Homework #3: Due Sept. 19, 2008

1. Show that the following sequences have the indicated limits using only the definition of a limit.
   
   (a) \( \lim_{n \to \infty} \left( \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2} \right) = \frac{1}{2}. \)
   
   (b) \( \lim_{n \to \infty} \left( \sqrt{n^2 - 1} - n \right) = 0. \)

2. For a given constant \( a > 0 \), show that \( a_n = \frac{a^n}{n!} \) is “eventually” strictly decreasing. In other words, show that there exists an \( N \) such that \( a_n > a_{n+1} \) for all \( n > N \). The value of \( N \) will depend on the value \( a \).

3. Show that if \( \{a_n\} \) is bounded, then \( \{|a_n|\} \) is bounded. Make sure you explain how the upper and lower bounds for \( |a_n| \) are related to the bounds for \( a_n \). Do not use zero as a lower bound for \( |a_n| \).

4. Show that if \( \lim_{n \to \infty} a_n = 2 \) then \( \lim_{n \to \infty} a_n^2 = 4 \) using only the definition of a limit. Hint: Since \( \{a_n\} \) converges, you know that \( \{a_n\} \) must be bounded.