^{(\epsilon - \mu)/T} \mp 1} = \left[ \exp\left( \frac{\sqrt{p^2 + m^2} - \mu}{T} \right) \mp 1 \right]^{-1} \tag{4}
\]

(a) First, prove that the pressure can be rewritten as

\[
P = \pm gT \int \frac{d^3p}{(2\pi)^3} \ln(1 \pm f_\mp). \tag{5}
\]

This is an equation of state, generalizing the ideal gas law \( P \approx nT \) that can be recovered when \( f_\mp \ll 1 \). (Hint: Use integration by part.)

(b) Differentiating Eq. (5) to verify following identities:

\[
\frac{\partial P}{\partial \mu} = n, \quad \frac{\partial P}{\partial T} = s, \tag{6}
\]
and prove
\[ \frac{dP}{dT} = s + n \left[ \frac{\mu}{T} + T \frac{d}{dT} \left( \frac{\mu}{T} \right) \right]. \tag{7} \]

Note that in the expanding Universe, temperature \( T \) can be used as a *time* variable, and therefore all variables can be written as a function of \( T \).

(c) Combining Eq. (1), Eq. (2) and Eq. (7) and prove following time evolution equation for the comoving entropy \( s a^3 \):
\[ \frac{d(sa^3)}{dT} = -\frac{\mu}{T} \frac{d(a^3 n)}{dT}. \tag{8} \]

What are the conditions to be satisfied in order to conserve the comoving entropy of a particle species?

(d) What does Eq. (8) imply for the total comoving entropy of the Universe in thermal equilibrium? (*Hint: entropy is an extensive quantity.*)

2. (20 pts) [Standard model of particle physics] The attached plot in the next page shows temperature evolution of the relativistic degrees of freedom in the early Universe including only elementary particles in the standard model of particle physics.

(a) Annotate events that contribute notable changes of \( g_* \).

(b) Calculate the \( g_* \) and \( g_{*s} \) (if different from \( g_* \)) for every plateau that you see in the plot.