Cosmology (ASTRO/PHYS 545)  
Spring 2019  

Problem Set 11  
Due 17 April 2019

Solving problem sets is one of the most efficient ways of learning the subject. You are encouraged to collaborate with fellow students and/or to consult senior students, local postdocs and me. But, please write the solution by yourself.

Homework is due at the end of the class on 17 April 2019. No homework will be accepted after I post the solution on the course webpage (usually right after the class following the due date).

1. (30 pts) [Super-horizon scale evolution during RD and MD] In class, we show that the large-scale evolution of Bardeen’s gravitational potential is described by

\[ u'' - \frac{\theta''}{\theta} u = 0, \]  

where

\[ u = \frac{\Phi}{\sqrt{\bar{\rho} + \bar{P}}}, \quad \theta = \frac{1}{a} \sqrt{\frac{\bar{\rho}}{\bar{\rho} + \bar{P}}}. \]  

(a) show that the following combination

\[ \zeta \equiv -\frac{2}{3} \frac{\sqrt{3}}{\pi G} \theta^2 \left( \frac{u'}{\theta} \right) \]  

is indeed a constant of motion of this problem on large scales.

(b) Using the background equations, show that \( \zeta \) is indeed the curvature perturbation in the uniform density gauge:

\[ \zeta = \Psi - H \frac{\delta \rho}{\bar{\rho}}. \]  

(c) Following the evolution of the Bardeen’s potential in the onset of matter-radiation equality, and plot \( \Phi(y)/\Phi_0 \) as a function of \( y = a/a_{eq} \) from \( y = 0.01 \) to \( y = 100 \). Set \( \Phi(y) = \Phi_0 \) at the initial time.

(d) How does density contrast \( \delta \rho/\bar{\rho} \) in Poisson gauge evolve in time on large-scales during the radiation era and matter era?

(e) How does density contrast \( \delta \rho/\bar{\rho} \) in synchronous comoving gauge [as defined in the problem 1-(c) in homework 9] evolve in time on large-scales during the radiation era and matter era?

2. (20 pts) [Angular integration of tensor] Using that for any function \( f(\hat{k} \cdot \hat{p}) \), the integration identity

\[ \int \frac{d^2 \Omega_\phi}{4\pi} f(\hat{k} \cdot \hat{p}) \hat{p}_i \hat{p}_j = \mathcal{A} \delta_{ij} + \mathcal{B} \hat{k}_i \hat{k}_j, \]  

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must hold with A and B being independent of \( \hat{k} \), prove the following angular integration relations:

\[
\int \frac{d^2 \Omega_{\hat{p}'}}{4\pi} \frac{\Theta(k, \hat{p}', t)}{2} \hat{p}'_k \hat{p}'_m = \frac{1}{6} \delta_{km} \left[ \Theta_0(k, t) + \Theta_2(k, t) \right] - \frac{1}{2} \hat{k}_k \hat{k}_m \Theta_2(k, t),
\]

(6)

\[
\int \frac{d^2 \Omega_{\hat{p}'}}{4\pi} P_{km}(k, p, \hat{p}', t)
= - \left( \frac{d\bar{f}_y}{d\ln p} \right) \left[ -\frac{1}{6} \delta_{km} \left( \Theta_{p0}(k, t) + \Theta_{p2}(k, t) \right) + \frac{1}{2} \hat{k}_k \hat{k}_m \left( \Theta_{p0}(k, t) + \Theta_{p2}(k, t) \right) \right].
\]

(7)

Here,

\[
\Theta_\ell = \frac{1}{(-i)^\ell} \int \frac{d^2 \Omega_{\hat{p}}}{4\pi} \Theta(k, \hat{p}, t) L_\ell^\ell(\hat{k} \cdot \hat{p}),
\]

(8)

are the partial-wave (or Legendre-expansion) coefficients.