5 Exercises

5.1 Generating Functions .................................................. 84
5.2 \(q\)-Counting ............................................................... 84
5.3 Partitions ................................................................. 85
5.4 Combinatorial Species .................................................. 86
1 Generating Functions

1.1 Formal Power Series

Given a field $F$, let $F[[x]]$ denote the ring of formal power series given by

$$
\left\{ \sum_{n \geq 0} c_n x^n \mid c_n \in F \right\}.
$$

Given two formal power series $f = \sum_{i \geq 0} c_i x^i$ and $g = \sum_{j \geq 0} d_j x^j$

$$
f + g = \sum_{n \geq 0} (c_n + d_n) x^n
$$

$$
fg = \sum_{n \geq 0} \left( \sum_{k=0}^{n} c_k d_{n-k} \right) x^n
$$

Given an infinite sequence of formal power series, $f_i(x)$ for $i = 0, 1, 2, 3, \ldots$, define

$$
\sum_{i \geq 0} f_i(x) = \lim_{n \to \infty} \sum_{i=0}^{n} f_i(x)
$$

$$
\prod_{i \geq 0} f_i(x) = \lim_{n \to \infty} \prod_{i=0}^{n} f_i(x)
$$

We say that $\lim_{n \to \infty} g_i(x) = g(x)$ if

$$
\forall n \exists i_n \text{ such that } g_i(x)|_{x^n} = g(x)|_{x^n} \text{ for all } i \geq i_n
$$

where $g|_{x^n}$ denotes the coefficient of $x^n$ in $g(x)$.

**Example 1.1** Compute the coefficient of $x^k$ in $\prod_{n \geq 1} \frac{1}{1-x^n}$

$$
\prod_{n \geq 1} \frac{1}{1-x^n} \bigg|_{x^k} = \prod_{n=1}^{k} \frac{1}{1-x^n} \bigg|_{x^k}
$$

$$
= \prod_{n=1}^{k} (1 + x^n + x^{2n} + x^{3n} + \ldots) \bigg|_{x^k}
$$

$$
= \prod_{n=1}^{k} (1 + x^n + x^{2n} + x^{3n} + \ldots + x^{kn}) \bigg|_{x^k}
$$

$$
= \sum_{0 \leq a_1, a_2, \ldots, a_k \leq k} x^{a_1+2a_2+3a_3+\ldots+ka_k} \bigg|_{x^k}
$$

$$
= \# \text{ of ways of writing } k \text{ as a sum of integers}.
$$

**Remark 1.1** What distinguishes a formal power series from a power (or Taylor) series is the question of convergence. In a Taylor series, one must consider an interval of convergence. For what values of $x$ does the given series converge? A formal power series should not be viewed as a function, but rather a set of placeholders for a sequence of numbers. In other words, the variable should be thought of as an indeterminant and not something that we will substitute with a specific value. In this setting, the question of convergence boils down to being able to compute the coefficient of any term in a finite number of steps.
1.2 Ordinary and Exponential Generating Functions

**Definition 1.1** Given an infinite sequence of numbers $a_0, a_1, a_2, \ldots$, the corresponding (ordinary) generating function is given by

$$\sum_{n \geq 0} a_n x^n$$

and the corresponding exponential generating function is given by

$$\sum_{n \geq 0} a_n \frac{x^n}{n!}.$$

**Example 1.2** The geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n \geq 0} x^n$$

is the generating function for the sequence $a_n = 1$ for $n \geq 0$. It can also be thought of as the exponential generating function for the sequence $b_n = n!$ for $n \geq 0$. The exponential generating function for the sequence $a_n$ is given by

$$\sum_{n \geq 0} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots = e^x.$$

**Example 1.3** The generating function for the sequence $a_n = n$ for $n \geq 0$ is given by

$$\sum_{n \geq 0} nx^n = x + 2x^2 + 3x^3 + x^4 + \cdots = x(1 + 2x + 3x^2 + 4x^3 + \cdots)$$

$$= \frac{d}{dx} \left( \frac{1}{1-x} \right)$$

$$= \frac{x}{(1-x)^2}.$$

**Example 1.4** Given a set $A$ and a function $f : A \to \mathbb{N}$, the generating function for the sequence $a_n = |\{a \in A \mid f(a) = n\}|$ is given by

$$\sum_{n \geq 0} a_n x^n = \sum_{a \in A} x^{f(a)}.$$

**Remark 1.2** Even though we will refer to them as generating “functions”, in the following notes, these series should be viewed as formal power series.
2 q-Counting

2.1 Introduction

Given a set \( S \), the simplest way of counting the number of elements in \( S \) is to assign a weight of “1” to each object and then evaluate the sum

\[
|S| = \sum_{s \in S} 1.
\]

Not an incredibly interesting formula at first glance, but from this we can easily deduce the Sum and Product rules of counting. For example, if \( S \) breaks up into the disjoint union of two sets, say \( A \) and \( B \), then

\[
|S| = \sum_{s \in S} 1 = \sum_{a \in A} 1 + \sum_{b \in B} 1 = |A| + |B|.
\]

Or if \( S \) is the Cartesian product, \( A \times B \), then

\[
|S| = \sum_{s \in S} 1 = \sum_{a \in A} \sum_{b \in B} 1 = \left( \sum_{a \in A} 1 \right) \left( \sum_{b \in B} 1 \right) = |A| \times |B|.
\]

The basic idea behind q-counting is to allow for the possibility of assigning different weights to different objects while still maintaining the use of the Sum and Product rules. In particular, given a set \( S \) and a weight function \( w : S \to \mathbb{N} \), the following generating function q-counts the elements of \( S \) by weight.

\[
\sum_{s \in S} q^{w(s)}
\]

Notice that by simply letting \( q = 1 \), we recover \( |S| \). However, we are also able to easily determine how many elements of \( S \) have weight \( n \). In other words, we have

\[
\sum_{s \in S} q^{w(s)} = \sum_{n \geq 0} c_n q^n
\]

where \( c_n \) is the number of elements of \( S \) with weight \( n \). Compare the above equality with Example 1.4.

Example 2.1 Let \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in S_n \). We say that \( i \) is a fixed point of \( \sigma \) if \( \sigma_i = i \). The number of fixed points for each permutation in \( S_3 \) is

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>123</th>
<th>132</th>
<th>213</th>
<th>231</th>
<th>312</th>
<th>321</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of fixed points</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Therefore the generating function (in this case, a polynomial) that q-counts elements of \( S_3 \) by number of fixed points is

\[ 2 + 3q + q^3. \]

Example 2.2 Let \( S = \mathbb{N} \) and \( w(s) = s \). Then the generating function which q-counts \( S \) by weight is

\[ 1 + q^1 + q^2 + q^3 + q^4 + \cdots = \frac{1}{1 - q}. \]
2.2 The Binomial Theorem

**Theorem 2.1** For all positive integers \( n \),

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}
\]

where

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

- \# of \( k \) element subsets of an \( n \) element set
- \# of rearrangements of \( k \) 1’s and \( n-k \) 0’s

The formula involving \( |R(1^k0^{n-k})| \) can be rewritten in the following form

\[
n! = k!(n-k)!|R(1^k0^{n-k})|
\]

which suggests that there is a bijection between \( S_n \) and \( S_k \times S_{n-k} \times R(1^k0^{n-k}) \). In fact, this bijection is quite simple to explain. Given \((\alpha, \beta, r) \in S_k \times S_{n-k} \times R(1^k0^{n-k})\), let \( \sigma \) be the permutation in \( S_n \) formed by placing the numbers \( \alpha_1 + n - k, \alpha_2 + n - k, \ldots, \alpha_k + n - k \) from left to right in the positions specified by the locations of the \( k \) 1’s in \( r \). Now place the numbers \( \beta_1, \beta_2, \ldots, \beta_{n-k} \) in \( \sigma \) from left to right in the positions specified by the locations of the \( n - k \) 0’s in \( r \).

**Example 2.3** Let \( n = 9, k = 4 \) and \( \sigma = (8, 5, 2, 4, 9, 3, 7, 6, 1) \). Then the corresponding element of \( S_k \times S_{n-k} \times R(1^k0^{n-k}) \) is given by

\[
(3421, 52431, 100010110).
\]

**Remark 2.1** It will prove most worth while to give a graphical representation of \( r \in R(1^m0^n) \). To this end, think of \( r \) as a set of directions in the sense that a “0” instructs you to move one unit NORTH while a “1” instructs you to move one unit EAST. In other wards, an element of \( R(1^m0^n) \) can be thought of as a lattice path from the origin to the point \((m, n)\) using only steps NORTH and EAST.

**Example 2.4** The path corresponding to \( r = 011100101111 \) is given by

![Lattice Path Diagram]
2.3 q-Analogues

**Definition 2.1** For every positive integer \( n \), we define the following q-analogues:

\[
\begin{align*}
[n] &= 1 + q + q^2 + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}, \\
[n]! &= [n][n-1] \cdots [2][1] = \prod_{i=1}^{n} \frac{1 - q^i}{1 - q}, \\
\begin{bmatrix} n \end{bmatrix}_k &= \frac{[n]!}{[k]![n-k]!} = \frac{\prod_{i=1}^{n-k} (1 - q^i) \prod_{i=1}^{n-k} (1 - q^i)}{\prod_{i=1}^{k} (1 - q^i)}, \\
\lim_{q \rightarrow 1} [n] &= n
\end{align*}
\]

**Example 2.5**

\[
[5]! = (1 + q)(1 + q + q^2)(1 + q + q^2 + q^3)(1 + q + q^2 + q^3 + q^4) \\
= 1 + 4q + 9q^2 + 15q^3 + 20q^4 + 22q^5 + 20q^6 + 15q^7 + 9q^8 + 4q^9 + q^{10}
\]

**Example 2.6**

\[
\begin{bmatrix} 5 \\ 2 \end{bmatrix} = (1 - q)(1 - q^2)(1 - q^3)(1 - q^4)(1 - q^5) \\
= 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6
\]

**Lemma 2.2**

\[
\begin{align*}
\begin{bmatrix} n \end{bmatrix}_k &= q^k \begin{bmatrix} n - 1 \end{bmatrix}_k + \begin{bmatrix} n - 1 \end{bmatrix}_k \\
&= \begin{bmatrix} n - 1 \end{bmatrix}_k + q^{n-k} \begin{bmatrix} n - 1 \end{bmatrix}_k
\end{align*}
\]

**Proof.**

\[
q^k \begin{bmatrix} n - 1 \end{bmatrix}_k + \begin{bmatrix} n - 1 \end{bmatrix}_k = \frac{q^k[n-1]!}{[k]![n-1-k]!} + \frac{[n-1]!}{[k-1]![n-k]!} \\
&= \frac{[n-1]!}{[k]![n-k]!} (q^k[n-k] + [k]) \\
&= \frac{[n-1]!}{[k]![n-k]!} (q^k(1 + q + \cdots + q^{n-k-1}) + 1 + q + \cdots + q^{k-1}) \\
&= \frac{[n-1]!}{[k]![n-k]!} (1 + q + \cdots + q^{n-1}) \\
&= \frac{[n]!}{[k]![n-k]!} = \frac{n}{k}
\]

The second identity follows similarly or by making the replacement \( k \rightarrow n - k \). \( \square \)
2.4 Inversions

**Definition 2.2** Let \( a = a_1, a_2, \ldots, a_n \) be a sequence of integers. The set of inversions of \( a \) is given by

\[
\text{Inv}(a) =\{(a_i, a_j) \mid i < j \text{ and } a_i > a_j\}
\]

with \( \text{inv}(a) = |\text{Inv}(a)| \).

**Example 2.7** \( \sigma = (3, 4, 6, 2, 7, 5, 1) \in S_7 \)

\[
\text{Inv}(\sigma) = \{(3, 2), (3, 1), (4, 2), (4, 1)(6, 2), (6, 5), (6, 1), (2, 1), (7, 5), (7, 1)(5, 1)\}.
\]

**Example 2.8** Compute \( \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} \).

<table>
<thead>
<tr>
<th>\sigma</th>
<th>123</th>
<th>132</th>
<th>213</th>
<th>231</th>
<th>312</th>
<th>321</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{inv}(\sigma)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>\text{Inv}(\sigma)</td>
<td>\emptyset</td>
<td>{(3,2)}</td>
<td>{(2,1)}</td>
<td>{(2,1),(3,1)}</td>
<td>{(3,1),(3,2)}</td>
<td>{(3,2),(3,1),(3,2)}</td>
</tr>
</tbody>
</table>

\[
q^0 + 2q^1 + 2q^2 + q^3 = (1 + q)(1 + q + q^2) = [3]!
\]

**Theorem 2.3** For every integer \( n > 0 \), the generating function that \( q \)-counts elements of \( S_n \) by inversions is given by

\[
\sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} = [n]!.
\]

**Proof.** Proceed by induction on \( n \). First notice that

\[
[n + 1]! = [n + 1][n]! = (1 + q + q^2 + \cdots + q^n) \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} = \sum_{\sigma \in S_{n+1}} q^{\text{inv}(\sigma)}
\]

To prove the last equality, consider what happens to the \( \text{inv} \) statistic when you insert an \( n + 1 \) into a permutation \( \sigma \in S_n \). Since each of the numbers appearing after the \( n + 1 \) will count as an inversion, the increase in \( \text{inv} \) is simply the position of \( \sigma \) into which \( n + 1 \) is inserted, read from right to left.

In other words, we have a natural bijection between \((i, \sigma)\) and \(\sigma'\), where \(0 \leq i \leq n\), \(\sigma \in S_n\) and \(\sigma' \in S_{n+1}\). Given \((i, \sigma)\), construct \(\sigma'\) by inserting \(n + 1\) into \(\sigma\) so that there are exactly \(i\) numbers to it’s right. Given \(\sigma'\), form \(\sigma\) by simply removing \(n + 1\) from \(\sigma'\) and let \(i = n + 1 - j\) if \(\sigma'_j = n + 1\). To complete the proof, we remark that \(\text{inv}(\sigma') = \text{inv}(\sigma) + i\). \(\square\)
Example 2.9 Consider inserting the number 5 into the permutation \((4, 1, 3, 2)\). The possible spaces into which 5 may be placed is given below:

\[
\begin{array}{cccccc}
4 & 3 & 1 & 2 & 1 & 0 \\
\end{array}
\]

Each space is labeled by the number of inversions created by the 5 if 5 were inserted in that position.

Example 2.10 From Example 2.5, we see that there are 22 permutations in \(S_5\) that have exactly 5 inversions.

Example 2.11 Let \(r = 0111001011011 \in R(1^50^4)\). \(inv(r) = 4 + 4 + 2 + 1 + 1 = 16\), where we have counted the number of inversions caused by each of the 1s in \(r\).

Remark 2.2 Given \(r \in R(1^n0^n)\), each 1 yields exactly \(i\) inversions if there are \(i\) 0’s appearing to it’s right. In the context of lattice paths, this means that there are exactly \(i\) vertical line segments to the right of the horizontal step represented by the 1. In other words, the number of inversions caused by the \(j^{th}\) 1 in \(r\) is exactly the number of cells above the \(j^{th}\) horizontal line segment in the corresponding lattice path. Therefore, \(inv(r)\) is simply the area of the \(n \times m\) rectangle which lies above the path specified by \(r\). Compare Example 2.4 with Example 2.11.

Theorem 2.4 For every pair of integers \(0 \leq k \leq n\), the generating function that \(q\)-counts elements of \(R(1^k0^{n-k})\) by inversions is given by:

\[
\sum_{r \in R(1^k0^{n-k})} q^{\text{inv}(r)} = \binom{n}{k}.
\]

Proof. We proceed by induction on \(n\), the length of \(r\). Notice that the left hand side of the above identity can be broken up into the following two summations:

\[
LHS = \sum_{r \in R(1^k0^{n-k})} q^{\text{inv}(r)} + \sum_{r \in R(1^k0^{n-k})} q^{\text{inv}(r)}
\]

Note that if \(r = r'0\) where \(r' \in R(1^k0^{n-k-1})\), then \(\text{inv}(r) = \text{inv}(r') + k\) since each 1 in \(r'\) counts the final 0 in \(r\) as an inversion. On the other hand, if \(r = r'1\) where \(r' \in R(1^{k-1}0^{n-k})\) then \(\text{inv}(r) = \text{inv}(r')\) since the final 1 is never counted as an inversion. Therefore we have:

\[
LHS = \sum_{r' \in R(1^k0^{n-k-1})} q^{\text{inv}(r')+k} + \sum_{r' \in R(1^{k-1}0^{n-k})} q^{\text{inv}(r')}
\]

and by induction on \(n\) we have:

\[
LHS = q^k \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}.
\]
2.5 Major Index

**Definition 2.3** Let \( a = a_1, a_2, \ldots, a_n \) be a sequence of integers. The descent set of \( a \) is given by

\[
\text{Des}(a) = \{ i \mid a_i > a_{i+1} \}
\]

with \( \text{des}(a) = |\text{Des}(a)| \). The major index of \( a \) is given by

\[
\text{maj}(a) = \sum_{i \in \text{Des}(a)} i
\]

**Example 2.12** \( \sigma = (3, 4, 6, 2, 7, 5, 1) \in S_7 \). The descent set of \( \sigma \) is \( \{3, 5, 6\} \) with \( \text{des}(\sigma) = 3 \) and \( \text{maj}(\sigma) = 14 \).

**Example 2.13** Compute \( \sum_{\sigma \in S_3} q^{\text{maj}(\sigma)} \).

\[
\begin{array}{c|cccccc}
\sigma & 123 & 132 & 213 & 231 & 312 & 321 \\
\hline
\text{Des}(\sigma) & 0 & \{2\} & \{1\} & \{2\} & \{1\} & \{1, 2\} \\
\text{des}(\sigma) & 0 & 1 & 1 & 1 & 1 & 2 \\
\text{maj}(\sigma) & 0 & 2 & 1 & 2 & 1 & 3 \\
\end{array}
\]

\[q^0 + 2q^1 + 2q^2 + q^3 = (1 + q)(1 + q + q^2) = [3]!\]

**Theorem 2.5** For every integer \( n > 0 \), the generating function that \( q \)-counts elements of \( S_n \) by major index is given by

\[
\sum_{\sigma \in S_n} q^{\text{maj}(\sigma)} = [n]!
\]

**Proof.** Again we will proceed by induction on \( n \).

\[
[n + 1]! = [n + 1][n]! = (1 + q + q^2 + \cdots + q^n) \sum_{\sigma \in S_n} q^{\text{maj}(\sigma)}
\]

\[
= \sum_{\sigma \in S_{n+1}} q^{\text{maj}(\sigma)} \quad (2.1)
\]

To prove the last equality, consider what happens to the \( \text{maj} \) statistic when you insert an \( n+1 \) into a permutation \( \sigma \in S_n \). This change in the major index can be determined by labeling the spaces between the entries of \( \sigma \) (including before \( \sigma_1 \) and after \( \sigma_n \)) in the following manner:

1. **Label the last space with a 0.** Inserting an \( n+1 \) at the end of \( \sigma \) clearly does not introduce any new descents, and does not affect the location of the existing descents.
2. **Label the spaces after a descent with the numbers 1 through des(σ) from right to left.** Inserting an $n+1$ immediately after a descent, say in position $i$, shifts the index of each descent $j \geq i$ by exactly one. Therefore the increase in the major index is exactly the number of descents to the right of $i$, including $i$ itself.

\[
\tau = \sigma_1 \cdots \sigma_i n + 1 \sigma_{i+1} \cdots \sigma_n \quad \text{where } \sigma_i > \sigma_{i+1}
\]

\[
\text{maj}(\tau) = \text{maj}(\sigma) + |\{j \mid j \geq i \text{ and } j \in \text{Des}(\sigma)\}|
\]

3. **Label the remaining spaces with the numbers des(σ) + 1 through n from left to right.** Inserting an $n+1$ immediately after an ascent (i.e. a position which is not a descent), say in position $i$, introduces a new descent in position $i+1$ and shifts the indices of each descent to its right by exactly one. To see how this affects the major index, find two ascents $i$ and $j$ such that $i+1$ through $j-1$ are in Des(σ). Now compare what happens when $n+1$ is inserted after $\sigma_i$ resulting in $\alpha \in S_{n+1}$ versus when $n+1$ is inserted after $\sigma_j$ resulting in $\beta \in S_{n+1}$:

\[
\sigma = \sigma_1 \cdots \sigma_i < \sigma_{i+1} > \cdots > \sigma_j < \sigma_{j+1} \cdots \sigma_n
\]

\[
\alpha = \sigma_1 \cdots \sigma_i < n+1 > \sigma_{i+1} > \cdots > \sigma_j < \sigma_{j+1} \cdots \sigma_n
\]

\[
\beta = \sigma_1 \cdots \sigma_i < \sigma_{i+1} > \cdots > \sigma_j < n+1 > \sigma_{j+1} \cdots \sigma_n
\]

Notice that the descent sets of $\alpha$ and $\beta$ are identical except that Des($\alpha$) contains $j$ and not $j+1$ whereas Des($\beta$) contains $j+1$ and not $j$. Thus

\[
\text{maj}(\beta) = \text{maj}(\alpha) - j + j + 1 = \text{maj}(\alpha) + 1.
\]

With this labeling in mind, we now know how to increase the major index of $\sigma \in S_n$ by a specified amount between 0 and $n$, resulting in a permutation in $S_{n+1}$. In other words, we have a bijection between $(i, \sigma)$ and $\sigma'$, where $0 \leq i \leq n$, $\sigma \in S_n$ and $\sigma' \in S_{n+1}$. Given $(i, \sigma)$, construct $\sigma'$ by inserting $n+1$ into $\sigma$ so that the major index increases by exactly $i$. Given $\sigma'$, form $\sigma$ by removing $n+1$ from $\sigma'$ and let $i = \text{maj}(\sigma') - \text{maj}(\sigma)$. This is precisely the bijection that proves 2.1.

**Example 2.14** Consider inserting the number 8 into the permutation $(4, 6, 1, 7, 3, 5, 2)$. The possible spaces into which 8 may be placed is given below

\[
\begin{array}{cccccccc}
4 & 5 & 6 & 1 & 7 & 3 & 5 & 2 \\
4 & 5 & 6 & 1 & 7 & 3 & 5 & 2
\end{array}
\]

Each space is labeled by how much the major index would increase if 8 were inserted into that position.
Remark 2.3 Combining Theorem 2.3 and Theorem 2.5 yields
\[
\sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} = \sum_{\sigma \in S_n} q^{\text{maj}(\sigma)}
\]
which suggests that there is a bijection \( \phi : S_n \to S_n \) such that
\[
\text{inv}(\sigma) = \text{maj}(\phi(\sigma)).
\]
Since we now know how to build up both the inversion statistic and the major index, it should be a trivial matter to construct such a bijection. In fact, all we need to do is specify at each stage how much we want to increase each statistic. Suppose we want to construct two permutations, one built up according to its inversion statistic and the other built up according to its major index. If we build each one using the sequence \( \{\Delta_1, \ldots, \Delta_n\} \), where \( 0 \leq \Delta_i \leq i \) represents how much we wish to increase the corresponding statistic in going from a permutation in \( S_i \) to \( S_{i+1} \), then ultimately both permutations will have their statistic equal to \( \Delta_1 + \Delta_2 + \ldots + \Delta_n \). An example should suffice in making this concept crystal clear. Given the sequence \( \{0, 2, 2, 1, 5, 3\} \), build up \( \sigma \) according to its inversion statistic and build up \( \phi(\sigma) \) according to its major index. The following table illustrates this process.

<table>
<thead>
<tr>
<th>( \Delta )</th>
<th>( \sigma )</th>
<th>0</th>
<th>2</th>
<th>2</th>
<th>1</th>
<th>5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi(\sigma) )</td>
<td>1</td>
<td>12</td>
<td>312</td>
<td>3412</td>
<td>34152</td>
<td>634152</td>
<td>6347152</td>
</tr>
</tbody>
</table>

Note that \( \text{inv}(6347152) = 0 + 2 + 2 + 1 + 5 + 3 = \text{maj}(4713652) \).

Example 2.15 Let \( r = 0111001101101 \in \mathcal{R}(1^50^4) \), \( \text{maj}(r) = 4 + 7 + 10 = 21 \)

Example 2.16 Compute \( \sum_{r \in \mathcal{R}(1^20^2)} q^{\text{maj}(r)} \).

\[
\begin{array}{cccccccc}
r & 0011 & 0101 & 0110 & 1001 & 1010 & 1100 \\
\text{Des}(r) & 0 & \{2\} & \{3\} & \{1\} & \{1,3\} & \{2\} \\
\text{des}(r) & 0 & 1 & 1 & 1 & 2 & 1 \\
\text{maj}(r) & 0 & 2 & 3 & 1 & 4 & 2 \\
\end{array}
\]

\[
q^0 + q^1 + 2q^2 + q^3 + q^4 = (1 + q^2)(1 + q + q^2) = \left[ \begin{array}{c} 4 \\ 2 \end{array} \right]
\]

Theorem 2.6 For every pair of integers \( 0 \leq k \leq n \), the generating function that \( q \)-counts elements of \( \mathcal{R}(1^k0^{n-k}) \) by major index is given by
\[
\sum_{r \in \mathcal{R}(1^k0^{n-k})} q^{\text{maj}(r)} = \left[ \begin{array}{c} n \\ k \end{array} \right]
\]
Proof. In this case, we will immediately present a bijection $\varphi : \mathcal{R}(1^k 0^{n-k}) \to \mathcal{R}(1^k 0^{n-k})$ such that

$$\text{inv}(\varphi(r)) = \text{maj}(r).$$

This together with Theorem 2.4 proves our result. The bijection can be defined recursively in the following manner

$$\varphi(r) = \begin{cases} 
\varphi(r')1 & \text{if } r = r'1 \\
0'1\varphi(r')0 & \text{if } r = r'10^{l+1}, \ l \geq 0
\end{cases}$$

with initial conditions $\varphi(1^n) = 1^n$ and $\varphi(0^n) = 0^n$. In the first case, if $r = r'1$, then $r$ cannot possibly have a descent in the next to last position and so $\text{maj}(r) = \text{maj}(r')$. Similarly, the last 1 causes no inversions in $r$ so $\text{inv}(r) = \text{inv}(r')$. This explains why $\varphi(r) = \varphi(r')1$ in this case. This concept is illustrated below.

In the second case, if $r = r'10^{l+1}$, then $r$ has a descent in position $n-l-1$ and therefore $\text{maj}(r) = \text{maj}(r') + n-l-1$. If $r = 0'1r'0$, then the first 1 causes exactly $n-k-l$ inversions and the last 0 causes exactly $k-1$ inversions with $r'$. Therefore $\text{inv}(r) = \text{inv}(r') + n-l-1$. This explains why $\varphi(r) = 0'1\varphi(r')0$ in this case.

Example 2.17 From Example 2.6, we see that there are 2 rearrangements of 2 1s and 3 0s whose major index is 3.

Example 2.18 In the diagrams below, the elements of $\text{Des}(r)$ label the corresponding horizontal line segments. The area above the path of $\varphi(r)$ is broken up according to the.descents of $r$.  

$\varphi(r) = 1101101001011$
2.6 Foata’s Bijection

Theorems 2.3 through 2.6 can easily be seen to be specific cases of a much more general theorem. For a given \( \alpha = (\alpha_1, \ldots, \alpha_n) \), let \( R_\alpha = R(1^{\alpha_1} \cdots n^{\alpha_n}) \) denote the set of rearrangements of \( \alpha_1 \) ones, \( \alpha_2 \) twos, \( \ldots \), and \( \alpha_n \) n’s. Often times we will refer to \( r \in R_\alpha \) as a word and specific numbers in \( r \) as letters.

**Theorem 2.7** For all sequences of positive integers \( \alpha = (\alpha_1, \ldots, \alpha_n) \),

\[
\sum_{r \in R_\alpha} q^{\text{inv}(r)} = \left[ \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{\alpha_1, \alpha_2, \ldots, \alpha_n} \right] = \sum_{r \in R_\alpha} q^{\text{maj}(r)}
\]  

(2.2)

where the \( q \)-multinomial coefficient is given by

\[
\left[ \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{\alpha_1, \alpha_2, \ldots, \alpha_n} \right] = \frac{\alpha_1! \alpha_2! \cdots \alpha_n!}{\alpha_1! \alpha_2! \cdots \alpha_n!}.
\]

**Proof.** The left hand side can be broken up according to the first letter of \( r \), namely

\[
LHS = \sum_{r \in R_\alpha} q^{\text{inv}(r)} = \sum_{i=1}^{n} \sum_{\substack{r \in R_\alpha \\text{starts with an } i}} q^{\text{inv}(r)}.
\]

Notice that if \( r = ir' \) where \( r' \in R(1^{\alpha_1} \cdots i^{\alpha_i-1} \cdots n^{\alpha_n}) \) then

\[
\text{inv}(r) = \text{inv}(r') + \alpha_1 + \alpha_2 + \cdots + \alpha_{i-1}
\]

since each \( 1, 2, \ldots, i-1 \) in \( r' \) will count the initial \( i \) in \( r \) as an inversion. Therefore

\[
LHS = \sum_{i=1}^{n} \sum_{r' \in R(1^{\alpha_1} \cdots i^{\alpha_i-1} \cdots n^{\alpha_n})} q^{\text{inv}(r') + \alpha_1 + \alpha_2 + \cdots + \alpha_{i-1}}
\]

\[
= \sum_{i=1}^{n} q^{\alpha_1 + \alpha_2 + \cdots + \alpha_{i-1}} \left[ \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n - 1}{\alpha_1, \alpha_2, \ldots, \alpha_i - 1, \ldots, \alpha_n} \right]
\]

\[
= \left[ \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{\alpha_1, \alpha_2, \ldots, \alpha_n} \right]
\]

The proof of the last equality is similar in flavor in the proof of Lemma 2.2 and is left to the reader. To complete the proof of the theorem, we will demonstrate a bijection, \( \phi \), from \( R_\alpha \) to itself such that

\[
\text{maj}(r) = \text{inv}(\phi(r)).
\]

(2.3)

This bijection is a generalization of the proof of the previous theorem.
**Definition 2.4** Let \( r \in \mathcal{R}_n \) and assume that the last letter of \( r \) is \( y \). The \( x \)-factorization of \( r \) breaks up into the following two cases. If \( y > x \) then the \( x \)-factorization of \( r \) is given by

\[
r = r_1y_1|r_2y_2|\cdots|r_ky_k|
\]

where each \( r_i \) is a word consisting of letters less than or equal to \( x \) and each \( y_i \) is a letter greater than \( x \). If \( y \cdot x \) then the \( x \)-factorization of \( r \) is given by

\[
r = r_1y_1|r_2y_2|\cdots|r_ky_k|
\]

where each \( r_i \) is a word consisting of letters greater than \( x \) and each \( y_i \) is a letter less than or equal to \( x \). Note that in each case, \( r_i \) is allowed to be the empty word.

**Example 2.19** Let \( r = 652134768563124 \). Then the 3-factorization of \( r \) is given by

\[
6|5|2134|7|6|8|5|6|3124|
\]

and the 5-factorization of \( r \) is given by

\[
65|2|1|3|4|7685|63|1|2|4|.
\]

The next component of our bijection is to define the following function based on the \( x \)-factorization of a word.

\[
\gamma_x(r) = \begin{cases} 
  r & \text{if } r = \emptyset \\
  y_1r_1y_2r_2\cdots y_kr_k & \text{if the } x \text{-factorization of } r \text{ is } r_1y_1|r_2y_2|\cdots|r_ky_k.
\end{cases}
\]

**Example 2.20** Let \( r = 652134768563124 \). Then

\[
\gamma_3(r) = 6|5|4213|7|6|8|5|6|4312 \\
\gamma_5(r) = 56|2|1|3|4|5768|36|1|2|4|.
\]

Finally, we define our bijection \( \varphi \), recursively as

\[
\varphi(r) = \begin{cases} 
  r & \text{if } |r| \leq 1 \\
  \gamma_x(\varphi(r'))x & \text{if } r = r'x.
\end{cases}
\]

**Example 2.21** Compute \( \varphi(423412) \):

\[
\varphi(4) = 4 \\
\varphi(42) = \gamma_2(\varphi(4))2 = 42 \\
\varphi(423) = \gamma_3(\varphi(42))3 = \gamma_3(42)3 = 243 \\
\varphi(4234) = \gamma_4(\varphi(423))4 = \gamma_4(243)4 = 2434 \\
\varphi(42341) = \gamma_1(\varphi(4234))1 = \gamma_1(2434)1 = 24341 \\
\varphi(423412) = \gamma_2(\varphi(42341))2 = \gamma_2(24341)2 = 214342.
\]

Notice that \( \text{maj}(423412) = 5 = \text{inv}(214342) \).
Our next step is to verify equation 2.3. To this end, we hold the following truths to be self-evident:

**Lemma 2.8** Let \( w \in X^* \) and \( x \in X \). Define \( r_x(w) = \) number of letters of \( w \) which are greater than \( x \) and \( l_x(w) = \) number of letters of \( w \) which are less than or equal to \( x \).

1. \( \text{inv}(wx) = \text{inv}(w) + r_x(w) \)
2. \( \text{inv}(\gamma_x(w)) = \text{inv}(w) - r_x(w) \) if the last letter of \( w \) is \( \leq x \).
3. \( \text{inv}(\gamma_x(w)) = \text{inv}(w) + l_x(w) \) if the last letter of \( w \) is \( > x \).
4. \( \text{maj}(wx) = \text{maj}(w) \) if the last letter of \( w \) is \( \leq x \).
5. \( \text{maj}(wx) = \text{maj}(w) + |v| \) if the last letter of \( w \) is \( > x \).

We proceed by induction on the number of letters in \( w \). If \( w = vx \) where \( v \in X^* \) and \( x \in X \) then we are forced into dealing with the following two cases.

**Case 1** The last letter of \( v \) is \( \leq x \)

\[
\text{inv}(\varphi(vx)) = \text{inv}(\gamma_x(\varphi(v))x) \\
= \text{inv}(\gamma_x(\varphi(v))) + r_x(\gamma_x(\varphi(v))) \\
= \text{inv}(\varphi(v)) - r_x(v) + r_x(v) \\
= \text{inv}(\varphi(v)) \\
= \text{maj}(v) \text{ by induction} \\
= \text{maj}(vx) \text{ since last letter of } v \text{ is } \leq x
\]

**Case 2** The last letter of \( v \) is \( > x \)

\[
\text{inv}(\varphi(vx)) = \text{inv}(\gamma_x(\varphi(v))x) \\
= \text{inv}(\gamma_x(\varphi(v))) + r_x(\gamma_x(\varphi(v))) \\
= \text{inv}(\varphi(v)) + l_x(v) + r_x(v) \\
= \text{inv}(\varphi(v)) + |v| \\
= \text{maj}(v) + |v| \text{ by induction} \\
= \text{maj}(vx) \text{ since last letter of } v \text{ is } > x
\]

The final step is to verify that \( \varphi \) is in fact a bijection. This is left as an exercise for the reader. \( \square \)
2.7 Catalan Words

**Definition 2.5** We say that \( r \in \mathcal{R}(1^n0^n) \) is a Catalan word if every prefix of \( r \) contains at least as many 0’s as it does 1’s. The set of all such Catalan words is denoted by \( \mathcal{C}_n \). In terms of lattice paths, these words represent paths from \((0,0)\) to \((n,n)\) using only steps NORTH and EAST that remain weakly above the line \( y = x \).

**Example 2.22** Let \( r = 001100101101 \) and \( r' = 011001110010 \). From the following diagrams, we can easily see that \( r \) is a Catalan word while \( r' \) is not.

![Diagram](image)

**Theorem 2.9** (MacMahon) The generating function that \( q \)-counts Catalan words by major index is given by

\[
\sum_{r \in \mathcal{C}_n} q^{\text{maj}(r)} = \frac{1}{[n+1]} \binom{2n}{n}
\]

**Proof.** Using Theorem 2.6, we know that

\[
\binom{2n}{n} = \sum_{r \in \mathcal{R}(1^n0^n)} q^{\text{maj}(r)} = \sum_{r \in \mathcal{C}_n} q^{\text{maj}(r)} + \sum_{r \notin \mathcal{C}_n} q^{\text{maj}(r)}
\]

and thus

\[
\sum_{r \in \mathcal{C}_n} q^{\text{maj}(r)} = \binom{2n}{n} - \sum_{r \notin \mathcal{C}_n} q^{\text{maj}(r)}.
\]

We will now focus our attention on non-Catalan words. As we will show, these paths are in one-to-one correspondence with arbitrary paths from \((0,0)\) to \((n - 1, n + 1)\). More specifically, we will present a bijection \( \phi : \mathcal{R}(1^n0^n) - \mathcal{C}_n \to \mathcal{R}(1^{n-1}0^n+1) \) such that

\[\text{maj}(\phi(r)) = \text{maj}(r) - 1.\]
To begin, given a word \( r \in \mathcal{R}(1^n0^n) - \mathcal{C}_n \) we identify the “first deepest horizontal segment” of \( r \) to be the minimum index \( j \) such that the \( j^{th} \) letter of \( r \) is a “1” and the difference
\[
\# \text{ of 1s in positions 1 through } j \text{ of } r - \# \text{ of 0s in positions 1 through } j \text{ of } r
\]
is maximized. Visually, this is the first horizontal segment that is furthest to the right of the line \( y = x \). Form \( \phi(r) \) by replacing \( r \)’s first deepest horizontal segment by a 0. Clearly \( \phi(r) \) is a path from \((0,0)\) to \((n-1,n+1)\). To reverse the process, identify the “last deepest vertical segment” of \( \phi(r) \) and replace it with a 1.

**Example 2.23** Conversion between elements of \( \mathcal{R}(1^n0^n) - \mathcal{C}_n \) and \( \mathcal{R}(1^{n-1}0^{n+1}) \). The first deepest horizontal segment and last deepest vertical segment are circled.

\[
\begin{align*}
\sum_{r \in \mathcal{C}_n} q^{maj(r)} &= \binom{2n}{n} - \sum_{r \in \mathcal{C}_n} q^{maj(r)} \\
&= \binom{2n}{n} - \sum_{r \in \mathcal{C}_n} q^{maj(\phi(r))} \\
&= \binom{2n}{n} - \sum_{r \in \mathcal{R}(1^{n-1}0^{n+1})} q^{maj(r)+1} \\
&= \binom{2n}{n} - q \left[ \frac{2n}{n-1} \right] = \frac{1}{n+1} \binom{2n}{n}
\end{align*}
\]

Among other things, this says the cardinality of \( \mathcal{C}_n \) is given by \( \frac{1}{n+1} \binom{2n}{n} \), the \( n^{th} \) Catalan number.
2.8 The $q$-Binomial Theorem

**Theorem 2.10**

$$(x + y)(x + qy)(x + q^2 y) \cdots (x + q^{n-1} y) = \sum_{k=0}^{n} q^{\binom{n-k}{2}} \binom{n}{k} x^k y^{n-k}$$

**Proof.** It suffices to show that

$$(x + y)(x + qy)(x + q^2 y) \cdots (x + q^{n-1} y) \bigg|_{x^k y^{n-k}} = q^{\binom{n-k}{2}} \binom{n}{k}$$

or equivalently

$$q^{\binom{n-k}{2}} (x + y)(x + qy)(x + q^2 y) \cdots (x + q^{n-1} y) \bigg|_{x^k y^{n-k}} = \binom{n}{k}$$

We can think of each weakly increasing sequence $0 \leq j_1 \leq j_2 \leq \cdots \leq j_{n-k} \leq k$ as defining a path from $(0,0)$ to $(k,n-k)$ such that the area above the path is given by $j_1 + j_2 + \cdots + j_{n-k}$. In light of Remark 2.2, the area above the path is exactly the same as the inversions of the path. Therefore we have

$$\sum_{0 \leq j_1 \leq j_2 \leq \cdots \leq j_{n-k} \leq k} q^{j_1 + j_2 + j_3 + \cdots + j_{n-k}} = \sum_{r \in \mathcal{R}(1^n \cup \cdot k)} q^{\text{inv}(r)}$$

and the result follows from Theorem 2.4. \qed
3 Partitions

3.1 Introduction

The general problem in the theory of partitions is to enumerate representations of a positive integer \( n \) as the sum

\[
n = a_1 + a_2 + \cdots + a_k
\]

where each \( a_i \) comes from a multiset of integers. A partition of \( n \) is a representation of \( n \) as a sum of integers where the order of the terms (or parts) is irrelevant. Therefore we can think of a partition as a weakly decreasing sequence of numbers.

**Definition 3.1** Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) be a weakly decreasing sequence of nonnegative integers. The number of positive terms of \( \lambda \) is called the *length* of \( \lambda \) and is denoted \( l(\lambda) \). The *size* of \( \lambda \), denoted by \( |\lambda| \), is the sum \( \lambda_1 + \cdots + \lambda_k \). If \( |\lambda| = n \) then \( \lambda \) is said to be a partition of \( n \), denoted \( \lambda \vdash n \).

The Ferrers diagram corresponding to a partition \( \lambda \) is a graphical representation of \( \lambda \). To construct the Ferrers diagram for \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), simply place \( \lambda_{i+1} \) blocks on top of \( \lambda_i \) blocks, for each \( i = 1, 2, \ldots, k - 1 \).

**Example 3.1** The Ferrers diagram for the partition \( \lambda = (6, 4, 3, 1, 1) \) is

\[
\begin{array}{cccccc}
\square & & & & & \\
\square & & & & & \\
\square & \square & \square & & & \\
\square & \square & \square & \square & & \\
\square & \square & \square & \square & \square & \\
\end{array}
\]

By the *conjugate partition* of \( \lambda \), denoted \( \lambda' \), we are referring to the partition whose \( j^{th} \) component is the number of parts of \( \lambda \) that are at least \( j \). In terms of the Ferrers diagram of \( \lambda \), \( \lambda' \) simply lists the number of blocks in each column and is formed by reflecting the Ferrers diagram of \( \lambda \) across the line \( y = x \).

**Example 3.2** The conjugate partition corresponding to \( \lambda = (6, 4, 3, 1, 1) \) is \( \lambda' = (4, 3, 3, 2, 1, 1) \) and its Ferrers diagram is

\[
\begin{array}{cccccc}
\square & \square & & & & \\
\square & \square & \square & \square & & \\
\square & \square & \square & \square & \square & \\
\square & & & & & \\
\square & & & & & \\
\end{array}
\]
The main tool we will use to $q$-count partitions by size is the geometric series
\[
\frac{1}{1-q} = 1 + q + q^2 + q^3 + \cdots \\
\frac{1}{1-q^n} = 1 + q^n + q^{2n} + q^{3n} + \cdots
\]
We can think of these series as building up Ferrers diagrams by rows
\[
\frac{1}{1-q} = 1 + \square + \square + \cdots \\
\frac{1}{1-q^n} = 1 + \square + \square + \cdots
\]
or by columns
\[
\frac{1}{1-q} = 1 + \square + \square + \cdots \\
\frac{1}{1-q^n} = 1 + \square + \square + \cdots
\]

**Theorem 3.1** The generating function that $q$-counts partitions by size is
\[
\sum_{n \geq 0} p(n)q^n = \prod_{n \geq 1} \frac{1}{1-q^n}
\]
where $p(n)$ is the number of partitions of $n$.

**Proof #1.** See Example 1.1. □

**Proof #2.**
\[
\prod_{n \geq 1} \frac{1}{1-q^n} = \frac{1}{1-q} \frac{1}{1-q} \frac{1}{1-q} \cdots
\]
\[
= \left( \sum_{n_1 \geq 0} \square \right) \left( \sum_{n_2 \geq 0} \square \right) \left( \sum_{n_3 \geq 0} \square \right) \cdots
\]
\[
= \sum_{n_1 \geq 0} \sum_{n_2 \geq 0} \sum_{n_3 \geq 0} \cdots
\]
\[
= \sum_{\lambda} q^{\lambda_1}
\]
\[
= \sum_{n \geq 0} \sum_{\lambda \vdash n} q^n
\]
\[
= \sum_{n \geq 0} p(n)q^n
\]
□
3.2 Restricted Parts

Often times we will be concerned with partitions whose parts satisfy certain conditions.

**Example 3.3** The generating function for partitions with at most $n$ parts is given by
\[
\frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}
\]

**Example 3.4** The generating function for partitions with exactly $n$ parts is given by
\[
\frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)}
\]

**Example 3.5** The generating function for partitions with at least $n$ distinct parts is given by
\[
\frac{q^{\binom{n}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)}
\]

**Example 3.6** The generating function for partitions with exactly $n$ distinct parts is given by
\[
\frac{q^{\binom{n+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)}
\]

**Example 3.7** The generating function for partitions with only odd parts is given by
\[
\frac{1}{(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots} = \prod_{n \geq 1} \frac{1}{1-q^{2n-1}} = \sum_{\lambda \text{ only odd parts}} q^{\lambda}
\]

**Example 3.8** The generating function for partitions with distinct parts is given by
\[
\prod_{n \geq 1} (1+q^n) = \sum_{\lambda \text{ no repeated parts}} q^{\lambda}
\]

**Example 3.9** The generating function for partitions with no part repeated more than twice is given by
\[
\prod_{n \geq 1} (1+q^n+q^{2n}) = \sum_{\lambda \text{ no part repeated more than twice}} q^{\lambda}
\]
Example 3.10 The generating function for partitions that fit in an $n \times m$ box is given by

$$\left[ \frac{n + m}{m} \right] = \sum_{\lambda \text{ at most } n \text{ parts} \atop \text{each part at most } m} q^{\lambda}$$

Given any generic subset, $S$, of positive integers, we can just as easily construct partitions that use only these parts. Consider the following examples:

Example 3.11 Let $S = \{1, 5, 10, 25, 50\}$. The generating function for partitions whose parts are elements of $S$ is given by

$$\frac{1}{(1-q)(1-q^5)(1-q^{10})(1-q^{25})(1-q^{50})}$$

Example 3.12 For any set $S \subseteq \{1, 2, 3, \ldots\}$, the generating function for partitions whose parts are elements of $S$ is given by

$$\prod_{n \in S} \frac{1}{1-q^n} = \sum_{\lambda \text{ parts in } S} q^{\lambda}$$

Example 3.13 For any set $S \subseteq \{1, 2, 3, \ldots\}$, the generating function for partitions whose parts are distinct elements of $S$ is given by

$$\prod_{n \in S} (1 + q^n) = \sum_{\lambda \text{ parts in } S \atop \text{no repeated parts}} q^{\lambda}$$
**Theorem 3.2** (Euler) For every integer \( n \geq 0 \) the number of partitions of \( n \) into distinct parts is equal to the number of partitions of \( n \) with only odd parts. Symbolically,

\[
\prod_{n \geq 1} (1 + q^n) = \prod_{n \geq 1} \frac{1}{1 - q^{2n-1}}.
\]

**Proof #1.**

\[
\prod_{n \geq 1} (1 + q^n) = \prod_{n \geq 1} \frac{(1 - q^n)(1 + q^n)}{1 - q^n} = \prod_{n \geq 1} \frac{1 - q^{2n}}{1 - q^n} = \prod_{n \geq 1} \frac{1}{1 - q^{2n-1}}.
\]

**Proof #2.** Glaisher (1883)

Let \( \lambda = (\lambda_1, \ldots, \lambda_k) \) be a partition of \( n \) with distinct parts. Construct a partition \( \mu \vdash n \) with only odd parts in the following manner. Write each \( \lambda_i = (2k + 1)2^l \) for unique \( k \) and \( l \). Let \( \mu \) have \( 2^l \) parts equal to \( 2k + 1 \). To reverse the process, assume that \( \mu \) has \( \alpha \) parts of length \( 2k + 1 \). Write \( \alpha \) in its binary expansion

\[
\alpha = \alpha_0 2^0 + \alpha_1 2^1 + \cdots + \alpha_m 2^m.
\]

**Example 3.14** Another way to describe Glaisher’s bijection is, starting with a partition with only odd parts, repeatedly combine parts of the same size, until only only distinct parts remain.
3.3 Number of Parts

Without much additional effort, we can also keep track of the length of the partitions.

Example 3.15

\[
\prod_{n \geq 1} \frac{1}{1 - zq^n} = \sum_{\lambda} z^{t(\lambda)} q^{|\lambda|}
\]

\[
\prod_{n \geq 1} (1 + zq^n) = \sum_{\lambda: \text{no repeated parts}} z^{t(\lambda)} q^{|\lambda|}
\]

**Theorem 3.3 (q-Binomial Theorem)**

\[
(1 + z)(1 + zq)(1 + zq^2) \cdots (1 + zq^{n-1}) = \sum_{k=0}^{n} \binom{n}{k} q^{k} z^{k}
\]

where \( \lambda \) is any partition with at most \( k \) parts and largest part at most \( n - k \).

**Theorem 3.4**

\[
\frac{1}{(1 - z)(1 - zq)(1 - zq^2) \cdots (1 - zq^{n-1})} = \sum_{k \geq 0} \binom{n + k - 1}{k} z^{k}
\]

where \( \lambda \) is any partition with at most \( k \) parts and largest part at most \( n - 1 \).
3.4 Durfee Square

Theorem 3.5 (Euler)

\[
\prod_{n \geq 1} \frac{1}{1 - q^n} = 1 + \sum_{m \geq 1} \frac{q^{m^2}}{(1 - q) \cdots (1 - q^m)^2}
\]

where \( \lambda \) is an arbitrary partition whose largest part is at most \( m \) and \( \mu \) is an arbitrary partition with at most \( m \) parts.

\[
\prod_{n \geq 1} \frac{1}{1 - q^n} = 1 + \sum_{m \geq 1} \frac{z^mq^{m^2}}{(1 - q)(1 - zq^2)(1 - zq^4) \cdots (1 - zq^{2m})}
\]

Theorem 3.6 (Euler) For every integer \( n \geq 0 \) the number of partitions of \( n \) into distinct odd parts is equal to the number of self-conjugate partitions of \( n \). Symbolically,

\[
\prod_{n \geq 1} (1 + q^{2n - 1}) = 1 + \sum_{m \geq 1} \frac{q^{m^2}}{(1 - q^2)(1 - q^4) \cdots (1 - q^{2m})}
\]

where \( \lambda \) is an arbitrary partition with at most \( m \) parts

\[
\prod_{n \geq 1} (1 + zq^{2n - 1}) = 1 + \sum_{m \geq 1} \frac{z^mq^{m^2}}{(1 - zq^2)(1 - zq^4) \cdots (1 - zq^{2m})}
\]
3.5 Distinct Parts

**Theorem 3.7 (Euler)**

\[
\prod_{n \geq 1} (1 + q^n) = 1 + \sum_{m \geq 1} \frac{q_{\frac{m+1}{2}}}{(1 - q) \cdots (1 - q^m)}
\]

where \( \lambda \) is an arbitrary partition with at most \( m \) parts.

\[
\prod_{n \geq 1} (1 + zq^n) = 1 + \sum_{m \geq 1} \frac{z^m q_{\frac{m+1}{2}}}{(1 - q) \cdots (1 - q^m)}
\]

**Theorem 3.8 (Sylvester)**

\[
\prod_{n \geq 1} (1 + q^n) = 1 + \sum_{m \geq 1} q^{\frac{m^2 + m}{2}} (1 + q^{2m}) \frac{(1 + q)(1 + q^2) \cdots (1 + q^{m-1})}{(1 - q)(1 - q^2) \cdots (1 - q^m)}
\]

where \( \lambda \) is an arbitrary partition with distinct parts whose largest part is at most \( m - 1 \) and \( \mu \) is an arbitrary partition with at most \( m \) distinct parts.

\[
\prod_{n \geq 1} (1 + zq^n) = 1 + \sum_{m \geq 1} z^m q^{\frac{m^2 + m}{2}} (1 + zq^{2m}) \frac{(1 + zq)(1 + zq^2) \cdots (1 + zq^{m-1})}{(1 - q)(1 - q^2) \cdots (1 - q^m)}
\]
3.6 Sylvester’s Decomposition of Jacobi Triple Product

**Theorem 3.9**

\[
\prod_{n \geq 1} (1 - q^{2n})(1 + z q^{2n-1})(1 + q^{2n-1}/z) = 1 + \sum_{n \geq 1} q^n (z^n + z^{-n}) = \sum_{n=\infty} \sum_{n=-\infty} q^n z^n
\]

Start by making the following replacements \( z \to z/q \) and \( q \to \sqrt[q]{q} \). This yields

\[
\prod_{n \geq 1} (1 - q^n)(1 + z q^{n-1})(1 + q^n/z) = \sum_{n=\infty} \sum_{n=-\infty} z^n q^{(1)}
\]

which can be rewritten in the following manner

\[
\prod_{n \geq 1} \frac{1}{(1 - q^n)} \sum_{\lambda} q^{\lambda} = \prod_{n \geq 1} (1 + z q^{n-1})(1 + q^n/z)
\]

\[
\left( \sum_{\lambda} q^{\lambda} \right) \left( \sum_{-\infty} z^n q^{(1)} \right) = \left( \sum_{\lambda \in D^*} z^{l(\lambda)} q^{\lambda} \right) \left( \sum_{\lambda \in \mathcal{D}} z^{-l(\lambda)} q^{\lambda} \right)
\]

where \( D \) denotes the set of partitions with distinct nonzero parts and \( D^* \) denotes the set of partitions with distinct parts allowing for a single part of size 0.

Need a bijection between \((\lambda, n)\) and \((\alpha, \beta)\) where \( \lambda \) is an arbitrary partition, \( n \) is any integer, \( \alpha \in D^* \) and \( \beta \in \mathcal{D} \).

**Case 1** \( n \geq 0 \)

**Subcase 1a:** last cell of extended diagonal is at the end of a column

\[
(\lambda, n) = (\lambda, n-1) \leftrightarrow (\alpha, \beta) = (\alpha, \beta)
\]
Subcase 1b: last cell of extended diagonal is \textit{not} at the end of a column

\[
(\lambda, n) = \begin{array}{c}
\lambda \\
n-1
\end{array} \quad \longleftrightarrow \quad (\alpha \cup \{0\}, \beta)
\]

Case 2 \( n < 0 \)

Subcase 2a: last cell of extended diagonal is \textit{at} the end of a column

\[
(\lambda, n) = \begin{array}{c}
\lambda \\
-n
\end{array} \quad \longleftrightarrow \quad (\alpha \cup \{0\}, \beta)
\]

Subcase 2b: last cell of extended diagonal is \textit{not} at the end of a column

\[
(\lambda, n) = \begin{array}{c}
\lambda \\
-n
\end{array} \quad \longleftrightarrow \quad (\alpha, \beta)
\]
3.7 Euler’s Pentagonal Number Theorem

**Theorem 3.10**

\[
\prod_{n \geq 1} (1 - q^n) = \sum_{m = -\infty}^{\infty} (-1)^m q^{\frac{3m^2 - m}{2}}
\]

**Proof #1.** Using Jacobi’s Triple Product

Starting with the formula

\[
\prod_{n \geq 1} (1 - q^n)(1 + zq^{n-1})(1 + q^n/z) = \sum_{n = -\infty}^{\infty} z^n q^{\binom{n}{2}}
\]

make the replacements \( q \to q^3 \) and \( z \to -q \)

\[
\prod_{n \geq 1} (1 - q^{3n})(1 - q^{3n-2})(1 - q^{3n-1}) = \sum_{n = -\infty}^{\infty} (-1)^n q^{3\binom{n}{2} + n}
\]

\[\square\]

**Proof #2.** Using Sylvester’s Identity with \( z = -1 \)

\[
\prod_{n \geq 1} (1 - q^n) = 1 + \sum_{m \geq 1} (-1)^m q^{\frac{3m^2 - m}{2}} \frac{1 - q^{2m}}{1 - q^m} \frac{(1 - q)(1 - q^2) \cdots (1 - q^{m-1})}{(1 - q)(1 - q^2) \cdots (1 - q^m)}
\]

\[
= 1 + \sum_{m \geq 1} (-1)^m q^{\frac{3m^2 - m}{2}} \frac{1 - q^{2m}}{1 - q^m}
\]

\[
= 1 + \sum_{m \geq 1} (-1)^m q^{\frac{3m^2 - m}{2}} (1 + q^m)
\]

\[
= 1 + \sum_{m \geq 1} (-1)^m q^{\frac{3m^2 - m}{2}} + \sum_{m \geq 1} (-1)^m q^{\frac{3m^2 + m}{2}} = \sum_{m = -\infty}^{\infty} (-1)^m q^{\frac{3m^2 - m}{2}}
\]

\[\square\]
3.7.1 Franklin’s Involution

Proof #3. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and assume that $\lambda$ starts with $s(\lambda)$ consecutive integers
Define a sign-reversing involution in the following manner

Case 1: If $s(\lambda) < k$ and $s(\lambda) < \lambda_k$

\[
\begin{array}{c}
\begin{array}{cccccccc}
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\end{array}
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\begin{array}{cccccccc}
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\end{array}
\end{array}
\]

or $s(\lambda) = k$ and $k < \lambda_k - 1$

\[
\begin{array}{c}
\begin{array}{cccccccc}
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\end{array}
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\begin{array}{cccccccc}
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\end{array}
\end{array}
\]

Case 2: If $s(\lambda) < k$ and $s(\lambda) \geq \lambda_k$

\[
\begin{array}{c}
\begin{array}{cccccccc}
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\end{array}
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\begin{array}{cccccccc}
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\end{array}
\end{array}
\]

or $s(\lambda) = k$ and $k \geq \lambda_k + 1$

\[
\begin{array}{c}
\begin{array}{cccccccc}
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\end{array}
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\begin{array}{cccccccc}
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\end{array}
\end{array}
\]

Case 3: Everything else is a fixed point. This includes $s(\lambda) = k$ and $k = \lambda_k$ or $k = \lambda_k - 1$

\[
\begin{array}{c}
\begin{array}{cccccccc}
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\end{array}
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\begin{array}{cccccccc}
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\end{array}
\end{array}
\]
3.7.2 An Extension of Franklin’s Involution

Using Sylvester’s Identity with \( z = -q^k \) for \( k \geq 0 \) yields

\[
\prod_{n>k} (1 - q^n) = 1 + \sum_{m \geq 1} (-1)^m q^{\frac{3m^2 - m}{2} + mk} (1 - q^{2m+k}) \frac{(1 - q^{k+1})(1 - q^{k+2}) \cdots (1 - q^{k+m-1})}{(1-q)(1-q^2) \cdots (1-q^m)}
\]

\[
= 1 + \sum_{m \geq 1} (-1)^m q^{\frac{3m^2 - m}{2} + mk} \frac{(1 - q^{2m+k})}{(1-q^k)} \left[ m + k - 1 \right]
\]

Define \( k \)-landing staircase

\[
\text{3-landing staircase} \quad \text{4-landing staircase} \quad \text{5-landing staircase}
\]

Define an involution on partitions with distinct parts greater than \( k \) in a manner similar to Franklin’s involution. In other words, move the \( k \)-landing staircase on top of the diagram, if possible, or move the top row of the diagram along side the \( k \)-landing staircase\(^1\), if possible. The fixed points are as follows

\[
\begin{array}{c}
\lambda \\
m \\
m \\
\lambda^* \\
m \\
k \\
k \\
k \\
m-1 \\
m \\
k \\
\end{array}
\]

where \( \lambda \) is any partition that fits in an \( m \times k \) box and \( \lambda^* \) is any partition that fits in an \( m \times k \) box with largest part equal to \( k \). Therefore the fixed points with exactly \( m \) rows is given by

\[
(-1)^m q^{m^2 + \binom{m}{2} + km} \left[ \frac{m+k}{m} \right] + (-1)^m q^{m^2 + \binom{m}{2} + km + m+k} \left[ \frac{m-1+k}{m} \right]
\]

\[
= (-1)^m q^{m^2 + \binom{m}{2} + km} \left[ \frac{m+k}{m} \right] + q^{m+k} \left[ \frac{m-1+k}{m} \right] - q^{m+k} \left[ \frac{m+1+k}{m} \right]
\]

\[
= (-1)^m q^{m^2 + \binom{m}{2} + km} \left[ \frac{m+k}{m} \right] + q^{m+k} \left[ \frac{m-1+k}{m} \right] - q^{m+k} \left[ \frac{m+1+k}{m} \right]
\]

\[
= (-1)^m q^{m^2 + \binom{m}{2} + km} \left[ \frac{m+k}{m} \right] + q^{m+k} \left[ \frac{m-1+k}{m} \right]
\]

\[
= (-1)^m q^{m^2 + \binom{m}{2} + km} \left[ \frac{m+k}{m} \right] + q^{m+k} \left[ \frac{m-1+k}{m} \right]
\]

\[
= (-1)^m q^{m^2 + \binom{m}{2} + km} \left[ \frac{m+k}{m} \right] (1 - q^{2m+k})
\]

\[
= (-1)^m q^{m^2 + \binom{m}{2} + km} \left[ \frac{m+k}{m} \right] \frac{1 - q^{2m+k}}{1 - q^k}
\]

\(^1\)Complete details regarding how this is done are available upon request.
3.7.3 Recursive formula for the number of partitions of $n$.

Euler’s Pentagonal Number Theorem implies that

$$1 = \prod_{n \geq 1} \frac{1}{1 - q^n} \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{3m^2 - m}{2}}$$

$$= \left( \sum_{n \geq 0} p(n)q^n \right) (1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots)$$

Take the coefficient of $q^n$ for $n > 0$ on both sides

$$0 = p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - \cdots$$

$$+ (-1)^m p \left( n - \frac{3m^2 - m}{2} \right) + (-1)^m p \left( n - \frac{3m^2 + m}{2} \right) + \cdots$$

and solve for $p(n)$.

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + \cdots$$

$$+ (-1)^{m-1} p \left( n - \frac{3m^2 - m}{2} \right) + (-1)^{m-1} p \left( n - \frac{3m^2 + m}{2} \right) + \cdots$$

Example 3.16

$$p(0) = 1$$
$$p(1) = p(0) = 1$$
$$p(2) = p(1) + p(0) = 2$$
$$p(3) = p(2) + p(1) = 3$$
$$p(4) = p(3) + p(2) = 5$$
$$p(5) = p(4) + p(3) - p(0) = 7$$
$$p(6) = p(5) + p(4) - p(1) = 11$$
$$p(7) = p(6) + p(5) - p(2) - p(0) = 15$$
3.8 Rogers-Ramanujan Identities

**Theorem 3.11** The number of partitions of \( n \) whose parts differ by at least two is equal to the number of partitions of \( n \) whose parts are equivalent to 1 or 4 mod 5. Symbolically,

\[
\prod_{n \geq 1} \frac{1}{(1 - q^{5n-1})(1 - q^{5n-4})} = \sum_{m \geq 0} \frac{q^{m^2}}{(1 - q)(1 - q^2) \cdots (1 - q^m)}. \tag{3.1}
\]

**Proof.** Schur (1917)

Multiplying both sides of (3.1) by \( \prod (1 - q^n) \) and using Jacobi’s triple product yields

\[
\prod_{n \geq 1} (1 - q^n) \prod_{n \geq 1} \frac{1}{(1 - q^{5n-1})(1 - q^{5n-4})} = \prod_{n \geq 1} (1 - q^{5n})(1 - q^{5n-2})(1 - q^{5n-3})
\]

\[
= \sum_{m = -\infty}^{\infty} (-1)^m q^{\frac{5m^2 - m}{2}} \tag{3.2}
\]

\[
= \left( \prod_{n \geq 1} 1 - q^n \right) \left( \sum_{m \geq 0} \frac{q^{m^2}}{(1 - q) \cdots (1 - q^m)} \right)
\]

Notice that the right hand side suggests looking at ordered pairs of partitions \((\lambda, \mu)\) where \( \lambda \) is a partition with distinct parts weighted by \((-1)^{(\lambda)} q^{\mid \lambda \mid}\) and \( \mu \) is a partition whose parts differ by at least two weighted by \( q^{\mid \mu \mid}\).

The set of all such pairs of partitions can be broken up into the following disjoint sets

\[
A = \{ (\lambda, \mu) \mid \lambda_1 > \mu_1 + 1 \text{ or } \lambda_1 < \mu_1 \} \\
B = \{ (\lambda, \mu) \mid \lambda_1 = \mu_1 \} \\
C = \{ (\lambda, \mu) \mid \lambda_1 = \mu_1 + 1 \}
\]

**Case #1** \((\lambda, \mu) \in A\)

\[
\lambda_1 > \mu_1 + 1:
\]

\[
\lambda_1 < \mu_1:
\]
Case #2 \((\lambda, \mu) \in B \cup C\)

Let \(\lambda = (\lambda_1, \ldots, \lambda_k)\) and \(\mu = (\mu_1, \ldots, \mu_l)\). Furthermore, assume that \(\lambda\) starts with \(s(\lambda)\) integers that differ by exactly one and that \(\mu\) starts with \(d(\mu)\) integers that differ by exactly two.

\[
\begin{align*}
B_1 &= \{(\lambda, \mu) \in B \mid \lambda_k \leq s(\lambda) \text{ and } \lambda_k \leq d(\mu)\} \\
B_2 &= \{(\lambda, \mu) \in B \mid d(\mu) < \lambda_k \text{ and } d(\mu) \leq s(\lambda)\} \\
B_3 &= \{(\lambda, \mu) \in B \mid s(\lambda) < \lambda_k \text{ and } s(\lambda) \leq d(\mu)\}
\end{align*}
\]

\[
\begin{align*}
C_1 &= \{(\lambda, \mu) \in C \mid \lambda_k \leq s(\lambda) \text{ and } \lambda_k \leq d(\mu)\} \\
C_2 &= \{(\lambda, \mu) \in C \mid s(\lambda) < \lambda_k \text{ and } s(\lambda) \leq d(\mu)\} \\
C_3 &= \{(\lambda, \mu) \in C \mid d(\mu) < \lambda_k \text{ and } d(\mu) < s(\lambda)\}
\end{align*}
\]

Subcase 2a: \((\lambda, \mu) \in B_1 \cup C_2\)

\[
\begin{align*}
&\text{fixed points of } B_1 : \\
&m \begin{cases} \text{m} \\ \text{m} \end{cases} \begin{cases} \text{m} \\ \text{m-1} \end{cases}
\end{align*}
\]

\[
\begin{align*}
&\text{fixed points of } C_2 : \\
&m \begin{cases} \text{m} \\ \text{m} \end{cases} \begin{cases} \text{m} \\ \text{2m-1} \end{cases}
\end{align*}
\]

The combined weight of the above fixed points is

\[
(-1)^m q^{m^2 + \binom{m}{2} + m^2} + (-1)^m q^{m^2 + \binom{m+1}{2} + m^2} = (-1)^m q^{(5m^2 - m)/2} + (-1)^m q^{(5m^2 + m)/2}
\]

which are precisely the terms showing up in 3.2.
Subcase 2b: \((\lambda, \mu) \in B_2 \cup C_1\)

Subcase 2c: \((\lambda, \mu) \in B_3 \cup C_3\)

The second of the Rogers-Ramanujan identities can be stated and proved in a similar manner.

**Theorem 3.12** The number of partitions of \(n\) whose parts are at least two and differ by at least two is equal to the number of partitions of \(n\) whose parts are equivalent to 2 or 3 mod 5. Symbolically,

\[
\prod_{n \geq 1} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})} = \sum_{m \geq 0} \frac{q^{m^2+m}}{(1-q)(1-q^2) \cdots (1-q^m)}.
\]
Theorem 3.13

\[ 1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \frac{q^4}{1 + \cdots}}}} = \prod_{n \geq 0} \frac{(1 - q^{3n+2})(1 - q^{5n+3})}{(1 - q^{5n+1})(1 - q^{5n+4})} \]

Proof. Let

\[ F(z) = 1 + \sum_{m \geq 1} \frac{z^m q^{m^2}}{(1 - q)(1 - q^2) \cdots (1 - q^m)} \]

which satisfies

\[ F(z) = F(zq) + zq F(zq^2) \]

The first term accounts for all those partitions whose smallest part is at least 2 and the second term accounts for all those partitions whose smallest part is exactly 1. Thus the ratio \( c(z) = \frac{F(z)}{F(zq)} \) satisfies

\[ c(z) = 1 + \frac{zq}{c(zq)} \]

Iterating yields

\[ c(z) = 1 + \frac{zq}{1 + \frac{zq^2}{c(zq^2)}} = 1 + \frac{zq}{1 + \frac{zq^2}{1 + \frac{zq^3}{c(zq^3)}}} = \cdots \]

Therefore

\[ 1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \frac{q^4}{1 + \cdots}}}} = c(1) = \frac{F(1)}{F(q)} = \frac{\sum_{m \geq 0} q^{n^2}}{\sum_{m \geq 0} q^{m^2 + m}} \]

and the result follows from the Rogers-Ramanujan identities.

\[ \square \]

Remark 3.1 Taking the limit as \( q \) tends to 1 yields

\[ \frac{1 + \sqrt{5}}{2} = \frac{2}{1} \cdot \frac{3}{4} \cdot \frac{7}{6} \cdot \frac{8}{9} \cdot \frac{12}{11} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{18}{19} \cdots \]
### 3.9 Involution Principle

Let $A$ and $B$ be finite sets such that

\[ A = A^+ \cup A^- \quad \text{and} \quad A^+ \cap A^- = \emptyset \]
\[ B = B^+ \cup B^- \quad \text{and} \quad B^+ \cap B^- = \emptyset \]

Let $\alpha : A \to A$ and $\beta : B \to B$ be a sign-reversing bijections with positive fixed points. That is, let

\[ F_{\alpha} = \{ a \in A \mid \alpha(a) = a \} \subseteq A^+ \quad F_{\beta} = \{ b \in B \mid \beta(b) = b \} \subseteq B^+ \]

be the set of fixed points of $\alpha$ and $\beta$ respectively. Then

\[ a \notin F_{\alpha} \Rightarrow \begin{cases} \alpha(a) \in A^- & \text{if } a \in A^+ \\ \alpha(a) \in A^+ & \text{if } a \in A^- \end{cases} \]
\[ b \notin F_{\beta} \Rightarrow \begin{cases} \beta(b) \in B^- & \text{if } b \in B^+ \\ \beta(b) \in B^+ & \text{if } b \in B^- \end{cases} \]

Lastly, let $f : A \to B$ be a sign-preserving bijection:

\[ f(A^+) = B^+ \quad \text{and} \quad f(A^-) = B^- \]

Now, starting with $a \in F_{\alpha}$, repeatedly apply the functions $\alpha$, $\beta$ and $f$ in the following manner:

\[ a \in F_{\alpha} \Rightarrow f(a) \xrightarrow{\in B^+} \beta(f(a)) \xrightarrow{\in A^-} f^{-1}(\beta(f(a))) \xrightarrow{\in A^+} \alpha(f^{-1}(\beta(f(a)))) \xrightarrow{\in B^-} \cdots \xrightarrow{\in B^-} b \in F_{\beta} \]

where $b$ is the first element of $F_{\beta}$ to appear.

**Theorem 3.14 (Garsia-Milne)** The above map is a bijection between $F_{\alpha}$ and $F_{\beta}$

**Proof.** Since $\alpha$, $\beta$ and $f$ are bijections, any composition of such functions is going to be a bijection. Furthermore, the sequence must terminate since each of the sets is finite.

---

**Remark 3.2** When the involution principle first appeared, the functions $\alpha$ and $\beta$ were assumed to be involutions. A function $f$ is said to be an involution if $f^{-1} = f$. It was later pointed out that these functions need only be bijections.
3.10 Multisets

In the following, we make use of multiset notation for partitions. In particular, we set

\[ \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0) = \{1^{m_1}, 2^{m_2}, \ldots, i^{m_i}, \ldots \} \]

where \( m_i = |\{j \mid \lambda_j = i\}| \). We also define the following operations on partitions:

\[
\begin{align*}
\{1^{m_1}2^{m_2}3^{m_3}\ldots\} & \cup \{1^{n_1}2^{n_2}3^{n_3}\ldots\} = \{1^{\max(m_1,n_1)}2^{\max(m_2,n_2)}3^{\max(m_3,n_3)}\ldots\} \\
\{1^{m_1}2^{m_2}3^{m_3}\ldots\} & \cap \{1^{n_1}2^{n_2}3^{n_3}\ldots\} = \{1^{\min(m_1,n_1)}2^{\min(m_2,n_2)}3^{\min(m_3,n_3)}\ldots\} \\
\{1^{m_1}2^{m_2}3^{m_3}\ldots\} & \subseteq \{1^{n_1}2^{n_2}3^{n_3}\ldots\} \text{ iff } m_i \leq n_i \ \forall i \\
|\{1^{m_1}2^{m_2}3^{m_3}\ldots\}| & = m_1 + 2m_2 + 3m_3 + \cdots
\end{align*}
\]

Let \( \mathcal{A} = \{A_1, A_2, \ldots, \} \) be a sequence of distinct multisets. We shall think of elements of \( \mathcal{A} \) as forbidden patterns. In other words, we will make the following definition

\[
P_n(\mathcal{A}) = \{ \lambda \vdash n \mid A_i \not\subseteq \lambda \text{ for all } i \}
\]

which can be thought of as the set of partitions of \( n \) that do not contain any of the patterns given in \( \mathcal{A} \).

**Example 3.17** \( \mathcal{A} = \{\{2\}, \{4\}, \{6\}, \ldots\} \) \( P_n(\mathcal{A}) \) = set of partitions of \( n \) with only odd parts.

**Example 3.18** \( \mathcal{A} = \{\{1^2\}, \{2^2\}, \{3^2\}, \ldots\} \) \( P_n(\mathcal{A}) \) = set of partitions of \( n \) with distinct parts.

**Theorem 3.15** (Remmel) Let \( \mathcal{A} = \{A_1, A_2, \ldots, \} \) and \( \mathcal{B} = \{B_1, B_2, \ldots, \} \) be two sequences of distinct multisets. If for all finite \( S \subseteq \{1, 2, 3, \ldots, \} \)

\[
\left| \bigcup_{i \in S} A_i \right| = \left| \bigcup_{i \in S} B_i \right|
\]

then for all \( n \),

\[
|P_n(\mathcal{A})| = |P_n(\mathcal{B})|.
\]

**Remark 3.3** If \( A_i \cap A_j = \emptyset = B_i \cap B_j \) for all \( i, j \) and \( |A_i| = |B_i| \) for all \( i \) then

\[
\left| \bigcup_{i \in S} A_i \right| = \sum_{i \in S} |A_i| = \sum_{i \in S} |B_i| = \left| \bigcup_{i \in S} B_i \right|.
\]
Corollary 3.16 The number of partitions of $n$ whose parts are only odd is equal to the number of partitions of $n$ whose parts are distinct.

Corollary 3.17 (Glaisher) The number of partitions of $n$ with no part divisible by $d$ is equal to the number of partitions of $n$ with no part repeated $d$ or more times.

Proof.

\[
A = \{ \{d\}, \{2d\}, \{3d\}, \ldots \} \\
B = \{ \{1d\}, \{2d\}, \{3d\}, \ldots \}
\]

Corollary 3.18 (Schur) The number of partitions of $n$ into parts equivalent to $\pm 1 \mod 6$ is equal to the number of partitions of $n$ into distinct parts equivalent to $\pm 1 \mod 3$.

Proof.

\[
A = \{ \{2\}, \{3\}, \{4\}, \{6\}, \{8\}, \{9\}, \{10\}, \ldots \} \\
B = \{ \{1\}, \{3\}, \{2\}, \{6\}, \{4\}, \{9\}, \{5\}, \ldots \}
\]

Corollary 3.19 Let $M \subseteq \{1, 2, 3, \ldots \}$. If $2M = \{2m \mid m \in M\} \subseteq M$ then the number of partitions of $n$ with distinct parts taken from $M$ is equal to the number of partitions of $n$ into parts taken from $M - 2M$.

Proof.

\[
A = \{ \{i\}_{i \in M}, \{2k\}_{k \in M}\} \\
B = \{ \{i\}_{i \in M}, \{k^2\}_{k \in M}\}
\]

Example 3.19 $M = \{1, 2, 3, 4, \ldots \}$, $2M = \{2, 4, 6, 8, \ldots \}$ and $M - 2M = \{1, 3, 5, 7, \ldots \}$

Corollary 3.20 Let $M \subseteq \{1, 2, 3, \ldots \}$ and $d > 1$. If $dM = \{dm \mid m \in M\} \subseteq M$ and $M' = M - dM$ then the number of partitions of $n$ whose parts are taken from $M$ with no part repeated $d$ or more times is equal to the number of partitions of $n$ into parts taken from $M'$.

Proof.

\[
A = \{ \{i\}_{i \in M}, \{dk\}_{k \in M}\} \\
B = \{ \{i\}_{i \in M}, \{kd\}_{k \in M}\}
\]

Corollary 3.21 The number of partitions of $n$ with no consecutive even parts is equal to the number of partitions of $n$ with no consecutive repeated parts.

Proof.

\[
A = \{ \{2, 4\}, \{4, 6\}, \{6, 8\}, \{8, 10\}, \ldots \} \\
B = \{ \{1, 1, 2, 2\}, \{2, 2, 3, 3\}, \{3, 3, 4, 4\}, \{4, 4, 5, 5\}, \ldots \}
\]
Proof of Theorem. Given $\lambda \vdash n$ and $A, B$ sequences of multisets of forbidden patterns, define

\[ S_A(\lambda) = \{ i \mid A_i \subseteq \lambda \} = \text{set of forbidden } A \text{ patterns contained in } \lambda \]

\[ S_B(\lambda) = \{ i \mid B_i \subseteq \lambda \} = \text{set of forbidden } B \text{ patterns contained in } \lambda \]

Furthermore, if $S_A(\lambda) \neq \emptyset$ then let $m_A(\lambda) = \min(S_A(\lambda))$ and similarly, if $S_B(\lambda) \neq \emptyset$ then let $m_B(\lambda) = \min(S_B(\lambda))$.

Example 3.20 Let $\lambda = \{1^23^1\}$ with

\[ A = \{\{2\}, \{4\}, \{6\}, \ldots\} \]

\[ B = \{\{1^2\}, \{2^2\}, \{3^3\}, \ldots\} \]

then $S_A(\lambda) = \emptyset$ and $S_B(\lambda) = \emptyset$.

Define the sets $A$ and $B$ as follows

\[
A = \{ (\lambda, S) \mid \lambda \vdash n \text{ and } S \subseteq S_A(\lambda) \} \quad B = \{ (\lambda, S) \mid \lambda \vdash n \text{ and } S \subseteq S_B(\lambda) \} \\
A^+ = \{ (\lambda, S) \in A \mid |S| \text{ is even} \} \quad B^+ = \{ (\lambda, S) \in B \mid |S| \text{ is even} \} \\
A^- = \{ (\lambda, S) \in A \mid |S| \text{ is odd} \} \quad B^- = \{ (\lambda, S) \in B \mid |S| \text{ is odd} \}
\]

Notice that $P_n(A) \times \emptyset$ is contained in $A^+$ and $P_n(B) \times \emptyset$ is contained in $B^+$. Define

\[
\alpha(\lambda, S) = \begin{cases} (\lambda, S) & \text{if } S = S_A(\lambda) = \emptyset \\ (\lambda, S - m_A(\lambda)) & \text{if } m_A(\lambda) \in S \\ (\lambda, S \cup m_A(\lambda)) & \text{if } m_A(\lambda) \notin S \end{cases}
\]

\[
\beta(\lambda, S) = \begin{cases} (\lambda, S) & \text{if } S = S_B(\lambda) = \emptyset \\ (\lambda, S - m_B(\lambda)) & \text{if } m_B(\lambda) \in S \\ (\lambda, S \cup m_B(\lambda)) & \text{if } m_B(\lambda) \notin S \end{cases}
\]

\[
f(\lambda, S) = \left( \lambda - \bigcup_{i \in S} A_i \right) \cup \left( \bigcup_{i \in S} B_i \cup S \right) \quad f^{-1}(\lambda, S) = \left( \lambda - \bigcup_{i \in S} B_i \right) \cup \left( \bigcup_{i \in S} A_i \right)
\]

Now simply apply the involution principle and the proof is complete since

\[ |P_n(A)| = |F_\alpha| = |F_\beta| = |P_n(B)|. \]

Example 3.21

\[ A = \{\{2\}, \{4\}, \{6\}, \{8\}, \{10\}, \ldots\} \quad B = \{\{1,1\}, \{2,2\}, \{3,3\}, \{4,4\}, \{5,5\}, \ldots\} \]

<table>
<thead>
<tr>
<th>$A^+$</th>
<th>$f$</th>
<th>$B^+$</th>
<th>$\beta$</th>
<th>$B^-$</th>
<th>$f^{-1}$</th>
<th>$A^-$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1^2, 5^1}, \emptyset$</td>
<td>$\rightarrow$</td>
<td>${1^2, 5^1}, \emptyset$</td>
<td>$\rightarrow$</td>
<td>${1^2, 5^1}, {1}$</td>
<td>$\rightarrow$</td>
<td>${2,5^1}, {1}$</td>
<td>$\leftarrow$</td>
</tr>
<tr>
<td>${2, 5^3}, \emptyset$</td>
<td>$\rightarrow$</td>
<td>${2, 5^3}, \emptyset$</td>
<td>$\rightarrow$</td>
<td>${2, 5^3}, {5}$</td>
<td>$\rightarrow$</td>
<td>${2, 510}, {5}$</td>
<td>$\leftarrow$</td>
</tr>
<tr>
<td>${2, 5^3, 1, 5} \rightarrow$</td>
<td>${1^2, 5^3}, {1, 5} \rightarrow$</td>
<td>${1^2, 5^3}, {5} \rightarrow$</td>
<td>${1^2, 5, 10}, {5}$</td>
<td>$\leftarrow$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>${2, 5, 10}, \emptyset$</td>
<td>$\rightarrow$</td>
<td>${2, 5, 10}, \emptyset$</td>
<td>$\rightarrow$</td>
<td>${2, 5, 10}, {1}$</td>
<td>$\rightarrow$</td>
<td>${2, 5, 10}, {1}$</td>
<td>$\leftarrow$</td>
</tr>
</tbody>
</table>

Note that this is precisely the same correspondence as Glaisher’s bijection, which was used in the proof of Theorem 3.2.
3.11 Plane Partitions

**Definition 3.2** We say that $\pi: \lambda \to \mathbb{Z}^+$ is a plane partition of shape $\lambda$ if for every pair of cells $a, b \in \lambda$

$$\pi(a) \geq \pi(b)$$

whenever $b$ is due EAST of $a$ or due NORTH of $a$. The size of $\pi$, denoted $|\pi|$, is given by

$$|\pi| = \sum_{c \in \lambda} \pi(c).$$

**Example 3.22** A plane partition of 33

Another way to visualize a plane partition is through a sequence of non-intersecting lattice paths.

**Example 3.23** Forming non-intersecting lattice paths from a plane partition.
Lemma 3.22 The number of plane partitions that fit in an $r \times s \times t$ box is given by

$$\det \left( \begin{array}{c} s + t \\ s + i - j \end{array} \right)_{i,j=1}^r.$$ 

Proof. Consider all collections of paths which start at the points $(0,t), (1,t), \ldots, (r-1, t+r-1)$ and end at the points $(s,0), (s+1,1), \ldots, (s+r-1, r-1)$ using only steps due EAST and due SOUTH. To each set of paths, we can associate the permutation $\sigma \in S_r$ if for each $1 \leq i \leq r$, the path that starts at point $(\sigma_i - 1, t + \sigma_i - 1)$ ends at point $(s + i - 1, i - 1)$.

Example 3.24 The permutation corresponding to the collection of paths displayed below is $\sigma = (3,4,1,2,5)$.

![Diagram of lattice paths]

If $\sigma_i = j$, then the total number of paths from $(j - 1, t + j - 1)$ to $(s + i - 1, i - 1)$ is

$$\begin{pmatrix} s + t \\ s + i - j \end{pmatrix}.$$ 

Thus for any fixed permutation $\sigma$, the total number of sets of paths that correspond to $\sigma$ is given by

$$\prod_{i=1}^r \begin{pmatrix} s + t \\ s + i - \sigma_i \end{pmatrix}.$$ 

If we weight each set of paths by $(-1)^{inv(\sigma)}$ then we have

$$\sum_{\sigma \in S_r} (-1)^{inv(\sigma)} \prod_{i=1}^r \begin{pmatrix} s + t \\ s + i - \sigma_i \end{pmatrix} = \sum_{\sigma \in S_r} (-1)^{inv(\sigma)} \prod_{i=1}^r \begin{pmatrix} s + t \\ s + i - \sigma_i \end{pmatrix} \text{ (3.4)}$$ 

where the left hand sum is over all sets of $r$ lattice paths. We will show that every single term corresponding to a set of lattice paths with at least one point of intersection cancels out with another such set of paths. To this end, given a set of lattice paths with at least one point of intersection, find the rightmost point of intersection. If there is more than one point of intersection in this column, take the one in the lowest row. Now consider what happens if you switch the tail ends of the two paths that meet at this rightmost point, as illustrated below.

![Diagram of lattice paths with switch]
Example 3.25  The set of paths that cancels with those shown in Example 3.24 is illustrated below.

![Diagram of lattice paths](image)

By virtue of choosing the rightmost point of intersection, we have guaranteed that the two lines that meet at this point will end next to each other. In other words, if lines $a$ and $b$ form the rightmost point of intersection then we necessarily have that $\sigma_i = a$ and $\sigma_{i+1} = b$ for some $i$. Therefore, after we have switched the tail ends of these two lines, the resulting permutation, call it $\sigma'$, will be identical to $\sigma$ except that $\sigma'_i = b$ and $\sigma'_{i+1} = a$. Therefore we must have

$$(-1)^{\text{inv}(\sigma)} = (-1)^{\text{inv}(\sigma')}$$

since the pair of numbers $(a, b)$ will count as an inversion in only one of these two permutations. Therefore, these two sets of lattice paths cancel each other out in (3.4). Consequently, the only sets of paths remaining are those that do not have any points of intersection, which are exactly the paths we were trying to count. Note that each of these paths must correspond to the identity permutation, which means that each one has a positive sign associated with it and cannot be canceled out with any other path. Therefore the quantity given in 3.4 is precisely the number of non-intersecting lattice paths that we sought. The proof is complete after noting that the right hand side of 3.4 is the definition of the determinant whose $(i,j)$ entry is $\binom{s+t}{i+j}$.

Example 3.26  Let $r = 3$ $s = t = 2$. The number of plane partitions that fit in an $r \times s \times t$ box is

$$\begin{bmatrix}
\binom{4}{3} & \binom{4}{4} & \binom{4}{3} \\
\binom{3}{1} & \binom{3}{4} & \binom{1}{1} \\
\binom{4}{4} & \binom{3}{1} & \binom{2}{2}
\end{bmatrix} = \begin{bmatrix} 6 & 4 & 1 \\
4 & 6 & 4 \\
1 & 4 & 6
\end{bmatrix} = 6(36 - 16) - 4(24 - 4) + (16 - 6) = 50$$

Example 3.27  Let $r = 2$ $s = 3$ and $t = 2$. The number of plane partitions that fit in an $r \times s \times t$ box is

$$\begin{bmatrix}
\binom{5}{4} & \binom{5}{2} \\
\binom{3}{1} & \binom{5}{5}
\end{bmatrix} = \begin{bmatrix} 10 & 10 \\
5 & 10
\end{bmatrix} = 50$$
**Theorem 3.23** The generating function for the number of plane partitions that fit in an $r \times s \times t$ box is given by

\[
\det \left( q^{i(i-j)} \begin{pmatrix} s + t \\ s + i - j \end{pmatrix} \right)_{i,j=1}^r.
\]

**Proof.** Since each set of lattice paths, non-intersecting or otherwise, can be thought of as an ordered set of partitions, we can define the size of a set of lattice paths to be the sum of the sizes of the corresponding partitions. Unfortunately, the size of a set of lattice paths is not preserved under the involution we used to proof Lemma 3.22. We can see why this is so in the following example.

![Diagram](image)

In fact, we can see that the difference in the total area is exactly $\sigma_i - \sigma_{i+1}$. For this reason, we are going to weight each path by the sum of the sizes of the corresponding partitions plus the following statistic

\[ f(\sigma) = \sum_{k=1}^{r} k(k - \sigma_k). \]

Notice that the difference between $f(\sigma)$ and $f(\sigma')$ is

\[ f(\sigma') - f(\sigma) = \sum_{k=1}^{r} k(k - \sigma'_k) - \sum_{k=1}^{r} k(k - \sigma_k) = \sum_{k=1}^{r} k(\sigma_k - \sigma'_k) = i(\sigma_i - \sigma'_i) + (i + 1)(\sigma_{i+1} - \sigma'_{i+1}) = i(\sigma_i - \sigma_{i+1}) + (i + 1)(\sigma_{i+1} - \sigma_i) = \sigma_{i+1} - \sigma_i. \]

In other words, if the total area changes by $\sigma_i - \sigma_{i+1}$, then our statistic changes by exactly $\sigma_{i+1} - \sigma_i$. Therefore, their sum is preserved under our involution. Now we can repeat the argument used to prove Lemma 3.22 to show that

\[
\sum_{\sigma \in S_r} (-1)^{inv(\sigma)} q^{i(i-j)} \left[ \begin{pmatrix} s + t \\ s + i - j \end{pmatrix} \right]_{i,j=1}^{r}
\]

$q$-counts by size the number of plane partitions that fit in an $r \times s \times t$ box. We should note that since the only terms that do not cancel out correspond to the identity permutation, we really are weighting our set of lattice paths by their size since $f(id) = 0.$
Lemma 3.24 (Krattenthaler)

\[
\det((x_i - a_i) \cdots (x_i - a_{i-1})(1 - x_i b_{i+1}) \cdots (1 - x_i b_n))_{i,j=1}^n = \prod_{1 \leq i < j \leq n} (x_j - x_i)(1 - a_i b_j)
\]

Proof (Sketch). Think of the left hand side as defining a polynomial \( f(x_1, x_2, \ldots, x_n) \) whose coefficients are polynomials in \( a_1, \ldots, a_{n-1} \) and \( b_2, \ldots, b_n \). Notice that if \( x_i = x_j \) for any \( i < j \), then there are two identical rows in the determinant, and therefore, the polynomial vanishes. Thus

\[
f(x_1, x_2, \ldots, x_n) = C \prod_{1 \leq i < j \leq n} (x_j - x_i)
\]

To determine \( C \), set \( x_i = a_i \) for \( i = 1, \ldots, n - 1 \).

\[
f(a_1, a_2, \ldots, a_{n-1}, x_n) = \prod_{i=1}^{n-1} (a_i - a_1)(a_i - a_2) \cdots (a_i - a_{i-1})(1 - a_i b_{i+1}) \cdots (1 - a_i b_n) \times (x_n - a_1)(x_n - a_2) \cdots (x_n - a_{n-1}) = C(x_n - a_1)(x_n - a_2) \cdots (x_n - a_{n-1}) \prod_{1 \leq i < j \leq n} (a_j - a_i)
\]

and therefore,

\[
C = \prod_{i=1}^{n-1} (1 - a_i b_{i+1}) \cdots (1 - a_i b_n) = \prod_{1 \leq i < j \leq n} (1 - a_i b_j)
\]

as desired. \( \square \)

Corollary 3.25

\[
\det(x_i^{-j}(1 - a_1 x_i) \cdots (1 - a_{j-1} x_i)(1 - b_{j+1} x_i) \cdots (1 - b_n x_i))_{i,j=1}^n = \prod_{1 \leq i < j \leq n} (1 - x_j / x_i)(1 - a_i b_j)
\]

Proof. Starting with Krattenthaler’s formula, factor out an \( x_j \) out of each \((x_j - x_i)\). This yields

\[
\prod_{1 \leq i < j \leq n} (x_j - x_i)(1 - a_i b_j) = x_2 x_3^2 \cdots x_j^{j-1} \prod_{1 \leq i < j \leq n} (1 - x_j / x_i)(1 - a_i b_j)
\]

Now replacing each \( x_i \) by \( 1/x_i \) yields

\[
\prod_{1 \leq i < j \leq n} (1 - x_j / x_i)(1 - a_i b_j)
\]

\[
= x_2 x_3^2 \cdots x_n^{n-1} \det((1/x_i - a_1) \cdots (1/x_i - a_{j-1})(1 - b_{j+1} / x_i) \cdots (1 - b_n / x_i))
\]

\[
= x_2 x_3^2 \cdots x_n^{n-1} \det(x_i^{-j+1}(1 - a_j x_i) \cdots (1 - a_{j-1} x_i)(1 - b_{j+1} / x_i) \cdots (1 - b_n / x_i))
\]

\[
= \det(x_i^{-j}(1 - a_j x_i) \cdots (1 - a_{j-1} x_i)(1 - b_{j+1} / x_i) \cdots (1 - b_n / x_i))
\]

The last equality follows from distributing the \( x_i^{-j} \) factor onto each term in the \( i \)th row of the determinant. \( \square \)
Theorem 3.26 The generating function for the number of plane partitions that fit in an \( r \times s \times t \) box is given by
\[
\prod_{i=1}^{r} \prod_{j=1}^{s} \frac{1 - q^{t+i+j-1}}{1 - q^{t+i+j-1}}.
\]

Proof. Using the following notation
\[(q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n),\]
the determinantal formula given in Theorem 3.23 can be rewritten as
\[
\det \left( q^{i-j} \begin{bmatrix} s + t \\ s + i - j \end{bmatrix} \right) = \det \left( q^{i-j} \frac{(q)_{s+t}}{(q)_{s+i-j}(q)_{t-i+j}} \right).
\]

For each \( i = 1, 2, \ldots, r \), factor out the following terms from row \( i \)
\[
\frac{(q)_{s+t}}{(q)_{s+i-1}(q)_{t-i+r}}.
\]
Doing so produces
\[
\prod_{i=1}^{r} \frac{(q)_{s+t}}{(q)_{s+i-1}(q)_{t-i+r}} \det \left( q^{i-j} \left( 1 - q^{s+i+j+1} \right) \cdots \left( 1 - q^{s+i-1} \right) \left( 1 - q^{t-i+j+1} \right) \cdots \left( 1 - q^{t-i+r} \right) \right).
\]
Applying Corollary 3.25 with \( x_i = q^i \), \( a_i = q^{s-i} \) and \( b_j = q^{t+j} \) yields
\[
\prod_{i=1}^{r} \frac{(q)_{s+t}}{(q)_{s+i-1}(q)_{t-i+r}} \prod_{1 \leq i < j \leq n} \left( 1 - q^{s-i} \right) \left( 1 - q^{t+j-1} \right).
\]
We shall break up the above product into two pieces. First consider the following.
\[
\prod_{i=1}^{r} \frac{(q)_{s+t}}{(q)_{t-i+r}} \prod_{1 \leq i < j \leq n} \left( 1 - q^{s+i+j-1} \right) = \left( \prod_{i=1}^{r} \frac{(q)_{s+t}}{(q)_{t-i+r}} \right) \left( \prod_{i=1}^{r-1} \prod_{j=i+1}^{r} \left( 1 - q^{s+i+j-1} \right) \right)
\]
\[
= \prod_{i=1}^{r} \frac{(q)_{s+t}}{(q)_{t-i+r}} \prod_{j=i+1}^{r} \left( 1 - q^{s+i+j-1} \right)
\]
\[
= \prod_{i=1}^{r} \frac{(q)_{s+t+r-i}}{(q)_{t-i+r}} \prod_{j=1}^{r-i} \left( 1 - q^{s+i+j} \right)
\]
\[
= \prod_{i=1}^{r} \frac{(q)_{s+t+r-i}}{(q)_{t-i+r}} \prod_{j=1}^{r-i} \left( 1 - q^{s+i+j} \right)
\]
\[
= \prod_{i=1}^{r} \left( q_{s+t+i-1} \right) \left( q_{s+i-1} \right) \prod_{i=1}^{r} \left( 1 - q^{t+i+j-1} \right)
\]
Now consider the remaining factors:

\[
\prod_{i=1}^{r} \frac{1}{(q)_{s+i-1}} \prod_{1 \leq i < j \leq n} (1 - q^{j-i}) = \left( \prod_{j=1}^{r} \frac{1}{(q)_{s+i-1}} \right) \left( \prod_{i=1}^{r} \prod_{j=i+1}^{r} (1 - q^{j-i}) \right) \\
= \prod_{i=1}^{r} \frac{1}{(q)_{s+i-1}} \prod_{j=1}^{r-i} (1 - q^{j}) \\
= \prod_{i=1}^{r} \frac{(q)_{r-i}}{(q)_{s+i-1}} \\
= \prod_{i=1}^{r} \frac{(q)_{r-i}}{(q)_{s+i-1}} \\
= \prod_{i=1}^{r} \prod_{j=1}^{s} \frac{1}{1 - q^{i+j-1}}.
\]

Putting everything back together again yields

\[
\det \left( q^{(i-j)} \begin{bmatrix} s + t \\ s + i - j \end{bmatrix} \right) = \prod_{i=1}^{r} \prod_{j=1}^{s} \frac{1 - q^{t+i+j-1}}{1 - q^{i+j-1}}
\]
as desired. \(\square\)

**Example 3.28** Let \(r = 3\), \(s = t = 2\). The generating function that \(q\)-counts by size the number of plane partitions that fit in an \(r \times s \times t\) box is given by

\[
\prod_{i=1}^{3} \prod_{j=1}^{2} \frac{1 - q^{i+j+1}}{1 - q^{i+j-1}} = \frac{1 - q^3}{1 - q} \frac{1 - q^4}{1 - q^2} \frac{1 - q^5}{1 - q^3} \frac{1 - q^6}{1 - q^4} \\
= 1 + q + 3q^2 + 4q^3 + 6q^4 + 6q^5 + 8q^6 + 6q^7 + 6q^8 + 4q^9 + 3q^{10} + q^{11} + q^{12}
\]
The 4 plane partitions of 3 that fit in a \(3 \times 2 \times 2\) box are displayed below

![Plane Partitions](image)

**Remark 3.4** Taking the limit \(q \to 1\) yields the following formula for the number of plane partitions that fit in an \(r \times s \times t\) box:

\[
\prod_{i=1}^{r} \prod_{j=1}^{s} \frac{t + i + j - 1}{i + j - 1}.
\]

In the case \(r = 3\) and \(s = t = 2\) we get

\[
\begin{array}{cccccc}
3 & 4 & 4 & 5 & 5 & 6 \\
1 & 2 & 2 & 3 & 3 & 4
\end{array}
= 50.
\]
Theorem 3.27  The generating function for the number of plane partitions of \( n \), \( pp(n) \), is given by

\[
\sum_{n \geq 0} pp(n) q^n = \prod_{n \geq 1} \frac{1}{(1 - q^n)^n}.
\]

Proof. Starting with Theorem 3.26, let \( r \) and \( s \) tend to infinity. Thus the factors of this infinite product can be written in the following matrix

\[
\begin{array}{cccccc}
1 - q^{s+1} & 1 - q^{s+2} & 1 - q^{s+3} & 1 - q^{s+4} & \cdots \\
1 - q^t & 1 - q^t & 1 - q^t & 1 - q^t & \cdots \\
1 - q^{t+2} & 1 - q^{t+3} & 1 - q^{t+4} & \cdots \\
1 - q^t & 1 - q^t & 1 - q^t & \cdots \\
1 - q^{t+3} & 1 - q^{t+4} & \cdots \\
1 - q^t & 1 - q^t & \cdots \\
\vdots \\
\end{array}
\]

As \( t \) tends to \( \infty \), each of the numerators \( 1 - q^{t+j} \) tends to 1 as a formal power series. Therefore, each of the above factors converges to

\[
\begin{array}{cccccc}
\frac{1}{1-q} & \frac{1}{1-q^2} & \frac{1}{1-q^3} & \frac{1}{1-q^4} & \cdots \\
\frac{1}{1-q^2} & \frac{1}{1-q^3} & \frac{1}{1-q^4} & \cdots \\
\frac{1}{1-q^3} & \frac{1}{1-q^4} & \cdots \\
\frac{1}{1-q^4} & \cdots \\
\frac{1}{1-q^4} & \cdots \\
\vdots \\
\end{array}
\]

as claimed. \( \square \)

Example 3.29

\[
\prod_{n \geq 1} \frac{1}{(1 - q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + 86q^7 + 160q^8 + 282q^9 + 500q^{10} + \cdots
\]

The six plane partitions of 3 are displayed below

![Six plane partitions of 3](image-url)
3.12 Reverse Plane Partitions

**Definition 3.3** We say that a function \( \pi : \lambda \to \mathbb{N} \) is a reverse plane partition of shape \( \lambda \) if for every pair of cells \( a, b \in \lambda \)

\[
\pi(a) \leq \pi(b)
\]

whenever \( b \) is due EAST of \( a \) or due NORTH of \( a \). The set of all reverse plane partitions of shape \( \lambda \) is denoted \( RPP(\lambda) \). We define the size of \( \pi \), denoted \( |\pi| \), to be

\[
|\pi| = \sum_{c \in \lambda} \pi(c).
\]

**Example 3.30** A reverse plane partition of shape \((5, 4, 3, 1)\).

**Definition 3.4** Given a partition \( \lambda \) and a cell \( c \) in the Ferrers diagram of \( \lambda \), let the hook length of \( c, h(c) \), refer to the number of cells in \( \lambda \) that are due North or due East of \( c \), including \( c \) itself. We shall also impose an order on the cells of \( \lambda \) from right to left, bottom to top.

**Example 3.31** Let \( \lambda = (5, 4, 3, 1) \). The hook lengths of each cell is given by

\[
\begin{array}{cccc}
1 & 4 & 2 & 1 \\
6 & 4 & 3 & 1 \\
8 & 6 & 5 & 3 & 1
\end{array}
\]

The order of the cells of \( \lambda \) is given as

\[
\begin{array}{cccc}
13 & 12 & 9 & 6 \\
11 & 8 & 5 & 3 \\
10 & 7 & 4 & 2 & 1
\end{array}
\]
Theorem 3.28 The generating function for the number of reverse plane partitions of \( n \) of shape \( \lambda \) is given by

\[
\sum_{\pi \in RPP(\lambda)} q^{\left| \pi \right|} = \prod_{c \in \lambda} \frac{1}{1 - q^{h(c)}}.
\]

Example 3.32 Let \( \lambda = (5, 4, 3, 1) \)

\[
\frac{1}{(1 - q)^4(1 - q^2)(1 - q^3)(1 - q^4)(1 - q^5)(1 - q^6)}
\]

Proof. Let \( M(\lambda) \) be the set of all multisets of cells of \( \lambda \). Each element \( m \in M(\lambda) \) can be thought of as a function from \( \lambda \) to \( \mathbb{N} \). These multisets can also be realized as a labeling of the cells of \( \lambda \) with nonnegative integers in much the same way that reverse plane partitions are, except that there are no conditions on the relative values between cells. We define the size of \( m \), denoted \( |m| \), to be

\[
\sum_{c \in \lambda} m(c) h(c)
\]

and the weight of \( m \in M(\lambda) \), denoted \( w(m) \), to be

\[
w(m) = q^{\left| m \right|}.
\]

The corresponding generating function is given by

\[
\sum_{m \in M(\lambda)} w(m) = \sum_{m(c_1) \geq 0} \cdots \sum_{m(c_{|\lambda|}) \geq 0} q^{m(c_1) h(c_1) + \cdots + m(c_{|\lambda|}) h(c_{|\lambda|})}
\]

\[
= \left( \sum_{m(c_1) \geq 0} q^{m(c_1) h(c_1)} \right) \cdots \left( \sum_{m(c_{|\lambda|}) \geq 0} q^{m(c_{|\lambda|}) h(c_{|\lambda|})} \right)
\]

\[
= \left( \frac{1}{1 - q^{h(c_1)}} \right) \cdots \left( \frac{1}{1 - q^{h(c_{|\lambda|})}} \right)
\]

\[
= \prod_{c \in \lambda} \frac{1}{1 - q^{h(c)}}.
\]

To complete our proof, we require a weight preserving bijection between elements of \( M(\lambda) \) and \( RPP(\lambda) \).
3.12.1 Hillman-Grassl Bijection, Part I

Step 1: Input $\pi \in \mathcal{RP}(\lambda)$ and let $m \in M(\lambda)$, $m(c) = 0$ for all $c \in \lambda$

Step 2: Let $c$ be the last cell in $\lambda$ such that $\pi(c) > 0$
    Let $p$ be a path starting at $c$

    Step 2a: Let $c_b$ be the cell immediately below $c$
        Let $c_r$ be the cell immediately to the right of $c$
        If $\pi(c) = \pi(c_b)$ then append $c_b$ to $p$ and set $c = c_b$
        else if $c_r \in \lambda$ then append $c_r$ to $p$ and set $c = c_r$
        else set $c = c_r$

    Step 2b: If $c \in \lambda$ then repeat Step 2a, otherwise go to Step 3.

Step 3: Decrease $\pi(c)$ by 1 for each $c$ in $p$.
    Increase $m(\text{pivot})$ by 1, where $\text{pivot}$ is the cell in the same column as the first cell in $p$ and
    in the same row as the last cell in $p$.
    If $|\pi| > 0$ then repeat Step 2. Otherwise, go to Step 4.

Step 4: Output $m \in M(\lambda)$.

Example 3.33 The path $p$ for each of the following reverse plane partitions is highlighted in green. The pivot is highlighted in yellow. The second reverse plane partition is the only one where the pivot is not actually part of the path.
3.12.2 Hillman-Grassl Bijection, Part II

**Step 1:** Input $m \in M(\lambda)$ and let $\pi \in \mathcal{RPP}(\lambda)$, $\pi(c) = 0$ for all $c \in \lambda$

**Step 2:** Let pivot be the first cell in $\lambda$ such that $m(\text{pivot}) > 0$ (say it’s in row $i$ and column $j$)

Let $c$ be the cell at the end of row $i$

Let $p$ be a path starting at $c$

**Step 2a:** Let $c_a$ be the cell immediately above $c$

Let $c_l$ be the cell immediately to the left of $c$

If $\pi(c) = \pi(c_a)$ then append $c_a$ to $p$ and set $c = c_a$

else append $c_l$ to $p$ and set $c = c_l$

**Step 2b:** If $c$ is not the last cell in column $j$ of $\lambda$ then repeat Step 2a, otherwise go to Step 3.

**Step 3:** Increase $\pi(c)$ by 1 for each $c$ in $p$.

Decrease $m(\text{pivot})$ by 1.

If $|m| > 0$ then repeat Step 2. Otherwise, go to Step 4.

**Step 4:** Output $\pi \in \mathcal{RPP}(\lambda)$.

Notice that the number of cells in each path $p$ is precisely the hook length of the corresponding pivot. Therefore the size of the plane partition $\pi$ is precisely the weight of the multiset $m$. This completes the proof of Theorem 3.28.

Example 3.34

Input $\rightarrow$

```
1 1
1 2
2 1 2 1
```

```
Input
```

```
Output
```

4 Combinatorial Species

4.1 Introduction

The goal of this section is to study combinatorial objects that can be built up from smaller pieces. For example, permutations can be written as a product of disjoint cycles. Each of these cycles can be thought of as a permutation in and of itself. Other objects that can be viewed in a similar manner are set partitions, graphs, trees, lists, etc. Turning the tables around, it’s natural to ask what kind of objects and how many of them can we construct given certain building blocks. For example, how many permutations are there in $S_n$ that can be written as the product of disjoint cycles of length 2 or 3? To get things started, we make the following definitions.

**Definition 4.1** A *card of weight* $m$ consists of a picture involving (among other things) $m$ circles numbered 1 through $m$.

**Example 4.1** A card of weight 5.

![Card of weight 5 diagram]

**Definition 4.2** A *deck of weight* $m$, denoted $D_m$, is a collection of cards of weight $m$.

**Example 4.2** A deck of weight 3.

![Deck of weight 3 diagrams]

**Definition 4.3** An *exponential family* $\mathcal{F}$ is a sequence of decks $D_1, D_2, D_3, \ldots$. 
Definition 4.4 A hand of weight \( n \) consisting of \( k \) cards is formed in the following manner.

1. Select a set partition of \([n] = \{1,2,\ldots,n\}\) into exactly \( k \) parts. Specifically, \( \{A_1, A_2, \ldots, A_k\} \) is a collection of nonempty subsets of \([n]\) such that
   - (a) \( A_i \cap A_j = \emptyset \) for each \( i, j \) and
   - (b) \( A_1 \cup A_2 \cup \cdots \cup A_k = [n] \).

   We will also assume that the sets \( A_i \) are ordered according to least element, that is
   - (c) \( \min A_1 < \min A_2 < \cdots < \min A_k \).

2. For each \( i = 1, 2, \ldots, k \) let \( m = |A_i| \)
   - (a) Select a card from \( D_m \). The same card may be picked more than once.
   - (b) For each \( j = 1, 2, \ldots, m \) label \( a_j \) with \( A_i = \{a_1, a_2, \ldots, a_m\} \).

For a given exponential family \( F \), we denote the set of hands of weight \( n \) consisting of exactly \( k \) cards by \( F_{n,k}(F) \).

Example 4.3 A permutation in \( S_n \) written in cycle notation can be thought of as a hand of cards. For example, the permutation \((1,3)(2,6,5)(4)\) can be represented by the following hand taken from the sequence of decks whose cards represent cycles of any given length.

Example 4.4 A “forest” of labeled trees.
4.2 The Exponential Formula

**Theorem 4.1** For any exponential family \( F \)

\[
1 + \sum_{n \geq 1} \sum_{k=1}^{n} \frac{|F_{n,k}(F)| t^n}{n!} = e^{t \sum_{m \geq 1} |P_m| \frac{t^m}{m!}}
\]  

(4.1)

Given a set partition \( \Pi = (A_1, \ldots, A_k) \), the type of \( \Pi \) is given by \( 1^{p_1} 2^{p_2} \cdots n^{p_n} \) where \( p_i \) is the number of elements of \( \Pi \) of cardinality \( i \). The following lemma concerns itself with how many set partitions there are of a given type.

**Lemma 4.2** The number of set partitions of \( [n] \) of type \( 1^{p_1} 2^{p_2} \cdots n^{p_n} \) is given by

\[
\frac{n!}{p_1! p_2! \cdots p_n! (1!)^{p_1} (2!)^{p_2} \cdots (n!)^{p_n}}.
\]

**Proof.** Consider constructing a set partition of \( \{1, 2, \ldots, n\} \) of type \( 1^{p_1} 2^{p_2} \cdots n^{p_n} \) by starting with a permutation in \( S_n \). The first \( p_1 \) numbers each correspond to 1-element subsets. The next \( p_2 \) pairs of numbers each correspond to 2-element subsets and so on. Now consider how many permutations correspond to this same set partition. The first \( p_1 \) numbers could have been rearranged in any one of \( p_1! \) ways and still yielded the same set partition. The next \( p_2 \) pairs could have been reordered in any one of \( p_2! \) ways while still maintaining the order in each pair. Also, each pair could be ordered in \( 2! \) ways. In general, the \( p_i \) sets of size \( i \) can be reordered in \( p_i! \) ways, and each of these sets can be reordered in \( i! \) ways. Thus a given collection of \( p_i \) subsets of size \( i \) can be represented in \( p_i! (i!)^{p_i} \) different ways.

\[\square\]

**Example 4.5** Consider the permutation \( \sigma = (4, 3, 1, 6, 5, 2) \). This leads to a particular set partition of type \( 1^2 2^2 \), namely

\[\{4\}, \{3\}, \{1, 6\}, \{5, 2\}\]

Notice that there are exactly \( 2! (1!)^2 2! (2!)^2 = 16 \) permutations that lead to the same set partition. These permutations are listed below:

\[
\begin{align*}
(3, 4, 1, 6, 2, 5) & \quad (3, 4, 6, 1, 2, 5) & \quad (3, 4, 1, 6, 5, 2) & \quad (3, 4, 6, 1, 5, 2) \\
(3, 4, 2, 5, 1, 6) & \quad (3, 4, 2, 5, 6, 1) & \quad (3, 4, 5, 2, 1, 6) & \quad (3, 4, 5, 2, 6, 1) \\
(4, 3, 1, 6, 2, 5) & \quad (4, 3, 6, 1, 2, 5) & \quad (4, 3, 1, 6, 5, 2) & \quad (4, 3, 6, 1, 5, 2) \\
(4, 3, 2, 5, 1, 6) & \quad (4, 3, 2, 5, 6, 1) & \quad (4, 3, 5, 2, 1, 6) & \quad (4, 3, 5, 2, 6, 1)
\end{align*}
\]
Proof of Theorem. Let $\mathcal{F} = D_1, D_2, \ldots$ with $d_n = |D_n|$. First we will compute the number of hands of weight $n$ with $k$ cards.

$$|F_{n,k}(\mathcal{F})| = \sum_{\substack{p_1, \ldots, p_n \\ p_1 + 2p_2 + \cdots + np_n = n}} \frac{n!}{p_1!p_2! \cdots p_n!(1!)^{p_1}(2!)^{p_2} \cdots (n!)^{p_n}} d_1^{p_1} d_2^{p_2} \cdots d_n^{p_n}$$

$$= n! \sum_{\substack{p_1, \ldots, p_n \\ p_1 + 2p_2 + \cdots + np_n = n}} \frac{1}{p_1!p_2! \cdots p_n!} \left( \frac{d_1}{1!} \right)^{p_1} \left( \frac{d_2}{2!} \right)^{p_2} \cdots \left( \frac{d_n}{n!} \right)^{p_n}$$

$$= \frac{n!}{k!} \sum_{\substack{p_1, \ldots, p_n \\ p_1 + 2p_2 + \cdots + np_n = k}} k! \left( \frac{d_1}{1!} + \frac{d_2}{2!} + \cdots + \frac{d_n}{n!} \right)^k$$

$$= \frac{n!}{k!} \left( \frac{d_1}{1!} + \frac{d_2}{2!} + \cdots + \frac{d_n}{n!} \right)^k$$

Plugging this back into the left hand side of 4.1 yields

$$1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{k=1}^{n} |F_{n,k}(\mathcal{F})| x^k = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{k=1}^{n} \frac{n!}{k!} \left( \sum_{m \geq 1} \frac{d_m t^m}{m!} \right)^k$$

$$= 1 + \sum_{n \geq 1} \frac{t^n}{n!} \left( \sum_{k=1}^{n} \frac{x^k}{k!} \left( \sum_{m \geq 1} \frac{d_m t^m}{m!} \right)^k \right)$$

$$= 1 + \sum_{k \geq 1} \frac{x^k}{k!} \left( \sum_{m \geq 1} \frac{d_m t^m}{m!} \right)^k$$

$$= e^{\sum_{m \geq 1} \frac{d_m t^m}{m!}}$$
4.3 Permutations with \( k \) cycles

**Corollary 4.3** Let \( s_{n,k} \) be the number of permutations of \( n \) with exactly \( k \) cycles. Then

\[
1 + \sum_{n \geq 1} \sum_{k=1}^{n} s_{n,k} x^k \frac{t^n}{n!} = \left( \frac{1}{1-t} \right)^x.
\]

\( s_{n,k} \) is referred to as a Stirling number of the first kind.

**Proof.** Let \( \mathcal{F} \) be the exponential family whose \( m^{th} \) deck consists of all cycles of length \( m \).

Then \(|\mathcal{D}_m| = (m-1)!\) and \( F_{n,k}(\mathcal{F}) \) is the set of permutations of \( n \) with exactly \( k \) cycles. Therefore

\[
1 + \sum_{n \geq 1} \sum_{k=1}^{n} s_{n,k} x^k \frac{t^n}{n!} = e^x \sum_{m \geq 1} \frac{1}{m!} t^m = e^x \ln \frac{1}{1-t}.
\]

\( \square \)

The first few terms of the series expansion of \((1 - t)^{-x}\) are

\[
1 + xt + (x + x^2) \frac{t^2}{2!} + (2x + 3x^2 + x^3) \frac{t^3}{3!} + (6x + 11x^2 + 6x^3 + x^4) \frac{t^4}{4!} + (24x + 50x^2 + 35x^3 + 10x^4 + x^5) \frac{t^5}{5!} + \cdots
\]

from which we can gather the following table of values of \( s_{n,k} \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td>24</td>
<td>50</td>
<td>35</td>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

It is interesting to note that the above series expansion can be written in the following form

\[
1 + xt + (x + x^2) \frac{t^2}{2!} + x(x+1)(x+2) \frac{t^3}{3!} + x(x+1)(x+2)(x+3) \frac{t^4}{4!} + x(x+1)(x+2)(x+3)(x+4) \frac{t^5}{5!} + \cdots
\]

The proof of this is left as an exercise for the reader.
4.4 Set Partitions into \( k \) parts

**Corollary 4.4** Let \( S_{n,k} \) be the number of set partitions of \([n]\) into \( k \) parts. Then

\[
1 + \sum_{n \geq 1} \sum_{k=1}^{n} S_{n,k} x^k \frac{t^n}{n!} = e^{x(e^t - 1)}.
\]

\( S_{n,k} \) is referred to as a Stirling number of the second kind.

**Proof.** Let \( \mathcal{F} \) be the exponential family whose \( m \)th deck, \( D_m \), consists of a single card with \( m \) numbered circles

```
  1  2  ...  m
```

Then \(|D_m| = 1\) and \( F_{n,k}(\mathcal{F}) \) is the collection of set partitions of \([n]\) with exactly \( k \) parts. Therefore

\[
1 + \sum_{n \geq 1} \sum_{k=1}^{n} S_{n,k} x^k \frac{t^n}{n!} = e^{x \sum_{m=1}^{\infty} \frac{t^m}{m!}}
\]

The first few terms of the series expansion of \( e^{x(e^t - 1)} \) are

\[
1 + xt + (x + x^2) \frac{t^2}{2!} + (x + 3x^2 + x^3) \frac{t^3}{3!} + (x + 7x^2 + 6x^3 + x^4) \frac{t^4}{4!} + (x + 15x^2 + 25x^3 + 10x^4 + x^5) \frac{t^5}{5!} + \cdots
\]

from which we can gather the following table of values of \( S_{n,k} \).

<table>
<thead>
<tr>
<th>( n )</th>
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<th>3</th>
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<td>1</td>
<td>15</td>
<td>25</td>
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</tr>
</tbody>
</table>

**Remark 4.1** Setting \( x = 1 \) in , yields

\[
\sum_{n \geq 0} B_n \frac{t^n}{n!} = e^{e^t - 1}
\]

where \( B_n \) is the \( n \)th Bell number and represents the number of set partitions of \([n]\). The first few terms of the series expansion of \( e^{e^t - 1} \) is given below.

\[
1 + t + \frac{2}{2!} t^2 + \frac{5}{3!} t^3 + \frac{15}{4!} t^4 + \frac{52}{5!} t^5 + \frac{203}{6!} t^6 + \frac{877}{7!} t^7 + \frac{4140}{8!} t^8 + \frac{21147}{9!} t^9 + \cdots
\]
4.5 Ordered Lists

**Corollary 4.5** Let $L_{n,k}$ be the number of ways of placing $n$ labeled balls into $k$ indistinguishable tubes. Then

$$1 + \sum_{n \geq 1} \sum_{k \geq 1} L_{n,k} x^k \frac{t^n}{n!} = e^x \frac{1}{1 - t}$$

**Proof.** Let $D_m$ be the set of linearly ordered lists of length $m$.

$$|D_m| = m!$$ and $F_{n,k}(F)$ is equal to the set of placements of $n$ labeled balls into $k$ indistinguishable tubes.

$$1 + \sum_{n \geq 1} \sum_{k \geq 1} L_{n,k} x^k \frac{t^n}{n!} = e^x \sum_{m \geq 1} \frac{t^m}{m!}$$

□

**Remark 4.2** In this case, we can actually use the above identity to find an explicit formula for $L_{n,k}$.

$$\sum_{k=1}^{n} L_{n,k} x^k = e^x \frac{1}{1 - t} \bigg|_{t^n/n!}$$

$$= 1 + \sum_{k \geq 1} \frac{x^k}{k!} \left( \frac{t}{1 - t} \right)^k \bigg|_{t^n/n!}$$

$$= 1 + \sum_{k \geq 1} \frac{x^k n!}{k!} \left( t + t^2 + t^3 + \cdots \right)^k \bigg|_{t^n}$$

$$= 1 + \sum_{k \geq 1} \frac{x^k n!}{k!} \left( \frac{n - 1}{k - 1} \right)$$

The last equality follows from the fact that the number of positive integer solutions to $a_1 + \cdots + a_k = n$ is equal to $\binom{n-1}{k-1}$. In other words, we have established that

$$L_{n,k} = \frac{n!}{k!} \binom{n-1}{k-1}.$$
4.6 Permutations with Even Lengthed Cycles

**Corollary 4.6** Let $n$ be any positive integer. The number of permutations of $S_{2n}$ with only cycles of even length is

$$1^2 \cdot 3^2 \cdot 5^2 \cdots (2n - 1)^2 = \prod_{i=1}^{n}(2i - 1)^2.$$ 

**Proof.** Let

$$D_m = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \text{set of ordered cycles} & \text{if } m \text{ is even} \end{cases}$$

Then $F_{n,k}$ is the set of permutations of $n$ with $k$ cycles all of even length. Therefore

$$1 + \sum_{n=1}^{2n} \frac{t^{2n}}{(2n)!} \sum_{k=1}^{2n} |F_{2n,k}| x^k = e^x \sum_{m \geq 1} \frac{t^{2m}}{2m!} = e^x \sum_{m \geq 1} \frac{t^{2m}}{2m} = e^x \ln \frac{1}{1-t^2} = \left( \frac{1}{1-t^2} \right)^{x/2}$$

Recall Newton’s Binomial Theorem

**Theorem 4.7** For any real number $\alpha$,

$$(x+y)^\alpha = \sum_{k \geq 0} \binom{\alpha}{k} x^k y^{\alpha-k} \quad \left( \frac{\alpha}{k} \right) = \frac{\alpha(\alpha-1) \cdots (\alpha-k+1)}{k!}$$

Therefore

$$\left( \frac{1}{1-t^2} \right)^{x/2} = (1-t^2)^{-x/2} = \sum_{k \geq 0} \binom{-x/2}{k} (-t^2)^k = \sum_{k \geq 0} \frac{(-1)^k t^{2k}}{k!} (-x/2)(-x/2-1) \cdots (-x/2-k+1) = \sum_{k \geq 0} \frac{t^{2k}}{k!} \frac{x/2(x/2+1) \cdots (x/2+k-1)}$$

Taking the coefficient of $t^{2n}/(2n)!$ yields

$$\sum_{k=1}^{2n} |F_{2n,k}| x^k = \frac{(2n)!}{n!} \frac{(x/2)(x/2+1) \cdots (x/2+n-1)} = \frac{(2n)!}{2^n n!} (x(x+2) \cdots (x+2n-2) = 135 \cdots (2n-1)(x+2) \cdots (x+2n-2)$$

Finally, letting $x = 1$ yields our result. \(\square\)
Corollary 4.8 Let $n$ be any positive integer. The number of permutations of $S_{2n}$ with only cycles of length 2 (i.e. involutions with no fixed points) is

$$1 \cdot 3 \cdot 5 \cdots (2n - 1) = \prod_{i=1}^{n} (2i - 1).$$

Proof. Let

$$D_m = \begin{cases} 1 \circ \cdots \circ 2 & \text{if } m = 2 \\ \emptyset & \text{if } m \neq 2 \end{cases}$$

$F_{n,k}$ is the set of permutations of $n$ with exactly $k$ cycles of length 2. Therefore

$$1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{k=1}^{n} [F_{n,k}] = e^{t^2/2} = 1 + \sum_{n \geq 1} \frac{x^n t^{2m}}{2^n m!}$$

Letting $x = 1$ yields

$$1 + \sum_{n \geq 1} \frac{t^{2n}}{(2n)!} |F_{2n,n}| = e^{t^2/2} = 1 + \sum_{n \geq 1} \frac{t^{2m}}{2^n m!}$$

Now take the coefficient of $t^{2n}$

$$\frac{|F_{2n,n}|}{(2n)!} = \frac{1}{2^n n!}$$

and therefore

$$|F_{2n,n}| = \frac{(2n)!}{2^n n!} = 1 \cdot 3 \cdot 5 \cdots (2n - 1).$$

Remark 4.3 Comparing the last two corollaries suggest that there is a bijection between permutations of $S_{2n}$ with only cycles of even length and ordered pairs of involutions with no fixed points. Given $\alpha$ and $\beta$ involutions with no fixed points, we can construct a permutation whose cycles are

$$(a, \alpha(a), \beta(\alpha(a)), \alpha(\beta(\alpha(a))), \cdots)$$

Example 4.6 Let $\sigma_1 = (1, 2)(3, 5)(4, 10)(6, 7)(8, 9)$ and $\sigma_2 = (1, 9)(2, 7)(3, 4)(5, 10)(6, 8)$. Then the corresponding permutation is given by

$$(1, 2, 7, 6, 8, 9)(3, 5, 10, 4)$$
4.7 Cayley Trees and Lagrange Inversion

**Definition 4.5** A tree is a connected graph with no cycles. A Cayley tree is a tree with vertices labeled 1 through \( n \), where \( n \) is the number of vertices in the graph. A rooted tree is a tree where one vertex is designated the root. A forest is a collection of trees. In particular, a \( k \)-forest as a collection of \( k \) trees.

**Example 4.7** Below are two representations of the same Cayley Tree. The only difference is that the tree on the right is rooted at node 12 while the other tree is not rooted.

![Cayley Tree Diagram](attachment:image.png)

**Theorem 4.9** The number of Cayley Trees on \( n \) vertices is

\[
n^{n-2}.
\]

**Proof.** Let \( D_n \) consist of the set of rooted Cayley Trees. The set \( F_{n,k}(\mathcal{F}) \) is the collection of \( k \)-forests of rooted Cayley trees on \( n \) nodes. This set is in one-to-one correspondence with the set of Cayley trees with \( n + 1 \) nodes where the smallest node has degree \( k \). As the picture below suggests, starting with Cayley trees \( T_1, T_2, \ldots, T_k \) rooted at \( r_1, r_2, \ldots, r_k \) respectively, we can easily form a single Cayley tree with one additional node. First insert a "0" node and then draw an edge from each of the roots \( r_1, r_2, \ldots, r \) to this new node.

![Diagram](attachment:image.png)
Let $c_m$ be the number of Cayley Trees on $n$ nodes. Then $D_m = mc_m$ and

$$1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{k=1}^{n} |F_{n,k}| x^k = e^x \sum_{m \geq 1} mc_m t^m (m_t)^m$$

Setting $x = 1$, yields

$$\sum_{n \geq 0} c_{n+1} \frac{t^n}{n!} = e^{\sum_{m \geq 1} c_m t^m (m_t)^m}$$

Letting $f(t) = \sum_{n \geq 1} c_n t^{n/(n-1)!}$ yields

$$\frac{1}{t} f(t) = e^{f(t)}$$

or in other words

$$f(t) = te^{f(t)}.$$ 

\textbf{Theorem 4.10 (Lagrange Inversion)} Suppose

$$R(x) = R_0 + R_1 x + R_2 x^2 + \cdots$$

with $R_0 \neq 0$ and $f(x) = f_1 x + f_2 x^2 + \cdots$ such that

$$f(x) = x R(f(x))$$

then for any pair of integers $k \leq n$

$$[f(x)]^k \bigg|_{x=n} = \frac{k}{n} \left[ R(x) \right]^n \bigg|_{x=n-k}.$$ 

Using the Lagrange Inversion formula with $k = 1$ and $f(x) = xe^{f(x)}$ (i.e. $R(x) = e^x$)

$$\frac{c_n}{(n-1)!} = f(x) \bigg|_{x=n} = \frac{1}{n} \left( e^x \right)^n \bigg|_{x=n-1} = \frac{1}{n} e^{xn} \bigg|_{x=n-1} = \frac{n^{n-1}}{n(n-1)!}$$

and therefore $c_n = n^{n-2}$, as desired.

\hfill \Box

\textbf{Remark 4.4} Suppose we are given a function $F(x) = x/R(x)$. We wish to determine the functional inverse of $F$. In particular, we wish to find the series expansion for the function $f(x)$ such that

$$F(f(x)) = x$$

By definition, this means that

$$f(x) \bigg|_{x=R(f(x))} = x$$

which of course happens if and only if $f(x) = xR(f(x))$. In other words, the function $f(x)$ in the Lagrange Inversion formula is the functional inverse of $x/R(x)$. 

4.7.1 Functional Digraphs

Let $\mathcal{F}_n = \{ f : [n] \to [n] \mid f(1) = 1 \ f(n) = n \}$. Clearly we have $|\mathcal{F}_n| = n^{n-2}$ and therefore it is natural to seek out a bijection between such functions and Cayley trees. For example, let $f \in \mathcal{F}_n$ be defined explicitly by

$$
\begin{array}{c|cccccccccccccc}
  x & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
  f(x) & 1 & 10 & 13 & 15 & 10 & 1 & 7 & 15 & 10 & 14 & 16 & 13 & 1 & 2 & 16 & 16 \\
\end{array}
$$

In general, we can build up a directed graph, or digraph, associated with $f$ in the following manner. For each $1 \leq i \leq n$, draw the following directed edge

$$
i \xrightarrow{} f(i)
$$

The resulting graph will necessarily contain a collection of disconnected graphs, each of which contains exactly one cycle. Now arrange the graphs from left to right according to the smallest element, $s_i$, in each cycle such that

$$
1 = s_1 < s_2 < \cdots < s_k = n
$$

To construct the corresponding Cayley tree, erase the edges between $s_i$ and $f(s_i)$ and add an edge between $s_i$ and $f(s_{i+1})$.

Notice that we can safely remove the arrows since each edge invariably points towards node $n$. In other words, we can easily recover the arrows on each edge by following the unique path from any vertex to node $n$. To reverse the rest of the process, all that remains is to determine the sequence $s_1, \ldots, s_k$. To this end, one can readily verify that

$$
s_i = \text{the smallest number on the path from } s_{i-1} \text{ to } n \text{ (not including } s_{i-1})
$$

with initial conditions $s_1 = 1$ and $s_k = n$. 

\[ \text{Diagram of digraphs and Cayley trees.} \]
4.7.2 Fall and Rise Edges

Given a function \( f \in \mathcal{F}_n \), we define the weight of a directed edge in the digraph of \( f \) by

\[
w( i \rightarrow j ) = \begin{cases} 
q & \text{if } i > j \text{ "fall edges"} \\
t & \text{if } i \leq j \text{ "rise edges"}
\end{cases}
\]

and the weight of \( f \) by

\[
w(f) = \prod_{i=2}^{n-1} w(i \rightarrow f(i)) = q^{\text{fall}(f)}t^{\text{rise}(f)}
\]

where \( \text{fall}(f) \) equals the number of fall edges and \( \text{rise}(f) \) equals the number of rise edges in the digraph of \( f \). Summing over all such functions, we have

\[
\sum_{f \in \mathcal{F}_n} w(f) = \prod_{i=2}^{n-1} ((i-1)q + (n+1-i)t)
\]

Now consider the Cayley tree \( T \) that is constructed using the function \( f \). Define the weight of \( T \) by

\[
w(T) = \prod w(c) = tw(f).
\]

The last equality is due the extra rise edge that is necessarily added in the construction of a Cayley tree from a function.

We next answer the following question: What is the expected number of fall edges in a Cayley Tree? Since for the moment we are not interested in rise edges, we can set \( t = 1 \), which yields

\[
\sum q^{\text{fall}(T)} = \prod_{i=2}^{n-1} ((i-1)q + n + 1 - i).
\]

Now take the logarithmic derivative of both sides.

\[
\frac{d}{dq} \left( \ln \sum q^{\text{fall}(T)} \right) \bigg|_{q=1} = \frac{\sum \text{fall}(T) q^{\text{fall}(T)-1}}{\sum q^{\text{fall}(T)}} \bigg|_{q=1} = \frac{\sum \text{fall}(T)}{n^{n-2}}
\]

\[
\frac{d}{dq} \left( \ln \prod_{i=2}^{n-1} ((i-1)q + n + 1 - i) \right) \bigg|_{q=1} = \frac{d}{dq} \left( \sum_{i=2}^{n-1} \ln((i-1)q + n + 1 - i) \right) \bigg|_{q=1}
\]

\[
= \sum_{i=2}^{n-1} \frac{i - 1}{(i-1)q + n + 1 - i} \bigg|_{q=1}
\]

\[
= \sum_{i=2}^{n-1} \frac{i - 1}{n} \sum_{i=1}^{n-2} i = \frac{(n-1)(n-2)}{2n}
\]

Therefore we have that the expected number of fall edges is given by

\[
\frac{\sum \text{fall}(T)}{n^{n-2}} = \frac{(n-1)(n-2)}{2n}.
\]
Similarly, we can ask the question: What is the expected number of rise edges in a Cayley Tree? Setting \( q = 1 \), we have

\[
\sum t^{\text{rise}(T)} = t \prod_{i=2}^{n-1} (i - 1 + (n + 1 - i)t).
\]

Again, taking the logarithmic derivative (with respect to \( t \)) of both sides produces

\[
\frac{d}{dt} \left( \ln \sum t^{\text{rise}(T)} \right) \bigg|_{t=1} = \frac{\sum \text{rise}(T) t^{\text{rise}(T)-1} \bigg|_{t=1}}{\sum q^{\text{rise}(T)} \bigg|_{t=1}} = \frac{\sum \text{rise}(T)}{n^{n-2}}
\]

\[
\frac{d}{dt} \left( \ln t \prod_{i=2}^{n-1} (i - 1 + (n + 1 - i)t) \right) \bigg|_{t=1} = \frac{d}{dt} \left( \ln t + \sum_{i=2}^{n-1} \ln(i - 1 + (n + 1 - i)t) \right) \bigg|_{t=1}
\]

\[
= \frac{1}{t} + \sum_{i=2}^{n-1} \frac{n + 1 - i}{i - 1 + (n + 1 - i)t} \bigg|_{t=1}
\]

\[
= 1 + \sum_{i=2}^{n-1} \frac{n + 1 - i}{n} = 1 + \frac{1}{n} \sum_{i=2}^{n-1} i = \frac{(n - 1)(n + 2)}{2n}
\]

And therefore the expected number of rise edges in a Cayley tree is

\[
\frac{\sum \text{rise}(T)}{n^{n-2}} = \frac{(n - 1)(n + 2)}{2n}.
\]

**Remark 4.5** Notice that

\[
\frac{(n - 1)(n - 2)}{2n} + \frac{(n - 1)(n + 2)}{2n} = n - 1
\]

which is precisely the number of edges in a tree on \( n \) vertices, as it must be.
4.8 Operations On Species

**Definition 4.6** Given a combinatorial species \( D = \{D_0, D_1, D_2, \ldots \} \), the **deck enumerator** is given by

\[
\mathcal{E}(D) = \sum_{m \geq 0} |D_m| \frac{x^m}{m!}
\]

and the **weighted deck enumerator** is given by

\[
\mathcal{E}_w(D) = \sum_{m \geq 0} w(D_m) \frac{x^m}{m!}
\]

where \( w \) is a weight function defined on cards and \( w(D_m) = \sum_{c \in D_m} w(c) \).

Given two combinatorial species

\[
\mathcal{A} = \{A_n\}_{n \geq 0} \quad \text{and} \quad \mathcal{B} = \{B_n\}_{n \geq 0}
\]

and corresponding weight functions \( w_{\mathcal{A}} \) and \( w_{\mathcal{B}} \), we define the following combinatorial species.

**Definition 4.7** The combinatorial species \( \mathcal{A} + \mathcal{B} \) is given by the sequence of disjoint unions \( \{A_n \cup B_n\}_{n \geq 0} \) with weight function \( w_{\mathcal{A} + \mathcal{B}} \) defined by

\[
w_{\mathcal{A} + \mathcal{B}}(c) = \begin{cases} 
  w_{\mathcal{A}}(c) & \text{if } c \in \mathcal{A} \\
  w_{\mathcal{B}}(c) & \text{if } c \in \mathcal{B}
\end{cases}
\]

**Remark 4.6** The disjoint union of \( \mathcal{A} \) and \( \mathcal{B} \) is formally defined as

\( \mathcal{A} \cup \mathcal{B} = \{(a, 1) \mid a \in \mathcal{A}\} \cup \{(b, 2) \mid b \in \mathcal{B}\} \).

In other words, if \( \mathcal{A} \) and \( \mathcal{B} \) contain the same object, then \( \mathcal{A} \cup \mathcal{B} \) contains two copies of that object. In terms of cards, this is equivalent to having all of the cards from the \( \mathcal{A} \) deck colored red and all of the cards from the \( \mathcal{B} \) deck colored blue. In what follows, we will simply place a “1” or a “2” in the lower right hand corner of the card to distinguish between the two decks.

\[
\begin{align*}
\text{\( \mathcal{A} \):} & \quad \begin{array}{c}
1 \\
2 \\
3
\end{array} \\
\text{\( \mathcal{B} \):} & \quad \begin{array}{c}
1 \\
2 \\
3
\end{array}
\end{align*}
\]
**Definition 4.8** The combinatorial species $\mathcal{A} \times \mathcal{B}$ is a sequence of decks where cards of weight $n \geq 0$ are constructed in the following manner

1. Select $k$ between 0 and $n$ and write $[n]$ as the disjoint union of two sets $S = \{s_1 < s_2 < \cdots < s_k\}$ and $T = \{t_1 < t_2 < \cdots < t_{n-k}\}$.
2. Select a card from $c_1 \in A_k$ and $c_2 \in B_{n-k}$
3. On card $c_1$, relabel the $i^{th}$ circle with $s_i$ and on card $c_2$, relabel the $j^{th}$ circle with $t_j$.
4. The card $c$ is formed by placing the relabeled $c_1$ and $c_2$ side by side.

The corresponding weight function $w_{\mathcal{A} \times \mathcal{B}}$ is defined by

$$w_{\mathcal{A} \times \mathcal{B}}(c) = w_A(c_1)w_B(c_2).$$

**Example 4.8** A card from $\mathcal{A} \times \mathcal{B}$ where $\mathcal{A}$ represents the collection of ordered lists and $\mathcal{B}$ represents the collection of cycles.

**Definition 4.9** The combinatorial species $\mathcal{A}'$ is given by $\{C_n\}_{n \geq 0}$ where cards of weight $n$ are constructed by taking a card $c' \in A_{n+1}$ and filling in the circle numbered $n+1$. The corresponding weight function $w_{\mathcal{A}'}$ defined by

$$w_{\mathcal{A}'}(c) = w_A(c').$$

**Example 4.9** A card of weight 4 in $\mathcal{A}'$ where $\mathcal{A}$ represents the collection of cycles.
**Definition 4.10** If $B_0 = \emptyset$, then the combinatorial species $A(B)$ is a sequence of decks where cards of weight $n \geq 0$ are constructed in the following manner. If $n = 0$, then the only card of weight $n$ is $A_0$. Otherwise,

1. Select $k$ between 1 and $n$.
2. Select a set partition of $[n]$ into $k$ parts, $\{S_1, S_2, \ldots, S_k\}$ where $S_i = \{s_{i1} < s_{i2} < \cdots < s_{in_i}\}$ and $1 = s_{11} < s_{21} < s_{31} < \cdots < s_{k1}$.
3. Select a card $a \in A_k$.
4. For each $i = 1, \ldots, k$, select a card $b_i \in B_{n_i}$. Relabel the $j^{th}$ circle on $b_i$ with $s_{ij}$.
5. The card $c$ is formed by placing the relabeled card $b_i$ into circle $i$ on $a$.

The corresponding weight function $w_{A(B)}$ is defined by

$$w_{A(B)}(c) = w_A(a)w_B(b_1) \cdots w_B(b_k).$$

**Example 4.10** Given the set partition $\{\{1, 3, 5, 6\}, \{2, 8\}, \{4, 7, 9\}\}$ the following card is an element of $A(B)$, where $A$ represents the collection of ordered lists and $B$ represents sets.
Proof.

\[ \mathcal{E}_w(A) + \mathcal{E}_w(B) = \sum_{n \geq 0} \sum_{c \in A} w_A(c) \frac{x^n}{n!} + \sum_{n \geq 0} \sum_{c \in B} w_B(c) \frac{x^n}{n!} = \sum_{n \geq 0} \sum_{c \in A + B} w_{A + B}(c) \frac{x^n}{n!} = \mathcal{E}_w(A + B) \]

\[ \mathcal{E}_w(A) \mathcal{E}_w(B) = \left( \sum_{r \geq 0} \sum_{c \in A_r} w_A(c) \frac{x^r}{r!} \right) \left( \sum_{s \geq 0} \sum_{c \in B_s} w_B(c) \frac{x^s}{s!} \right) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} \sum_{c_1 \in A_k} w_A(c_1) \sum_{c_2 \in B_{n-k}} w_B(c_2) \frac{x^n}{n!} \right) = \sum_{n \geq 0} \sum_{c \in A \times B} w_{A \times B}(c) \frac{x^n}{n!} = \mathcal{E}_w(A \times B) \]

\[ \mathcal{E}_w(A') = \sum_{n \geq 0} w(A'_n) \frac{x^n}{n!} = \sum_{n \geq 0} w(A_{n+1}) \frac{x^n}{n!} = \sum_{n \geq 0} (n+1)w(A_{n+1}) \frac{x^n}{n!} = \frac{d}{dx} \sum_{n \geq 0} w(A_{n+1}) \frac{x^{n+1}}{(n+1)!} = \frac{d}{dx} \mathcal{E}_w(A) \]

To prove the last claim, we start by computing \( w_{A \times B}(C_n) \).

\[ w_{A \times B}(C_n) = \sum_{k=1}^{n} \sum_{\substack{p_1, \ldots, p_n \geq 0 \ \text{such that} \ p_1 + 2p_2 + \cdots + np_n = k \ \text{and} \ p_1 + \cdots + p_n = n}} \frac{n! w(A_k) w(B_1)^{p_1} \cdots w(B_n)^{p_n}}{p_1! p_2! \cdots p_n! (1!)^{p_1} (2!)^{p_2} \cdots (n!)^{p_n}} \]

\[ = \sum_{k=1}^{n} \sum_{\substack{p_1, \ldots, p_n \geq 0 \ \text{such that} \ p_1 + 2p_2 + \cdots + np_n = n \ \text{and} \ p_1 + \cdots + p_n = k}} \frac{n! w(A_k)}{p_1! p_2! \cdots p_n!} \left( \frac{w(B_1)}{1!} \right)^{p_1} \cdots \left( \frac{w(B_n)}{n!} \right)^{p_n} \]

\[ = n! \sum_{k=1}^{n} \frac{w(A_k)}{k!} \sum_{\substack{p_1, \ldots, p_n \geq 0 \ \text{such that} \ p_1 + 2p_2 + \cdots + np_n = n \ \text{and} \ p_1 + \cdots + p_n = k}} \frac{k!}{p_1! p_2! \cdots p_n!} \left( \frac{w(B_1)}{1!} \right)^{p_1} \cdots \left( \frac{w(B_n)}{n!} \right)^{p_n} \]
\[ \mathcal{E}_w(\mathcal{A})(\mathcal{E}_w(\mathcal{B})) \bigg|_{x^n} = w(A_0) + \sum_{k \geq 1} \frac{w(A_k)(\mathcal{E}_w(\mathcal{B}))^k}{k!} x^n \]

\[ = w(A_0) + \sum_{k \geq 1} \frac{w(A_k)}{k!} \left( \frac{w(B_1)}{1!} x + \frac{w(B_2)}{2!} x^2 + \cdots + \frac{w(B_n)}{n!} x^n \right)^k \bigg|_{x^n} \]

\[ = \sum_{k=1}^{n} \frac{w(A_n)}{k!} \sum_{p_1, p_2, \ldots, p_n \text{: } p_1 + 2p_2 + \cdots + np_n = n} \frac{k!}{p_1!p_2! \cdots p_n!} \left( \frac{w(B_1)}{1!} \right)^{p_1} \cdots \left( \frac{w(B_n)}{n!} \right)^{p_n} \]

\[ = w_{\mathcal{A}(\mathcal{B})}((C_n)/n! \]

\[ = \mathcal{E}_w(\mathcal{A}(\mathcal{B})) \bigg|_{x^n} \]

\[ \square \]

**Example 4.11** A combinatorial proof of

\[ \frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{1}{(1-x)^2} \]

would proceed as follows. Let \( \mathcal{A}_n \) be the set of linearly ordered lists of length \( n \).

\[
\begin{array}{c}
\sigma_1 \rightarrow \sigma_2 \rightarrow \cdots \rightarrow \sigma_n \\
| \sigma \in S_n
\end{array}
\]

Then \( \mathcal{A}_n' \) consists of cards from \( \mathcal{A}_{n+1} \) with circle \( n+1 \) filled in.

\[
\begin{array}{c}
\sigma_1 \rightarrow \cdots \rightarrow \sigma_{i-1} \rightarrow \sigma_i \rightarrow \cdots \rightarrow \sigma_n \\
| \sigma \in S_n
\end{array}
\]

Notice that elements of \( \mathcal{A}_n' \) can be easily put into a one-to-one correspondence with cards of the form

\[
\begin{array}{c}
\sigma_1 \rightarrow \cdots \rightarrow \sigma_i \rightarrow \sigma_{i+1} \\
\sigma_{i+2} \rightarrow \cdots \rightarrow \sigma_n \\
| \sigma \in S_n
\end{array}
\]

\[ \text{which are elements of } \mathcal{A} \times \mathcal{A}. \] Therefore

\[ \frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{d}{dx} \mathcal{E}(\mathcal{A}) = \mathcal{E}(\mathcal{A}') = \mathcal{E}(\mathcal{A} \times \mathcal{A}) = \mathcal{E}(\mathcal{A})\mathcal{E}(\mathcal{A}) = \left( \frac{1}{1-x} \right)^2 \]
4.9 Derangements

Permutations with no fixed points are called derangements. We can construct a formula for the number of derangements in $S_n$ by considering the following decks of cards. First, let $P_n$ consist of cards with permutations in $S_n$ written in cycle notation. Second, let $N_n$ consist of a single card with no structure and finally, let $D_n$ consist of cards with derangements in $S_n$ with written in cycle notation. By considering the following example, we can easily see that $P$ is isomorphic to $N \times D$.

**Example 4.12** Let $\sigma = (1, 6, 3)(2)(4, 9, 8, 5)(7)$

Applying Lemma 4.11 results in

$$\frac{1}{1-x} = \sum_{n \geq 0} n! \frac{x^n}{n!} = \mathcal{E}(P)$$

$$= \mathcal{E}(N)\mathcal{E}(D)$$

$$= \left( \sum_{n \geq 0} |N_n| \frac{x^n}{n!} \right) \mathcal{E}(D)$$

$$= \left( \sum_{n \geq 0} \frac{x^n}{n!} \right) \mathcal{E}(D) = e^x \mathcal{E}(D)$$

or in other words

$$\frac{e^{-x}}{1-x} = \mathcal{E}(D) = \sum_{n \geq 0} \frac{|D_n| x^n}{n!}.$$

Taking the coefficient of $x^n$ on both sides yields

$$\frac{|D_n|}{n!} = \left( \sum_{k \geq 0} \frac{(-1)^k x^k}{k!} \right) \left( \sum_{m \geq 0} \frac{x^m}{m!} \right) \bigg|_{x^n} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

and therefore

$$|D_n| = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \approx \frac{n!}{e}.$$
4.10 Tangent and Secant Numbers

**Definition 4.11** A permutation $\sigma \in S_n$ is said to be *alternating* if
\[ \sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots \]

**Example 4.13** $\sigma = (2,4,1,6,3,5)$

Define the following decks of cards

\[ A_{2n+1} = \left\{ \begin{array}{c} \sigma_1 \quad \cdots \quad \sigma_{2n} \\ \sigma_1 \quad \sigma_3 \quad \cdots \quad \sigma_{2n+1} \end{array} \right\} \mid \sigma \in S_{2n+1} \text{ alternating} \]

\[ B_{2n} = \left\{ \begin{array}{c} \sigma_1 \quad \cdots \quad \sigma_{2n} \\ \sigma_1 \quad \cdots \quad \sigma_{2n-1} \end{array} \right\} \mid \sigma \in S_{2n} \text{ alternating} \]

\[ C_0 = \left\{ \begin{array}{} \end{array} \right\} \]

where for all $n \geq 0$, the decks $A_{2n}$, $B_{2n+1}$ and $C_{n+1}$ are empty. Note that $B_0$ consists of a single empty card, like $C_0$. Elements of $A_{2n+1}$ are referred to as odd alternating permutations and elements of $B_{2n}$ are referred to as even alternating permutations.
In a manner similar to Example 4.11, we can put cards from $A'$ into a one-to-one correspondence with cards from $A \times A + C$.

Therefore

$$A' \approx A \times A + C$$

Letting $f(x) = \mathcal{E}(A)$, we have

$$f'(x) = 1 + f(x)^2$$

with initial condition $f(0) = 0$. In other words,

$$\frac{f'(x)}{1 + f(x)^2} = 1.$$

Integrating both sides reveals $f(x) = \tan x$. For this reason, the values $|A_{2n+1}|$ are referred to as tangent numbers. We can also put cards from $B'$ into a one-to-one correspondence with cards from $A \times B$.

Therefore

$$B' \approx A \times B$$

Letting $g(x) = \mathcal{E}(B)$, we have

$$g'(x) = f(x)g(x) = g(x)\tan x$$

with initial condition $g(0) = 1$. In other words,

$$\frac{g'(x)}{g(x)} = \tan x.$$

Integrating both sides yields $g(x) = \sec x$. For this reason, the values $|B_{2n}|$ are referred to as secant numbers.
Example 4.14 The trigonometric identity $1 + \tan^2 x = \sec^2 x$ can be proved combinatorially by providing a bijection between $\mathcal{C} + \mathcal{A} \times \mathcal{A}$ and $\mathcal{B} \times \mathcal{B}$. The main idea is to move the larger of the two last elements to the other side of the card. For example

Example 4.15 Similarly, we can show that $\frac{d}{dx} \sec^2 x = 2 \sec^2 x \tan x$ by providing a bijection between $(\mathcal{B} \times \mathcal{B})^c$ and $\mathcal{A} \times \mathcal{B} \times \mathcal{B} \times T$ where $T = \{T_n\}$ is given by

$$T_n = \begin{cases} \begin{pmatrix} 1 & 2 \end{pmatrix} & \text{if } n = 0 \\ \emptyset & \text{if } n > 0 \end{cases}$$
**Remark 4.7** The tangent and secant numbers can be constructed in a manner similar to that of Pascal’s triangle. Start by placing a 1 on top (row 0) and then fill in row \( i > 0 \) using the following rules:

1. if \( i \) is odd, then place a 0 at the left end of row \( i \) and let the \( j \)th number (from left to right) in row \( i \) be the sum of the \((j-1)\)st number in row \( i \) and the \((j-1)\)st number in row \( i - 1 \). For example:

   \[
   \cdots \quad x \\
   \cdots \quad y \quad x + y
   \]

2. if \( i \) is even, then place a 0 at the right end of row \( i \) and let the \( j \)th number (from right to left) in row \( i \) be the sum of the \((j-1)\)st number in row \( i \) and the \((j-1)\)st number in row \( i - 1 \). For example:

   \[
   x \cdots \\
   x + y \quad y \cdots
   \]

The first 6 rows are as follows:

<table>
<thead>
<tr>
<th></th>
<th>Secant #’s</th>
<th>Tangent #’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>16</td>
<td>16</td>
</tr>
</tbody>
</table>

Now the Taylor series expansions for \( \tan x \) and \( \sec x \) can be computed using the appropriate numbers from the above triangle.

\[
\tan x = 1 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{16}{5!}x^5 + \cdots = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots
\]

\[
\sec x = 1 + \frac{1}{0!}x + \frac{0}{1!}x^2 + \frac{1}{2!}x^3 + \frac{0}{3!}x^4 + \frac{5}{4!}x^5 + \cdots = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots
\]
4.11 Proof of the Lagrange Inversion Formula

It is well known that the set of formal power series, $F(x)$, that satisfy
\[
\begin{align*}
&i) \quad F(0) = 0 \\
&ii) \quad F'(0) \neq 0
\end{align*}
\]
form a group under the operation of formal substitution. Note that every such formal power series can be written in the form
\[F(x) = x/R(x)\]
where $R(x)$ is a formal power series with nonzero constant term. It is natural then to ask, given $R(x)$, what is the inverse of $F(x)$ in this group? The Lagrange Inversion formula gives us an explicit method for calculating the coefficients of the inverse power series.

Let us recall the statement of the Lagrange Inversion formula. Given two formal power series $R(x) = R_0 + R_1 x + R_2 x^2 + \cdots$ with $R_0 \neq 0$ and $f(x) = f_1 x + f_2 x^2 + \cdots$ such that
\[f(x) = x R(f(x))\] (4.2)
then the coefficients of $f(x)^k$ can be computed in the following manner
\[
[f(x)]^k \bigg|_{x^n} = \frac{k}{n} [R(x)]^n \bigg|_{x^{n-k}}.
\]

We begin the proof by defining the following decks of cards. Let $C$ be the combinatorial species whose $n^{th}$ deck consists of the set of rooted Cayley trees on $n$ nodes.

\[
\begin{array}{c}
12 \\
6 \quad 2 \\
5 \quad 9 \\
3 \quad 8 \\
11 \quad 13
\end{array}
\]

where $d_i$ is the number of children of vertex $i$ in $c$. The weight of the above tree is
\[R_3! R_0 R_0 R_1 R_0 R_0 R_2 R_0 R_2! R_0 R_0 R_2! R_0 R_3! R_0 R_2! R_0 R_2! = 288 R_0^6 R_1 R_2^3 R_3^2.
\]

Next, let $\mathcal{R}$ be the combinatorial species whose $n^{th}$ deck consists of the single card

\[r_n =
\begin{array}{cccc}
\circ \\
1 \\
\circ \\
2 \\
\cdots \\
\circ \\
n \\
\end{array}
\]

and therefore
\[
\mathcal{E}_w(\mathcal{R}) = \sum_{n \geq 0} n! R_n \frac{x^n}{n!} = \sum_{n \geq 0} R_n x^n = R(x).
\]

And finally, let $\mathcal{X}$ be the combinatorial species which consists of a single deck with one card

\[x_1 =
\begin{array}{c}
\circ \\
1 \\
\end{array}
\]

\[w(x_1) = 1\]
and therefore
\[ E_w(\mathcal{X}) = x. \]

Our first task is to use these species to construct \( f(x) \), the inverse of \( x/R(x) \). To this end, we point out that
\[ C \approx \mathcal{X} \times R(\mathcal{C}) \]
as illustrated in the following example, which corresponds to the rooted Cayley tree shown on the previous page.

\[ 1 \quad 2 \quad 3 \]
\[ \begin{array}{ccc}
  1 & 10 & 4 \\
  3 & 6 & 7 \\
  8 & 2 & 11 \\
  9 & 13 & 5 \\
\end{array} \]

Therefore
\[ E_w(C) = E_w(\mathcal{X} \times R(\mathcal{C})) = E_w(\mathcal{X})E_w(R(\mathcal{C})) = xE_w(R(\mathcal{C}))(E_w(\mathcal{C})) = xR(E_w(\mathcal{C})) \]

and thus \( E_w(C) \) satisfies Equation (4.2). It remains to show that
\[ (E_w(C))^k \Big|_{x^n} = \frac{k}{n} (R(x))^n \Big|_{x^{n-k}}. \]

(4.3)

To do so, we first consider the set \( \mathcal{F}_{n,k} \) of \( k \)-forests of rooted Cayley trees on \( n \) nodes with one node distinguished. The weight of \( f \in \mathcal{F}_{n,k} \) is defined as
\[ w(f) = \prod_{i=1}^{k} w(c_i) \]

where \( c_i \) is the \( i^{th} \) rooted Cayley tree in \( f \) and \( w(c_i) \) is defined in the same way as for the combinatorial species \( \mathcal{C} \). Therefore
\[ \sum_{f \in \mathcal{F}_{n,k}} w(f) = \frac{n}{k!} [E_w(\mathcal{C})]^k \bigg|_{x^n} = \frac{n}{k!} [E_w(\mathcal{C})]^k \bigg|_{x^n}. \]

(4.4)

The first equality follows from the fact that
\[ [E_w(\mathcal{C})]^k = E_w(\mathcal{C} \times \cdots \times \mathcal{C}) \]
k times
which can be thought of as the deck enumerator for ordered $k$-forests of rooted Cayley trees. Dividing by $k!$ removes the order from these forests and multiplying by $n$ is equivalent to selecting the distinguished node.

**Example 4.16** Let $n = 13$ and $k = 3$. A sample of $\mathcal{F}_{n,k}$ is displayed below. The root of each Cayley tree is drawn at the top. The distinguished element is marked by a $\blacktriangleright$.

Now consider the set $\mathcal{M}_{n,k}$ of maps, $m$, from a $n-k$ element subset of $[n]$ to $[n]$ with one of the elements of $[n] - \text{dom}(m)$ distinguished. The weight of $m$ is defined to be

$$\prod_{i=1}^{n} R_{v_i} v_i!$$

where $v_i$ is the number of elements of the domain of $m$ that get mapped to $i$. Therefore

$$\sum_{f \in \mathcal{M}_{n,k}} w(f) = k \binom{n}{k} \sum_{0 \leq v_1, v_2, \ldots, v_n} \left( \begin{array}{c} n-k \\ v_1, v_2, \ldots, v_n \end{array} \right) R_{v_1} v_1! R_{v_2} v_2! \cdots R_{v_n} v_n!$$

$$= k \frac{n!}{k!(n-k)!} \sum_{0 \leq v_1, v_2, \ldots, v_n} \frac{(n-k)!}{v_1!v_2!\cdots v_n!} R_{v_1} v_1! R_{v_2} v_2! \cdots R_{v_n} v_n!$$

$$= k \frac{n!}{k!} \sum_{0 \leq v_1, v_2, \ldots, v_n} R_{v_1} R_{v_2} \cdots R_{v_n}$$

$$= k \frac{n!}{k!} [R(x)]^n \bigg|_{x^{n-k}}.$$ 

The first equality follows from the fact that in order to construct an arbitrary element $m \in \mathcal{M}_{n,k}$, we first choose the domain of $m$ in $\binom{n}{k}$ ways. Then we distinguish one of the elements of $[n] - \text{dom}(m)$ in $k$ ways. And finally, summing over all possible sequences $v_1, \ldots, v_n$, we choose the $v_1$ preimages of 1, $v_2$ preimages of 2, \ldots and the $v_n$ preimages of $n$ in exactly $\binom{n-k}{v_1, v_2, \ldots, v_n}$ ways.
Example 4.17 Let $n = 15$ and $k = 3$. A sample of $\mathcal{M}_{n,k}$ is displayed below. As before, the distinguished element is marked by a $\blacktriangleright$.

It remains to show that
\[
\sum_{f \in \mathcal{F}_{n,k}} w(f) = \sum_{m \in \mathcal{M}_{n,k}} w(m).
\]

This will be accomplished by the following weight preserving bijection. First, take an arbitrary element $m \in \mathcal{M}_{n,k}$ and draw the trees first followed by the graphs with one cycle. Make sure the tree with the distinguished root is the rightmost tree and that the graphs are drawn from left to right according to the smallest element in the cycle, $s_i$. In other words, we have
\[ s_1 < s_2 < \cdots < s_l. \]

Now remove all of the edges from $s_i$ to $m(s_i)$ and add an edge from $s_i$ to $m(s_{i+1})$, including an edge from $r_k$ to $m(s_1)$.

The rightmost tree is now rooted at $s_l$ instead of $r_k$. This process insures that the weight of the resulting element of $\mathcal{F}_{n,k}$ is precisely the same as $m \in \mathcal{M}_{n,k}$. To reverse the process, we only need to point out that
\[ s_i = \text{the smallest number on the unique path from } s_{i-1} \text{ to } s_l \text{ (not including } s_{i-1}) \]
where $s_0 = r_k$. And this concludes the proof.
4.12 Formal Languages and Planar Trees

Let \( X = \{x_1, x_2, \ldots, x_n\} \) be an alphabet of noncommutative variables (i.e. \( x_i x_j \neq x_j x_i \) for \( i \neq j \)). We will also use the additive notation \( X = x_1 + x_2 + \cdots + x_n \) to represent our alphabet. Let \( X^* \) represent the set of all finite words with letters in \( X \). Using additive notation, we can formally define \( X^* \) as follows:

\[
X^* = \sum_{k \geq 0} \sum_{\substack{w \in X^* \mid |w| = k}} w
\]

\[
= \sum_{k \geq 0} (x_1 + x_2 + \cdots + x_n)^k
\]

\[
= \frac{1}{1 - X}
\]

A language is defined to be any subset of \( X^* \). Given two languages, \( \mathcal{L} \) and \( \mathcal{M} \), we formally define \( \mathcal{L} + \mathcal{M} \) and \( \mathcal{L} \mathcal{M} \) as follows:

\[
\mathcal{L} + \mathcal{M} = \sum_{w \in \mathcal{L}} w + \sum_{u \in \mathcal{M}} u
\]

\[
\mathcal{L} \mathcal{M} = \sum_{w \in \mathcal{L}} \sum_{u \in \mathcal{M}} wu
\]

We now consider planar trees in terms of formal languages. A planar tree is simply a tree drawn in the plane. Unlike Cayley trees however, planar trees have a root and the nodes are unlabeled. Throwing all conventional wisdom into the wind, we will draw the root node at the top of our tree with all edges emanating downwards. An internal node is a node which branches off, or in other words, has at least one edge emanating downward from it, leading towards what we will refer to as a child node. An external node is a node with no children.

Example 4.18 Below is an example of a planar tree. Internal and external nodes are marked by a \( \bullet \) and \( \circ \), respectively.

![Planar Tree Example](image)

The word of a tree is a sequence of letters from the alphabet \( X = \{x_0, x_1, x_2, \ldots\} \) where the letter \( x_i \) corresponds to a node in the tree with \( i \) children. In general, the word of a tree \( T \), denoted \( w(T) \), is formed as follows.
The word corresponding to the tree displayed in Example 4.18 is

\[ x_4 x_2 x_0 x_3 x_0 x_0 x_2 x_1 x_0 x_2 x_0 x_0 x_0. \]

A complete binary tree is a planar tree in which every node has either 0 or 2 children.

**Example 4.19** Below is an example of a complete binary tree.

Let \( B \) equal the set of all complete binary trees and define the language \( L \) to be

\[ L = \sum_{T \in B} w(T) \]

Since the root of any complete binary tree has either 0 or 2 children, \( L \) must satisfy the following functional equation

\[ L = x_0 + x_2 L L. \]

Setting \( x_0 = t \) and \( x_2 = x \) and allowing \( x \) and \( t \) to commute, we have

\[ L = \sum_{T \in B} e(T) x^i(T) = t + x L^2 \quad (4.5) \]

where \( i(T) \) is the number of internal nodes in \( T \) and \( e(T) \) is the number of external nodes in \( T \). Furthermore, rearranging the terms in the following manner

\[ L - t = x L^2 = x R(L - t) \]
where $R(x) = (x + t)^2$, allows us to apply the Lagrange inversion formula.

$$
\mathcal{L} - t \bigg|_{x^n} = \frac{1}{n} (x + t)^{2n} \bigg|_{x^{n-1}} = \frac{1}{n} \sum_{k=0}^{n} \binom{2n}{k} x^{k} t^{2n-k} \bigg|_{x^{n-1}} = \frac{1}{n} \binom{2n}{n-1} t^{n+1} = \frac{1}{n+1} \binom{2n}{n} t^{n+1}
$$

The above computations produce several results. The first being that any complete binary tree with $n$ internal nodes must have precisely $n + 1$ external nodes. The second result is that the number of complete binary trees with $n$ internal nodes is given by the $n$th Catalan number. Recall that this is also the number of Catalan words and thus we can expect there to be a bijection between these two sets. Given a complete binary tree, $T$, we can form a Catalan word $\varphi(T)$ in the following manner

$$
T = T_1 \quad \longleftrightarrow \quad \varphi(T) = 0\varphi(T_1)1\varphi(T_2)
$$

with initial condition $\varphi(\varnothing) = \emptyset$. The tree shown in Example 4.19 corresponds to the following word, which we have depicted in it’s lattice path form.

And finally, setting $t = 1$ in Equation 4.5 yields

$$
\mathcal{L} = 1 + x \mathcal{L}^2.
$$

Using the quadratic formula we can solve for $\mathcal{L}$ as

$$
\mathcal{L}(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.
$$

Since we have

$$
\frac{1 + \sqrt{1 - 4x}}{2x} = \frac{2}{1 - \sqrt{1 - 4x}} \quad \text{and} \quad \frac{1 - \sqrt{1 - 4x}}{2x} = \frac{2}{1 + \sqrt{1 - 4x}}
$$
our initial condition of $L(0) = 1$ forces

$$L(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$ 

Combining our last two results, we have

**Corollary 4.12** The generating function for the Catalan numbers is

$$\sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n = \frac{1 - \sqrt{1 - 4x}}{2x}.$$
5 Exercises

5.1 Generating Functions

1. Find a generating function for the sequence 1, 4, 9, 16, 25, ....

2. Show that the generating function for the sequence 1, 2, 3, 4, 5, ..., is of the form.

\[ \frac{R_n(q)}{(1-q)^{n+1}} \]

where \( R_n(q) \) satisfies the recursion

\[ R_{n+1}(q) = q(1-q)R_n'(q) + (n+1)qR_n(q) \]

with \( R_0(q) = 1 \).

5.2 \( q \)-Counting

1. Using the bijection shown in class between \( S_n \) and \( S_k \times S_{n-k} \times R(1^k 0^{n-k}) \) that sends \( \sigma \) to \( (\alpha, \beta, r) \), prove that \( \text{inv}(\sigma) = \text{inv}(\alpha) + \text{inv}(\beta) + \text{inv}(r) \). Use this fact to prove

\[ \sum_{r \in R(1^k 0^{n-k})} q^{\text{inv}(r)} = \binom{n}{k}. \]

2. (a) Show that \( \text{inv}(\sigma^{-1}) = \text{inv}(\sigma) \).

(b) Let \( \sigma \in S_n \). Find and prove a relationship between \( \text{maj}(\sigma) \) and \( \text{maj}(\sigma^c) \) where \( \sigma^c = (n+1-\sigma_1, n+1-\sigma_2, \ldots, n+1-\sigma_n) \).

(c) Let \( \sigma \in S_n \). Find and prove a relationship between \( \text{inv}(\sigma) \) and \( \text{inv}(\sigma^r) \) where \( \sigma^r = (\sigma_n, \sigma_{n-1}, \ldots, \sigma_2, \sigma_1) \).

3. Give combinatorial proofs of the following identities

(a) \[ \sum_{i=0}^{n} \binom{m+i}{i} = \binom{n+m+1}{n} \]

(b) \[ \sum_{i=0}^{n} i \binom{n}{i} = n2^{n-1} \]

(c) \[ \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k+r} = \binom{n+x}{n+r} \]

(d) \[ \sum_{i=0}^{n} q^i \left\lceil \frac{m+i}{i} \right\rceil = \frac{n+m+1}{n} \]

(e) \[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} (1-q)(1-q^2)(1-q^3) \cdots (1-q^{n-1}) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \]

(f) \[ \sum_{r \geq 0} q^{(N-r)(M-r-m)} \left[ \begin{array}{c} M-m \\ M + N \end{array} \right] \left[ \begin{array}{c} m+n+r \\ m + r \end{array} \right] = \left[ \begin{array}{c} m+n \\ M \end{array} \right] \left[ \begin{array}{c} n \\ N \end{array} \right] \]
4. (a) Prove that for any finite \( X \subseteq \{1, 2, 3, \ldots \} \) that the Foata bijection \( \varphi : X^* \to X^* \) defined by

\[
\varphi(vx) = \gamma_x(\varphi(v))x \quad \text{with} \quad \varphi(\emptyset) = \emptyset
\]

where \( x \in X \) and \( v \in X^* \) is a bijection from \( X^* \) to \( X^* \).

(b) Let \( X = \{1, 2, 3, 4\} \). Compute \( \varphi(3241234121) \) and \( \varphi^{-1}(3241234121) \).

5.3 Partitions

1. For any integer \( m > 1 \), the number of partitions of \( n \) whose parts are repeated at most \( m - 1 \) times is equal to the number of partitions of \( n \) with parts not divisible by \( m \).

2. The number of partitions of \( n \) in which only odd parts may be repeated is equal to the number of partitions of \( n \) in which no part appears more than three times.

3. The number of partitions of \( n \) with parts greater than 1 and no consecutive parts is equal to the number of partitions of \( n \) with no part appearing exactly once.

4. The absolute value of the difference between the number of partitions of \( n \) with an odd number of parts and the number of partitions of \( n \) with an even number of parts equals the number of partitions of \( n \) with distinct odd parts.

5. Use the recursion for \( p(n) = \) number of partitions of \( n \) from Euler’s pentagonal number theorem to compute \( p(13) \).

6. Find the preimages of the following pairs of partitions in the Sylvester bijection which was used in the proof of the Jacobi Triple Product identity

(a) \( \langle (10, 8, 7, 6, 4, 2, 0), (4, 3, 1) \rangle \)

(b) \( \langle (9, 8, 7, 5, 4, 1), (5, 3, 2, 1) \rangle \)

7. Let \( e(n) = \) number of partitions of \( n \) with an even number of even parts and let \( o(n) = \) number of partitions of \( n \) with an odd number of even parts. Show that \( e(n) - o(n) = \) number of self-conjugate partitions

8. (a) Show that the number of partitions of \( n \) with no part repeated 4 or more times and no consecutive repeated parts is equal to the number of partitions of \( n \) with no repeated even parts and no consecutive even parts.

(b) Use the Involution Principle to find the partition with no repeated even parts and no consecutive even parts that corresponds to the partition \( \{1, 2^2, 3, 4^3\} \) with no part repeated 4 or more times and no consecutive repeated parts.

9. Give a simple partition theoretic proof for each of the following identities:

(a) \( \prod_{n \geq 0} (1 + q^{2n}) = \frac{1}{1 - q} \)

(b) \( \prod_{n \geq 0} (1 + q^{3n} + q^{2\cdot3^n}) = \frac{1}{1 - q} \)

(c) \( \sum_{n \geq 0} nq^n = \frac{q}{(1 - q)^2} \)

(d) \( \left\lfloor n \right\rfloor = q^k \left\lfloor \frac{n - 1}{k} \right\rfloor + \left\lfloor \frac{n - 1}{k - 1} \right\rfloor \)
10. (a) Show that the number of partitions of \( n \) with no part repeated 4 or more times and no consecutive repeated parts is equal to the number of partitions of \( n \) with no repeated even parts and no consecutive even parts.

(b) Use the Involution Principle to find the partition with no repeated even parts and no consecutive even parts that corresponds to the partition \( \{1, 2^2, 3, 4^3\} \) with no part repeated 4 or more times and no consecutive repeated parts.

11. Using the Hillman-Grassl bijection

(a) construct a multiset of cells of \( \lambda = (5, 4, 4, 2) \) from the following reverse plane partition

\[
\begin{array}{cccccc}
3 & 4 & & & & \\
2 & 2 & 3 & 5 & & \\
1 & 2 & 3 & 4 & & \\
1 & 1 & 2 & 3 & 4 & \\
\end{array}
\]

(b) construct a reverse plane partition from the following multiset of cells of \( \lambda = (5, 4, 4, 2) \).

\[
\begin{array}{cccccc}
0 & 0 & & & & \\
1 & 0 & 1 & 0 & & \\
2 & 1 & 0 & 2 & & \\
1 & 3 & 1 & 1 & 0 & \\
\end{array}
\]

5.4 Combinatorial Species

1. The cycle structure of a permutation, \( \sigma \), is given by \( 1^{p_1}2^{p_2}\cdots n^{p_n} \) if \( \sigma \) has \( p_i \) cycles of length \( i \), for each \( i = 1, 2, \ldots, n \). Show that the number of permutations in \( S_n \) with cycle structure \( \{1, 2, \ldots, n\} \) of type \( 1^{p_1}2^{p_2}\cdots n^{p_n} \) is given by

\[
\frac{n!}{p_1!p_2!\cdots p_n!1^{p_1}2^{p_2}\cdots n^{p_n}}.
\]

2. Show that Stirling numbers of the first kind satisfy the following recursion

\[
s_{n+1,k} = s_{n,k-1} + ns_{n,k}
\]

3. Show that

\[
\sum_{k=1}^{n} s_{n,k} x^k = x(x + 1)(x + 2)\cdots(x + n - 1)
\]

These polynomials are referred to as the Rising Factorial Polynomials.

4. Show that

\[
\sum_{k=1}^{n} (-1)^{n-k} s_{n,k} x^k = x(x - 1)(x - 2)\cdots(x - n + 1)
\]

These polynomials are referred to as the Falling Factorial Polynomials.

5. Show that

\[
\sum_{n \geq 0} s_{n,k} \frac{x^n}{n!} = \frac{1}{k!} \left( \ln \frac{1}{1-x} \right)^k.
\]
6. Show that Stirling numbers of the second kind satisfy the following recursion

\[ S_{n+1,k} = S_{n,k-1} + kS_{n,k} \]

7. Show that

\[ x^n = \sum_{k=1}^{n} S_{n,k}x(x-1)\cdots(x-k+1). \]

8. Show that

\[ \sum_{n \geq 0} S_{n,k}x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}. \]

Use this to show that

\[ S_{n,k} = \sum_{p_1,\ldots,p_n \geq 1 \atop p_1 + p_2 + \cdots + p_k = n} \frac{1}{p_1!p_2!\cdots p_n!}. \]

9. Show that

\[ \sum_{n \geq 0} S_{n,k}x^n \frac{x^n}{n!} = \frac{1}{k!}(e^x - 1)^k. \]

10. Show that the Stirling numbers of the second kind can be computed as follows:

\[ S_{n,k} = \frac{n!}{k!} \sum_{p_1,\ldots,p_n \geq 1 \atop p_1 + p_2 + \cdots + p_k = n} \frac{1}{p_1!p_2!\cdots p_n!}. \]

11. Show that the Stirling numbers of the second kind can be computed as follows:

\[ S_{n,k} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n. \]

Hint: Try expanding \((e^x - 1)^k\) using the binomial theorem.

12. Show that the Bell numbers satisfy the recursion

\[ B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}. \]

13. Let \(D_n\) equal the number of derangements in \(S_n\). Show that

(a) \(D_{n+1} = (n+1)D_n - (-1)^n\)

(b) \(D_{n+1} = n(D_n + D_{n-1})\)

14. Use the operations on combinatorial species to prove the following

(a) \(\frac{d}{dx}(\tan^2 x) = 2\tan x + 2\tan^3 x\)

(b) \(\frac{d}{dx}(\sec x \tan x) = \sec x + 2\sec x \tan^2 x\)

(c) \(\frac{d}{dx}(\sec^2 x \tan x) = 3\sec^2 x \tan^2 x + \sec^2 x\)

15. Use combinatorial species to find an exponential generating function and a formula for
(a) the number of permutations of $S_n$ all of whose cycles are of length 2 or 3.
(b) the number of permutations of $S_n$ all of whose cycles are of length 4.
(c) the number of involutions of $S_n$.

16. Use the Lagrange inversion formula to find the number of planar trees with $n$ internal nodes such that each node has either 0, 3, or 4 children.

17. A ternary tree is a tree in which every node has either 0 or 3 children. Determine how many ternary trees there are with exactly $n$ internal nodes. Also determine the number of external nodes on a ternary tree with $n$ internal nodes.

18. Determine the series $f(x) = \sum_{n \geq 1} a_n x^n$ such that $f(x) = x(1 - f(x)^2)$.

19. Find the functional inverses of the following power series
   (a) $xe^{-x}$
   (b) $x(x + 1)^3$
   (c) $x(2 - x^3)^{-1}$

20. Calculate the number of planar trees with exactly 6 external nodes and 4 internal nodes, 3 of which have 2 children and 1 of which has 3 children.

21. (a) Find the unique tree word $w$ which is a circular rearrangement of the following word

   $x_0x_2x_0x_0x_2x_0x_2x_0x_3x_0x_0$

   (b) Construct the tree which corresponds to $w$. 

References


Index

Alternating permutation, 72
  even, 72
  odd, 72

Bell number, 57
Binomial Theorem, 4
  $q$-binomial, 17
  Newton’s, 59

Card, 52
Catalan
  generating function, 83
  number, 16, 82
  word, 15

Combinatorial species, 66

Deck, 52
Deck enumerator, 66

Derangement, 71
Descent, 8

Exponential family, 52

Ferrers diagram, 18
Fixed point, 3

Generating function
  exponential, 2
  ordinary, 2

Geometric series, 2, 19

Hand, 53

Inversion, 6
Involution, 36

Jacobi Triple Product, 26

Lattice path, 4

Major index, 8

Node
  child, 80
  external, 80
  internal, 80

Partition
  conjugate, 18
  definition, 18
  length, 18

plane, 40
reverse plane, 48
set, 53
size, 18

Pentagonal Number Theorem, 28

Secant number, 73
Stirling number
  first kind, 56
  second kind, 57

Tangent number, 73
Tree, 61
  binary, 81
  Cayley, 61
  forest, 61
  planar, 80
  rooted, 61
  word, 80