Classical $q$-Series Identities via Tilings

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**Pell Tilings**

**Definition**

A Pell tiling is a covering of an infinitely long board:

```
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ...
```

using three different types of tiles:
**Pell Tilings**

**Definition**
A Pell tiling is a covering of an infinitely long board:

```
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ...
```

using three different types of tiles:

![Tiles](image)

**Example**
```
...
```
**Pell Tilings**

**Definition**
A Pell tiling is a covering of an infinitely long board:

```
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ...  
```

using three different types of tiles:

- White square
- Black square
- Gray square

**Example**

```
[Black square] [White square] [White square] [White square] [White square] [White square] [White square] [White square] [White square] [White square] [White square] [White square] [White square] [White square] [White square] [White square] [White square] ... 
```
**Pell Tilings**

**Definition**
A Pell tiling is a covering of an infinitely long board:

\[1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ \ldots\]

using three different types of tiles:

![Tiles](image)

**Example**

![Tiling Example](image)
Pell Tilings

**Definition**
A Pell tiling is a covering of an infinitely long board:

```
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ...
```

using three different types of tiles:

- ![Tile 1](image1.png)
- ![Tile 2](image2.png)
- ![Tile 3](image3.png)

**Example**

- ![Example Tiling](example.png)
Pell Tilings

Definition

A Pell tiling is a covering of an infinitely long board:

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \ldots
\end{array}
\]

using three different types of tiles:

Example

\[
\begin{array}{cccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \ldots
\end{array}
\]
A Pell tiling is a covering of an infinitely long board:

\[
1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ \ldots
\]

using three different types of tiles:
**Pell Tilings**

**Definition**

A Pell tiling is a covering of an infinitely long board:

```
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ...
```

using three different types of tiles:

```
[Diagram showing three types of tiles]
```

**Example**

```
[Example diagram showing a Pell tiling]
```
**Pell Tilings**

**Definition**
A Pell tiling is a covering of an infinitely long board:

```
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ...
```

using three different types of tiles:

![Tiles](image)

**Example**

```
...
```

![Example](image)
A Pell tiling is a covering of an infinitely long board:

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \ldots \\
\end{array}
\]

using three different types of tiles:
Weighted Pell Tilings

The weight of tile $t$:

$$w(t) = \begin{cases} 
q^i & \text{if } t \text{ is a black square in position } i \\
zq^i & \text{if } t \text{ is a gray domino in position } i \\
1 & \text{if } t \text{ is a white square in position } i 
\end{cases}$$

The weight of tiling $T$:

$$w(T) = \prod_{t \in T} w(t)$$

Generating Function:

$$P(z; q) = \sum_{T \in \mathcal{T}} w(T)$$
By constructing tilings in two different ways, we obtained:

**Theorem** (Lebesgue)

\[
\sum_{n \geq 0} \frac{(-z; q)_n}{(q; q)_n} q^{n+1} = \prod_{n \geq 1} (1 + q^n)(1 + zq^{2n-1})
\]

where \((z; q)_n = (1 - z)(1 - zq) \cdots (1 - zq^{n-1})\).

**Theorem**

*The generating function for Pell tilings where at least \(m \geq 0\) white squares appear before the first domino, if any, is given by*

\[P(zq^m; q).\]
A Theorem of Euler

**Theorem**

\[
\prod_{n \geq 1} (1 + q^n) = \prod_{n \geq 1} \frac{1}{1 - q^{2n-1}}
\]

**Proof.**

Let \( z = -1 \) in the product side of Lebesgue:

\[
\prod_{n \geq 1} (1 + q^n)(1 - q^{2n-1})
\]
A Theorem of Euler

**Theorem**

\[
\prod_{n \geq 1} (1 + q^n) = \prod_{n \geq 1} \frac{1}{1 - q^{2n-1}}
\]

**Proof.**

Let \( z = -1 \) in the product side of Lebesgue:

\[
\prod_{n \geq 1} (1 + q^n)(1 - q^{2n-1})
\]

Find first occurrence of: 

- 😲□□□□ or □□□□□□
A Theorem of Euler

**Theorem**

\[
\prod_{n \geq 1} (1 + q^n) = \prod_{n \geq 1} \frac{1}{1 - q^{2n-1}}
\]

**Proof.**

Let \( z = -1 \) in the product side of Lebesgue:

\[
\prod_{n \geq 1} (1 + q^n)(1 - q^{2n-1})
\]

Find first occurrence of:  

\[
\begin{array}{c}
\text{or }
\end{array}
\]

and replace with:  

\[
\begin{array}{c}
\text{or }
\end{array}
\]

(respectively)
Theorem of Euler

**Theorem**

\[
\prod_{n \geq 1} (1 + q^n) = \prod_{n \geq 1} \frac{1}{1 - q^{2n-1}}
\]

**Proof.**

Let \( z = -1 \) in the product side of Lebesgue:

\[
\prod_{n \geq 1} (1 + q^n)(1 - q^{2n-1})
\]

Find first occurrence of: \( \begin{array}{|c|c|c|} \hline \text{black} & \text{white} & \text{white} \\ \hline \end{array} \) or \( \begin{array}{|c|c|c|} \hline \text{white} & \text{white} & \text{white} \\ \hline \end{array} \)

and replace with: \( \begin{array}{|c|c|c|} \hline \text{gray} & \text{gray} & \text{gray} \\ \hline \end{array} \) or \( \begin{array}{|c|c|c|} \hline \text{black} & \text{white} & \text{white} \\ \hline \end{array} \) (respectively)

Only remaining tiling consists of all white squares (weight 1).
A Theorem of Gauss

**Theorem**

\[ \sum_{n \geq 0} q^{(n+1)/2} = \prod_{n \geq 1} (1 + q^n)(1 - q^{2n}) \]

**Proof.**

Let \( z = -q \) in the product side of Lebesgue:

\[ \prod_{n \geq 1} (1 + q^n)(1 - q^{2n}) \]
A Theorem of Gauss

Theorem

\[ \sum_{n \geq 0} q^{\left(\frac{n+1}{2}\right)} = \prod_{n \geq 1} (1 + q^n)(1 - q^{2n}) \]

Proof.

Let \( z = -q \) in the product side of Lebesgue:

\[ \prod_{n \geq 1} (1 + q^n)(1 - q^{2n}) \]

Find first occurrence of: 

- ![Image](https://example.com/image1.png)
- ![Image](https://example.com/image2.png)
A Theorem of Gauss

**Theorem**

\[
\sum_{n \geq 0} q^{\left(\frac{n+1}{2}\right)} = \prod_{n \geq 1} (1 + q^n)(1 - q^{2n})
\]

**Proof.**

Let \( z = -q \) in the product side of Lebesgue:

\[
\prod_{n \geq 1} (1 + q^n)(1 - q^{2n})
\]

Find first occurrence of: 

\[
\begin{array}{c}
\text{or} \\
\end{array}
\]

and replace with: 

\[
\begin{array}{c}
\text{or} \\
\end{array}
\]

(respectively)
**Theorem**

\[ \sum_{n \geq 0} q^{\frac{n+1}{2}} = \prod_{n \geq 1} (1 + q^n)(1 - q^{2n}) \]

**Proof.**

Let \( z = -q \) in the product side of Lebesgue: \( \prod_{n \geq 1} (1 + q^n)(1 - q^{2n}) \)

Find first occurrence of: \( \text{ } \square \text{ } \square \text{ } \) or \( \text{ } \square \text{ } \text{ } \text{ } \)

and replace with: \( \text{ } \square \text{ } \text{ } \text{ } \) or \( \text{ } \text{ } \square \text{ } \text{ } \) (respectively)

Remaining tilings consist of black squares in positions 1 through \( n \) (weight \( q^{\frac{n+1}{2}} \)) for some \( n \geq 0 \) and white squares everywhere else.
A Generalization

**Theorem** For all $m \geq 0$,
\[
\sum_{n \geq 0} q^{\binom{n+1}{2}} \left[ \begin{array}{c} n + m - 1 \\ m - 1 \end{array} \right] = \prod_{n \geq 1} (1 + q^n)(1 - q^{2n+m-1})
\]

**Proof.**

Let $z = zq^m$ in the product side of Lebesgue: \( \prod_{n \geq 1} (1 + q^n)(1 + zq^{2n+m-1}) \)
(at least $m$ white squares appear before first domino.) Let $z = -1$. 

A Generalization

**Theorem** For all $m \geq 0$,

$$\sum_{n \geq 0} q^{\binom{n+1}{2}} \left[ \frac{n + m - 1}{m - 1} \right] = \prod_{n \geq 1} (1 + q^n)(1 - q^{2n+m-1})$$

**Proof.**

Let $z = zq^m$ in the product side of Lebesgue: $\prod_{n \geq 1} (1 + q^n)(1 + zq^{2n+m-1})$

(at least $m$ white squares appear before first domino.) Let $z = -1$.

Find first occurrence of:  

Find first occurrence of:  

or  

after $m$th white square.
**A Generalization**

**Theorem** For all $m \geq 0$,

$$\sum_{n \geq 0} q^{\binom{n+1}{2}} \left[ \binom{n + m - 1}{m - 1} \right] = \prod_{n \geq 1} (1 + q^n)(1 - q^{2n+m-1})$$

**Proof.**

Let $z = zq^m$ in the product side of Lebesgue: $\prod_{n \geq 1} (1 + q^n)(1 + zq^{2n+m-1})$

(at least $m$ white squares appear before first domino.) Let $z = -1$.

Find first occurrence of: $\blacklozenge\square$ or $\blacklozenge\blacklozenge\blacklozenge\square$ after $m$th white square.

and replace with: $\blacklozenge\blacklozenge\blacklozenge$ or $\blacklozenge\square\blacklozenge$ (respectively)
A Generalization

**Theorem** For all $m \geq 0$,

$$\sum_{n \geq 0} q^{\binom{n+1}{2}} \left( \begin{array}{c} n + m - 1 \\ m - 1 \end{array} \right) = \prod_{n \geq 1} (1 + q^n)(1 - q^{2n+m-1})$$

**Proof.**

Let $z = zq^m$ in the product side of Lebesgue: $\prod_{n \geq 1} (1 + q^n)(1 + zq^{2n+m-1})$

(at least $m$ white squares appear before first domino.) Let $z = -1$.

Find first occurrence of: ■□ or □□□ after $m$th white square.

and replace with: □□ or ■□□ (respectively)

Remaining tilings consist of $n$ black squares in positions 1 through $n + m - 1$ for some $n \geq 0$ and white squares everywhere else.
**Fibonacci Tilings**

The weight of tile $t$, denoted $w(t)$, is defined as follows:

$$w(t) = \begin{cases} 
0 & \text{if } t \text{ is a black square at position } i \\
 zq^i & \text{if } t \text{ is a gray domino at position } i \\
 1 & \text{if } t \text{ is a white square at position } i.
\end{cases}$$

The weight of a tiling $T$ is defined as

$$w(T) = \prod_{t \in T} w(t)$$

Generating Function:

$$F(z; q) = \sum_{T \in \mathcal{T}} w(T).$$
Theorem

\[ F(z; q) = \sum_{n \geq 0} \frac{z^n q^{n^2}}{(q; q)_n} \]
Theorem

\[ F(z; q) = \sum_{n \geq 0} \frac{z^n q^{n^2}}{(q; q)_n} \]

Theorem

The generating function for Fibonacci tilings where at least \( m \geq 0 \) white squares appear before the first domino, if any, is given by

\[ F(zq^m; q) . \]
## Domino Parity and Sliding

**Definition**

A domino is said to be even (odd) if its position is even (resp. odd).
Domino Parity and Sliding

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A domino is said to be even (odd) if its position is even (resp. odd).

Define a slide operation on dominoes that:

- can be performed on a domino that is followed by a white square,
- preserves number of even/odd dominoes, and
- increases the weight of the tiling by a factor of $q^2$. 
Domino Parity and Sliding

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Example
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**Example**

\[ \begin{array}{cccccccc}
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
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\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
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\hline
\end{array} \]
**Domino Parity and Sliding**

**Definition**
A domino is said to be even (odd) if its position is even (resp. odd).

Define a slide operation on dominoes that:
- can be performed on a domino that is followed by a white square,
- preserves number of even/odd dominoes, and
- increases the weight of the tiling by a factor of $q^2$.

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**Example**

```
[ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ]...
```

Arrow pointing to the right.
Domino Parity and Sliding

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**Example**

![Example Image]
**Domino Parity and Sliding**

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**Example**

![Example Image](image-url)
**Domino Parity and Sliding**

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Define a slide operation on dominoes that:

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- increases the weight of the tiling by a factor of $q^2$.

**Example**

```plaintext
e 

e 

e 

e 

e 

e 

e 

e 

e 
```

...
Domino Parity and Sliding

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Define a slide operation on dominoes that:

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- preserves number of even/odd dominoes, and
- increases the weight of the tiling by a factor of $q^2$.

Example

···
A Theorem of Sylvester

**Theorem**

\[
\sum_{n \geq 0} \frac{z^n q^{n^2}}{(q^2; q^2)_n} = \prod_{n \geq 1} (1 + zq^{2n-1})
\]
A Theorem of Sylvester

Theorem

\[ \sum_{n \geq 0} \frac{z^n q^{n^2}}{(q^2; q^2)_n} = \prod_{n \geq 1} (1 + zq^{2n-1}) \]

Proof.

RHS: Generating function for Fibonacci tilings that use only white squares and odd dominoes.
**A Theorem of Sylvester**

**Theorem**

\[ \sum_{n \geq 0} \frac{z^n q^{n^2}}{(q^2; q^2)_n} = \prod_{n \geq 1} (1 + zq^{2n-1}) \]

**Proof.**

RHS: Generating function for Fibonacci tilings that use only white squares and odd dominoes.

LHS: We can construct the same objects in the following manner:

1. Place \( n \) odd dominoes in positions 1, 3, 5, \ldots, \( 2n - 1 \).
2. Arbitrarily slide odd dominoes.

\[
\frac{1}{(1 - q^2)(1 - q^4) \cdots (1 - q^{2n})} = \frac{1}{(q^2; q^2)_n}
\]
Theorem

\[ \sum_{n \geq 0} \frac{z^n q^{n^2}}{(q; q)_n} = (-zq^2; q^2)_{\infty} \sum_{n \geq 0} \frac{z^n q^{n^2}}{(q^2; q^2)_n (-zq^2; q^2)_n} \]
Theorem

\[ \sum_{n \geq 0} \frac{z^n q^{n^2}}{(q; q)_n} = (-zq^2; q^2)_\infty \sum_{n \geq 0} \frac{z^n q^{n^2}}{(q^2; q^2)_n (-zq^2; q^2)_n} \]

Proof.

Count all Fibonacci tilings according to number of odd dominoes.

1. Place \( n \) odd dominoes in positions \( 1, 3, 5, \ldots, 2n - 1 \).
2. Arbitrarily place even dominoes in positions \( 2n + 2, 2n + 4, 2n + 6, \ldots \)

\[ z^n q^{n^2} \prod_{j \geq n+1} (1 + zq^{2j}) = (-zq^2; q^2)_\infty \frac{z^n q^{n^2}}{(-zq^2; q^2)_n} \]

3. Arbitrarily slide odd dominoes.
Rogers Identities, Part I

\[
\sum_{n \geq 0} \frac{z^n q^{n^2}}{(q; q)_n} = (-zq^2; q^2)_{\infty} \sum_{n \geq 0} \frac{z^n q^{n^2}}{(q^2; q^2)_n (-zq^2; q^2)_n}
\]

Setting \( z = 1 \) yields

\[
\sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n} = (-q^2; q^2)_{\infty} \sum_{n \geq 0} \frac{q^{n^2}}{(q^4; q^4)_n}
\]

Setting \( z = q \) yields

\[
\sum_{n \geq 0} \frac{q^{n^2+n}}{(q; q)_n} = (-q^3; q^2)_{\infty} \sum_{n \geq 0} \frac{q^{n^2+n}}{(q^2; q^2)_n (-q^3; q^2)_n}
\]

\[
= (-q; q^2)_{\infty} \sum_{n \geq 0} \frac{q^{n^2+n}}{(q^2; q^2)_n (-q; q^2)_{n+1}}
\]
Theorem

\[
\sum_{n \geq 0} \frac{z^n q^{n^2+n}}{(q; q)_n} = (-zq^2; q^2) \sum_{n \geq 0} \frac{z^n q^{n^2+2n}}{(q^2; q^2)_n (-zq^2; q^2)_n}
\]
**Theorem**

\[
\sum_{n \geq 0} \frac{z^n q^{n^2+n}}{(q; q)_n} = (-zq^2; q^2)\infty \sum_{n \geq 0} \frac{z^n q^{n^2+2n}}{(q^2; q^2)_n (-zq^2; q^2)_n}
\]

**Proof.**

LHS: \( F(zq; q) \) is generating function for weighted Fibonacci tilings that have a white square in position one.
**Theorem**

\[
\sum_{n \geq 0} \frac{z^n q^{n^2+n}}{(q; q)_n} = (-zq^2; q^2)_\infty \sum_{n \geq 0} \frac{z^n q^{n^2+2n}}{(q^2; q^2)_n (-zq^2; q^2)_n}
\]

**Proof.**

LHS: \(F(zq; q)\) is generating function for weighted Fibonacci tilings that have a white square in position one.

RHS: Count same collection of tilings based on number of odd dominoes.

1. Place \(n\) odd dominoes in positions 1, 3, 5, \ldots, \(2n - 1\).
2. Arbitrarily place even dominoes in positions \(2n + 2, 2n + 4, 2n + 6, \ldots\)

\[
z^n q^{n^2} \prod_{j \geq n+1} (1 + zq^{2j}) = (-zq^2; q^2)_\infty \frac{z^n q^{n^2}}{(-zq^2; q^2)_n}
\]

3. Arbitrarily slide every odd domino at least once.
Rogers Identities, Part II

\[
\sum_{n \geq 0} \frac{z^n q^{n^2+n}}{(q; q)_n} = \frac{(-zq^2; q^2)_\infty}{(q^2; q^2)_n} \sum_{n \geq 0} \frac{z^n q^{n^2+2n}}{(q^2; q^2)_n(-zq^2; q^2)_n}
\]

Setting \( z = 1 \) yields

\[
\sum_{n \geq 0} \frac{q^{n^2+n}}{(q; q)_n} = \frac{(-q^2; q^2)_\infty}{(q^2; q^2)_n(-q^2; q^2)_n} \sum_{n \geq 0} \frac{q^{n^2+2n}}{(q^2; q^2)_n(-q^2; q^2)_n}
\]

Setting \( z = 1/q \) yields

\[
\sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n} = \frac{(-q; q^2)_\infty}{(q^2; q^2)_n(-q; q^2)_n} \sum_{n \geq 0} \frac{q^{n^2+n}}{(q^2; q^2)_n(-q; q^2)_n}
\]
Signed Even Weighted Fibonacci Tilings

Weight tiles in the following manner:

\[
w(t) = \begin{cases} 
-zq^{2i} & \text{if } t \text{ is an even domino in position } 2i \\
zq^{2i} & \text{if } t \text{ is an odd domino in position } 2i - 1 \\
1 & \text{if } t \text{ is a white square covering position } i.
\end{cases}
\]
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\end{cases} \]

How does this affect sliding?

- Increasing position of domino by two increases its weight by \( q^2 \).
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\end{cases} \]

**How does this affect sliding?**

- Increasing position of domino by two increases its weight by \( q^2 \).
- Increasing position of odd domino by one does not change its weight.
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\end{cases}$$

**How does this affect sliding?**

- Increasing position of domino by two increases its weight by $q^2$.
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- Increasing position of even domino by one increases its weight by $q^2$. 
Signed Even Weighted Fibonacci Tilings

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\end{cases}$$

How does this affect sliding?

- Increasing position of domino by two increases its weight by $q^2$.
- Increasing position of odd domino by one does not change its weight.
- Increasing position of even domino by one increases its weight by $q^2$.

Thus, slide operation still increases the weight of a tiling by $q^2$. 
**Theorem**

\[
\sum_{n \geq 0} \frac{z^{2n} q^{4n^2 + 2n}}{(q^4; q^4)_n} = (zq^2; q^2) \infty \sum_{n \geq 0} \frac{z^n q^{n^2 + n}}{(q^2; q^2)_n (zq^2; q^2)_n}
\]

**Proof.**

Step I: RHS counts Fibonacci tilings according to new weights.

1. Place \( n \) odd dominoes in positions \( 1, 3, 5, \ldots, 2n - 1 \).
2. Arbitrarily place even dominoes in positions \( 2n + 2, 2n + 4, 2n + 6, \ldots \)

\[
z^n q^{2+4+\cdots+2n} \prod_{j \geq n+1} (1 - zq^2j) = (zq^2; q^2) \infty \frac{z^n q^{n^2 + n}}{(zq^2; q^2)_n}
\]

3. Arbitrarily slide odd dominoes
Theorem

\[
\sum_{n \geq 0} \frac{z^{2n} q^{4n^2 + 2n}}{(q_4 \cdot q_4)^n} = (zq^2; q^2)_\infty \sum_{n \geq 0} \frac{z^n q^{n^2 + n}}{(q^2 \cdot q^2)_n (zq^2; q^2)_n}
\]

Proof.

Step II: Cancel out terms via involution
Find first occurrence of:

\[
\boxed{\text{even } \# \text{ of consecutive odd dominoes}}
\]
THEOREM

\[
\sum_{n\geq 0} \frac{z^{2n} q^{4n^2+2n}}{(q^4; q^4)_n} = (zq^2; q^2)_{\infty} \sum_{n\geq 0} \frac{z^n q^{n^2+n}}{(q^2; q^2)_n (zq^2; q^2)_n}
\]

PROOF.

Step II: Cancel out terms via involution
Find first occurrence of:

\[
\square \quad \text{even \ # of consecutive odd dominoes} \quad \square \quad \square
\]

and replace with:

\[
\square \quad \text{even \ # of consecutive odd dominoes} \quad \square \quad \square \quad \square
\]

and vice versa.
**Theorem**

\[
\sum_{n \geq 0} \frac{z^{2n} q^{4n^2 + 2n}}{(q^4; q^4)_n} = (zq^2; q^2)_\infty \sum_{n \geq 0} \frac{z^n q^{n^2 + n}}{(q^2; q^2)_n (zq^2; q^2)_n}
\]

**Proof.**

Step II: Cancel out terms via involution
Find first occurrence of:

- even # of consecutive odd dominoes

and replace with:

- even # of consecutive odd dominoes

and vice versa.

Remaining tilings cannot have any even dominoes nor an odd number of consecutive odd dominoes.
**Theorem**

\[
\sum_{n \geq 0} \frac{z^{2n} q^{4n^2+2n}}{(q^4; q^4)_n} = (zq^2; q^2)_\infty \sum_{n \geq 0} \frac{z^n q^{n^2+n}}{(q^2; q^2)_n(zq^2; q^2)_n}
\]

**Proof.**

Step III: Count fixed points of involution
Fixed points consist of sequences of consecutive odd dominoes that contain an even number of dominoes.
**Theorem**

\[
\sum_{n \geq 0} \frac{z^{2n} q^{4n^2 + 2n}}{(q^4; q^4)_n} = (zq^2; q^2) \infty \sum_{n \geq 0} \frac{z^n q^{n^2 + n}}{(q^2; q^2)_n (zq^2; q^2)_n}
\]

**Proof.**

Step III: Count fixed points of involution
Fixed points consist of sequences of consecutive odd dominoes that contain an even number of dominoes.

1. Place \(2n\) odd dominoes in positions \(1, 3, 5, \ldots, 4n - 1\).

\[
z^{2n} q^{2+4+6+\cdots+4n} = z^{2n} q^{4n^2 + 2n}
\]

2. Arbitrarily slide odd dominoes in pairs

\[
\frac{1}{(1 - q^4)(1 - q^8) \cdots (1 - q^{4n})} = \frac{1}{(q^4; q^4)_n}
\]
Rogers Identities, Part III

\[
\sum_{n \geq 0} \frac{z^{2n} q^{4n^2 + 2n}}{(q^4; q^4)_n} = (zq^2; q^2)_\infty \sum_{n \geq 0} \frac{z^n q^{n^2 + n}}{(q^2; q^2)_n (zq^2; q^2)_n}
\]

Setting \( z = 1/q \) yields

\[
\sum_{n \geq 0} \frac{q^{4n^2}}{(q^4; q^4)_n} = (q; q^2)_\infty \sum_{n \geq 0} \frac{q^{n^2}}{(q^2; q^2)_n (q; q^2)_n}
\]

Setting \( z = q \) yields

\[
\sum_{n \geq 0} \frac{q^{4n^2 + 4n}}{(q^4; q^4)_n} = (q^3; q^2)_\infty \sum_{n \geq 0} \frac{q^{n^2 + 2n}}{(q^2; q^2)_n (q^3; q^2)_n}
\]

\[
= (q; q^2)_\infty \sum_{n \geq 0} \frac{q^{n^2 + 2n}}{(q^2; q^2)_n (q; q^2)_{n+1}}
\]
Rogers Identities, Part IV

\[ \sum_{n \geq 0} \frac{z^n q^{2n^2 + n}}{(q^2; q^2)_n} = (zq^2; q^2)_{\infty} \sum_{n \geq 0} \frac{z^n q^{(3n^2 + n)/2}}{(q; q)_n (zq^2; q^2)_n} \]

Setting \( z = 1/q \) yields

\[ \sum_{n \geq 0} \frac{q^{2n^2}}{(q^2; q^2)_n} = (q; q^2)_{\infty} \sum_{n \geq 0} \frac{q^{(3n^2 - n)/2}}{(q; q)_n (q; q^2)_n} \]

Setting \( z = q \) yields

\[ \sum_{n \geq 0} \frac{q^{2n^2 + 2n}}{(q^2; q^2)_n} = (q^3; q^2)_{\infty} \sum_{n \geq 0} \frac{q^{(3n^2 + 3n)/2}}{(q; q)_n (q^3; q^2)_n} \]

\[ = (q; q^2)_{\infty} \sum_{n \geq 0} \frac{q^{(3n^2 + 3n)/2}}{(q; q)_n (q; q^2)_{n+1}} \]
Rogers Identities, Recap

\[
\sum_{n \geq 0} \frac{z^n q^{n^2}}{(q; q)_n} = (-z q^2; q^2)_\infty \sum_{n \geq 0} \frac{z^n q^{n^2}}{(q^2; q^2)_n (-z q^2; q^2)_n}
\]

\[
\sum_{n \geq 0} \frac{z^n q^{n^2+n}}{(q; q)_n} = (-z q^2; q^2)_\infty \sum_{n \geq 0} \frac{z^n q^{n^2+2n}}{(q^2; q^2)_n (-z q^2; q^2)_n}
\]

\[
\sum_{n \geq 0} \frac{z^{2n} q^{4n^2+2n}}{(q^4; q^4)_n} = (z q^2; q^2)_\infty \sum_{n \geq 0} \frac{z^n q^{n^2+n}}{(q^2; q^2)_n (z q^2; q^2)_n}
\]

\[
\sum_{n \geq 0} \frac{z^n q^{2n^2+n}}{(q^2; q^2)_n} = (z q^2; q^2)_\infty \sum_{n \geq 0} \frac{z^n q^{(3n^2+n)/2}}{(q; q)_n (z q^2; q^2)_n}
\]
**Future Work**

**Theorem** *(Watson’s q-analog of Whipple’s theorem)*

\[
8 \phi_7 \left( \frac{z, q\sqrt{z}, -q\sqrt{z}, a, b, c, d, q^{-N}}{\sqrt{z}, -\sqrt{z}, zq/a, zq/b, zq/c, zq/d, zq^{N+1}; q, \frac{z^2 q^{N+2}}{abcd}} \right)
= \frac{(zq)_N (zq/cd)_N}{(zq/c)_N (zq/d)_N} 4 \phi_3 \left( \frac{zq/ab, c, d, q^{-N}}{zq/a, zq/b, cdq^{-N}/z; q, q} \right)
\]

Letting \(a, b, c, d, N \to \infty\) yields

**Theorem** *(Rogers and Ramanujan)*

\[
(zq; q)_\infty \sum_{n \geq 0} \frac{z^n q^{n^2}}{(q; q)_n} = 1 + \sum_{n \geq 1} \frac{(-1)^n (zq; q)_{n-1} (1 - zq^{2n}) z^{2n} q^{(5n^2 - n)/2}}{(q; q)_n}
\]
**Theorem (Rogers-Fine)**

\[
(1 - t) \sum_{n \geq 0} \frac{(a; q)_n t^n}{(b; q)_n} = \sum_{n \geq 0} \frac{(a; q)_n (atq/b)_n b^n t^n q^{n^2} - n(1 - atq^{2n})}{(b; q)_n (tq; q)_n}
\]

**Theorem (Jacobi Triple Product)**

\[
\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n \geq 1} (1 - q^{2n})(1 + zq^{2n-1})(1 + z^{-1}q^{2n-1})
\]

**Theorem (Gauss)**

\[
\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n \geq 1} \frac{(1 - q^n)}{(1 + q^n)}
\]
**Theorem (Cauchy)**

\[ 1 + \sum_{n \geq 1} \frac{(z; q)_n t^n}{(q; q)_n} = \prod_{n \geq 0} \frac{(1 - z t q^n)}{(1 - t q^n)} \]

**Theorem (Sylvester)**

\[ \sum_{n \geq 0} \frac{(-1)^n z^n q^{(3n^2+n)/2} (1 - z q^{2n+1}) (zq; q)_n}{(q; q)_n} = \prod_{n \geq 1} (1 - z q^n) \]