Perfect Coverings of Chessboards

A perfect covering of a chessboard with dominoes (i.e., a rectangle that covers exactly two adjacent squares of a chessboard) is an arrangement of identical dominoes where each square of the chessboard is covered by exactly one domino. An example of a perfect covering of a 4–by–7 chessboard is shown to the right.

The following questions guide you through the process of counting the number of perfect coverings for chessboards of varying dimensions.

1. Find a formula for $G_n$, the number of perfect coverings of a 1–by–$n$ chessboard with dominoes, for all integers $n \geq 1$. How should we define $G_0$, the number of perfect coverings of a 1–by–0 chessboard, so that it is consistent with your formula? Try to make sense of this definition.

2. Compute $F_n$, the number of perfect coverings of a 2–by–$n$ chessboard with dominoes, for $n = 1, 2, 3, 4, 5$. 
3. Based on the data you’ve collected so far, conjecture a relationship between \( F_n \), \( F_{n-1} \), and \( F_{n-2} \) for all \( n \geq 3 \). Use this relationship to compute \( F_{10} \).

4. Find a way to break up the collection of perfect coverings of a 2–by–5 chessboard into two categories. All of the perfect coverings that are in the same category should have something very specific in common. One category should be based on perfect coverings of a 2–by–3 board while the other category should be based on perfect coverings of a 2–by–4 board.

5. Explain how to break up the collection of perfect coverings of a 2–by–\( n \) chessboard into two categories, one category based on perfect coverings of a 2–by–(\( n - 1 \)) chessboard and the other based on perfect coverings of a 2–by–(\( n - 2 \)) chessboard. Explain why the relationship you conjectured in Problem #3 is true.
6. How should we define $F_0$, the number of perfect coverings of a 2-by-0 chessboard, so that the relationship you found in Problem #3 holds for $F_0$, $F_1$, and $F_2$? Try to make sense of this definition.

7. How many perfect coverings of a 3-by-3 chessboard are there? Justify your answer.

8. Find a necessary and sufficient condition on the integers $m$ and $n$ so that there exists at least one perfect covering of an $m$-by-$n$ chessboard. Recall that “necessary”, in this case, means that if $m$ and $n$ do not have this condition, then there is no perfect covering. “Sufficient” means that if $m$ and $n$ do have this condition, then there is at least one perfect covering. Justify your answer.
9. How many perfect coverings are there of an 8–by–8 chessboard where opposite corners have been removed from the board? Justify your answer.

10. A pruned chessboard is a chessboard obtained by removing any number of squares from a rectangular chessboard without separating the board into two or more smaller chessboards. For example, the board on the left is a pruned chessboard while the board on the right is not pruned since the two left–most squares do not have an edge in common with the rest of the board.

Remark: This is slightly different from how our textbook defines a pruned chessboard.

Is it possible to have a pruned chessboard that has an even number of squares but does not have a perfect covering? Is it possible to have a pruned chessboard that has the same number of white and black squares but does not have a perfect covering? Justify your answers.

11. Suppose that one region of a pruned chessboard does not have a perfect covering. Does that mean the entire board does not a perfect covering? Justify your answer.