1. **Lecture 1.** Firstly, let’s just get some notation out of the way.

**Notation.** $\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}, \mathbb{N}, \in, \subset, \subseteq, \{\}, \emptyset, A \times B.$

Everyone in high school should have studied equations of the form $y = mx + b.$ What does such an equation describe? It’s a line. Indeed, this is essentially where the “linear” in linear algebra comes from. Now, instead of thinking of this as the graph of the function

$$f(x) = mx + b$$

we want to write it as $x_1 - mx_2 = b$ and consider this as a linear equation in two variables.

**Definition 1.** (Linear equation). A linear equation in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = a_0$$

where the coefficients $a_i \in \mathbb{R}$ (i.e., the $a_i$’s are real numbers).

**Examples.** (Give examples and nonexamples of linear equations)

Once we view $y = mx + b$ as the linear equation $x_1 - mx_2 = b$, we can talk about its solution set. Whenever you have an equation in some variables, the solution set of the equation is just the collection of assignments to the variables that make the equation true. In this case, we can also plot the solution set on a graph with two axes, one for $x_1$ and one for $x_2$.

**Example.** Consider $2x_1 + 3x_2 = 0$

Now, suppose we wanted to solve two such equations simultaneously. For example, suppose we had:

$$
\begin{align*}
2x_1 + 3x_2 &= 0 \\
x_1 - x_2 &= 1
\end{align*}
$$

How could we solve this? Now, if we plot the solution sets of these equations on a graph, what does the solution correspond to on the graph? Now consider an arbitrary system of linear equations in 2 variables:

$$
\begin{align*}
a_1x_1 + a_2x_2 &= a \\
b_1x_1 + b_2x_2 &= b
\end{align*}
$$

Some natural questions: Does it always have a solution, is the solution always unique? Examples?

**Theorem 2.** Any linear system has either 0, 1, or infinitely many solutions.

**Definition 3.** We say a system of linear equations (i.e., a linear system) is *consistent* if it has at least 1 solution, and *inconsistent* if it has no solutions.

**Example.** A good example of an inconsistent linear system is $x = 1, x = 2$.

A key object of study in this course is a matrix.

**Definition 4.** An $m \times n$ matrix is a rectangular array of numbers with $m$ rows and $n$ columns. Here, $m \times n$ is the *size* of the matrix.
Example. The following is a $2 \times 3$ matrix
\[
\begin{bmatrix}
1 & \pi & 0 \\
-3 & -2 & \frac{1}{3}
\end{bmatrix}
\]

Now let's return to discussing systems of linear equations. The rest of the class will be spent discussing an algorithm for solving such systems. In fact, since systems can have 0, 1, or infinitely many solutions, instead of "solving", we are instead “determining the solution set” of the system. Consider this system:

\[
\begin{align*}
x_2 + 2x_3 &= 0 \\
-x_1 + 2x_2 + x_3 &= 1 \\
2x_1 + 5x_2 - x_3 &= -3
\end{align*}
\]

To solve this system, we need to make a few observations.

(a) (Interchange) If we interchange the positions of two equations, the resulting system has the same solution set as the original.

(b) (Scaling) If we multiply any equation by a nonzero constant, the resulting system has the same solution set.

(c) (Replacement) If we add a multiple of one equation to another equation, the resulting system again has the same solution set.

Proof. The first one is obvious. For (b), clearly it suffices to show that it doesn’t change the solution set of the equation we’re scaling. To see this, suppose our original equation was:

\[a_1x_1 + a_2x_2 + a_3x_3 = a\]

then after multiplying by the nonzero constant $c$, we get

\[c(a_1x_1 + a_2x_2 + a_3x_3) = ca\]

We want to compare the solution sets of these two equations. Suppose $(b_1, b_2, b_3)$ is a solution for the first equation, then we see that $a_1b_1 + a_2b_2 + a_3b_3 - a = 0$, but that also means that $c(a_1b_1 + a_2b_2 + a_3b_3 - a) = 0$, so $(b_1, b_2, b_3)$ is a solution to the second equation. Similarly, if $(b_1, b_2, b_3)$ is a solution to the second equation, then we know that $c(a_1b_1 + a_2b_2 + a_3b_3 - a) = 0$, so since $c \neq 0$, we can multiply by $1/c$ to see that $a_1b_1 + a_2b_2 + a_3b_3 - a = 0$, so $(b_1, b_2, b_3)$ is also a solution to the first. Thus, we’ve shown that the two solution sets are the same, so scaling does not change the solution set of the equation.

Lastly, for (c), since the operation only affects two equations, it suffices to show that the solution set of the “subsystem” consisting of those two equations is unchanged by this operation. Let those equations be

\[
\begin{align*}
a_1x_1 + a_2x_2 + a_3x_3 &= a \\
b_1x_1 + b_2x_2 + b_3x_3 &= b
\end{align*}
\]

and suppose we want to add $c$-times the first equation to the second, so the resulting system would be:

\[
\begin{align*}
a_1x_1 + a_2x_2 + a_3x_3 &= a \\
c(a_1x_1 + a_2x_2 + a_3x_3) + b_1x_1 + b_2x_2 + b_3x_3 &= b + ca
\end{align*}
\]

Suppose $(d_1, d_2, d_3)$ is a solution to this system, then...
Now, with these observations, we can define an algorithm for computing solution set of any system of linear equations. First, to simplify matters, we’ll represent the system in a more compact form using matrices.

**Definition 5.** Suppose the system has $n$ variables and $m$ equations, then the *coefficient matrix* of the linear system is the $m \times n$ matrix whose $(i, j)$th entry is the coefficient of $x_j$ in the $i$th equation.

The *augmented matrix* of the system is coefficient matrix with an extra column added, where the $j$th entry of this column is the constant term in the $j$th linear equation.

**Example.** For the example given earlier, the coefficient and augmented matrices are

$$
\begin{bmatrix}
0 & 1 & 2 \\
-1 & 2 & 1 \\
2 & 5 & -1
\end{bmatrix}
\quad
\begin{bmatrix}
0 & 1 & 2 & 0 \\
-1 & 2 & 1 & 1 \\
2 & 5 & -1 & -3
\end{bmatrix}
$$

The algorithm is difficult to describe, so we’ll just do it for a few examples.

**Examples.**

$$
\begin{align*}
x_2 + 2x_3 &= 0 \\
-x_1 + 2x_2 + x_3 &= 1 \\
2x_1 + 5x_2 - x_3 &= -3
\end{align*}
$$
2. Lecture 2.

Quick review. Elementary row operations. Fundamental questions: existence and uniqueness.

**Definition 6.** A matrix is in *echelon form* if it has the following three properties:

(a) All nonzero rows are above any rows of all zeros.
(b) Each leading entry of a row is in a column to the right of the leading entry of the row above it.
(c) All entries in a column below a leading entry are zeros.

A matrix in echelon form is said to be in *reduced echelon form* if it satisfies the additional conditions

(d) The leading entry in each nonzero row is 1.
(e) Each leading 1 is the only nonzero entry in its column.

**Definition 7.** If a matrix $A$ can be transformed into a matrix $B$ via a sequence of elementary row operations, then we say that $A$ is *row-equivalent* to $B$.

Note that any matrix that is row equivalent to the augmented matrix of some linear system describes a system with the same solution set.

**Definition 8.** A *pivot position* in a matrix $A$ is a location/position in $A$ that corresponds to a leading 1 in the reduced echelon form of $A$. A *pivot column* is a column of $A$ that contains a pivot position.

Note that pivots are not numbers. They are just locations in a matrix. The pivots of a matrix are properties of its row-equivalence class, not of any particular matrix.

Algorithm for obtaining reduced row-echelon form. Let $A$ be a matrix.

(a) Begin with leftmost nonzero column of $A$. This is a pivot column. This pivot position is at the top.
(b) If the pivot position has a 0, interchange some rows to make it nonzero.
(c) Use row replacements to kill everything below the pivot position.
(d) Now let $A$ be the matrix obtained by removing the first row and first column. Repeat steps 1-3.
(e) Starting with the rightmost pivot and moving left, use row replacements to kill everything above each pivot.
3. Lecture 3.

Last time:

**Definition 9.** A matrix is in *echelon form* if it has the following three properties:

(a) All nonzero rows are above any rows of all zeros.
(b) Each leading entry of a row is in a column to the right of the leading entry of the row above it.
(c) All entries in a column below a leading entry are zeros.

A matrix in echelon form is said to be in *reduced echelon form* if it satisfies the additional conditions

(d) The leading entry in each nonzero row is 1.
(e) Each leading 1 is the only nonzero entry in its column.

**Definition 10.** If a matrix $A$ can be transformed into a matrix $B$ via a sequence of elementary row operations, then we say that $A$ is *row-equivalent* to $B$. We’ll use the symbol $\sim$ to denote row-equivalence. I.e., $A \sim B$.

**Definition 11.** A *pivot position* in a matrix $A$ is a location/position in $A$ that corresponds to a leading 1 in the reduced echelon form of $A$. A *pivot column/row* is a column/row of $A$ that contains a pivot position.

**Theorem 12.** Each matrix is row-equivalent to exactly one reduced echelon matrix.

**Definition 13.** If $A$ is the augmented matrix of some linear system, the *basic variables* of the system are the variables that correspond to the pivot columns of $A$. The *free variables* of the system are the variables that are not basic.

**Definition 14.** A linear system is *consistent* if it has at least one solution, and *inconsistent* otherwise.

**Theorem 15.** A linear system either has 0, 1, or infinitely many solutions.

- It is inconsistent if and only if the last column of its augmented matrix is a pivot column.
- It has a unique solution if and only if it is consistent and has no free variables.
- It has infinitely many solutions if and only if it is consistent and has at least one free variable.

**Example.** Solve the systems:

\[
\begin{bmatrix}
1 & 1 & 3 & 0 \\
0 & 1 & 2 & 0 \\
1 & 1 & 4 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & 3 & 0 \\
0 & 1 & 2 & 0 \\
1 & 0 & 1 & 1
\end{bmatrix}
\]

A general or *parametric solution* to a linear system identifies the free variables, and writes the basic variables in terms of the free variables.

From the example, we can observe:

**Theorem 16.** Let $A$ be a matrix, then the positions of the row leaders of $A$ are the pivot positions of $A$.

**VECTOR SPACES**

Idea of a vector, vector spaces over a field of coefficients/scalars $F$.

**Definition 17.** A vector space over $F$ is a nonempty set $V$ of objects called vectors, on which is defined two operations, called addition and multiplication by scalars (elements of $F$), such that the following are true:
We will mostly restrict our attention to vector spaces over \( \mathbb{R} \).

(a) For all \( u, v \in V \), \( u + v \in V \).
(b) For all \( u, v \in V \), \( u + v = v + u \).
(c) \( (u + v) + w = u + (v + w) \).
(d) There is zero vector \( 0 \in V \), such that \( u + 0 = u \) for all \( u \in V \).
(e) For each \( u \in V \), there is a vector \( v \in V \) such that \( u + v = 0 \). We call this \( -u \).
(f) For every \( u \in V \) and \( c \in F \), \( cu \in V \).
(g) \( (c + d)u = cu + du \).
(h) \( c(du) = (cd)u \).
(i) \( 1u = u \).

We will mostly restrict our attention to vector spaces over \( \mathbb{R} \), i.e., where \( F = \mathbb{R} \). If a vector space is mentioned without mention of its field of coefficients, we’ll assume that it’s \( \mathbb{R} \).

Example. Prove uniqueness of \( 0 \), uniqueness of inverses, \( 0 = u \), and \( (-1)u = -u \).

Example. Last wednesday we discussed the set \( \mathbb{R}^2 \). These are the ordered pairs of real numbers (so \((1, 2) \neq (2, 1)) \). We can represent each element in many ways, though often when we think of \( \mathbb{R}^2 \) as a vector space, we’ll want to write them as \( 2 \times 1 \) matrices. Thus, elements of \( \mathbb{R}^2 \) will look like \[
\begin{bmatrix}
a \\
b
\end{bmatrix}
\] It’s important however to note that \((a, b) \) is just as good a way to represent a vector in \( \mathbb{R}^2 \).

To make \( \mathbb{R}^2 \) into a vector space, we’ll define addition and multiplication as follows:

\[
\begin{bmatrix}
a \\
b
\end{bmatrix} + \begin{bmatrix}
c \\
d
\end{bmatrix} = \begin{bmatrix}
a + c \\
b + d
\end{bmatrix}
\text{ and for any } c \in \mathbb{R},
\begin{bmatrix}
a \\
b
\end{bmatrix} \cdot \begin{bmatrix}
c \\
0
\end{bmatrix} = \begin{bmatrix}
ca \\
0
\end{bmatrix}
\]

Note that to talk about a set as a vector space, you need three pieces of information: its field of coefficients, and what addition and multiplication are on this set. For example, I can’t really ask if the set \([0, 1] \) is a vector space, because I haven’t defined its field of coefficients, addition, or multiplication. However, we can turn it into a vector space over itself if we define addition/multiplication mod 2.

Verify this is a vector space. View these vectors as points in the plane. Can view vectors also as arrows. Talk about adding arrows.

Example. Let \( u = (3, 1) \). What does the set \( \{ cu : c \in \mathbb{R} \} \) look like? What about \( \{ cu + v : c \in \mathbb{R} \} \) where \( v = (0, 1) \) ?

Let's relate these to linear equations. The first one contains all points that satisfy \( y = c, x = 3c \). Substituting, we get \( x = 3y \), or \( y = x/3 \). Repeat for other example.

Example. Introduce \( \mathbb{R}^3 \), then \( \mathbb{R}^n \).

Definition 18. For some vectors \( v_1, \ldots, v_n \) in an vector space, a linear combination of \( v_1, \ldots, v_n \) is an expression of the form:

\[
c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \quad c_i \in \mathbb{R}
\]

Note that by properties (a) and (f), this is a vector, and property (c) allows us to omit parentheses.

Now let \( u, v \in \mathbb{R}^3 \). Given a vector \( w \), is \( w \) a linear combination of \( u, v \)?

Theorem 19. Let \( v_1, v_2, \ldots, v_n, w \) be vectors, then the equation

\[
x_1 v_1 + \cdots + x_n v_n = w
\]

has the same solution set as the linear system whose augmented matrix is

\[
\begin{bmatrix}
v_1 \\
v_2 \\
\cdots \\
v_n \\
w
\end{bmatrix}
\]

Definition 20. Let \( v_1, \ldots, v_n \in V \), then the set of all linear combinations of \( v_1, \ldots, v_n \) is denoted by \( \text{Span} \{ v_1, \ldots, v_n \} \) and is called the subset of \( V \) spanned/generated by \( v_1, \ldots, v_n \).
Thus, asking if a vector $w$ is a linear combination of $u, v$ is the same as asking if $w \in \text{Span} \{ u, v \}$.

**Exercise.** Let $A$ be a set of vectors in $V$, then must $0 \in \text{Span} \ A$?

Prove the following:

1. $\text{Span} \ \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} = \mathbb{R}^3$.
2. $\text{Span} \ \{(1, 0, 0), (0, 1, 0), (2, -5, 0)\} \neq \mathbb{R}^3$. 
Lecture 4

ANNOUNCEMENTS: Theory Quiz next Tuesday, definitions list online.

QUICK REVIEW OF LAST TIME.

Consider $\mathbb{R}^3$. Let $u, v \in \mathbb{R}^3$. What does $\text{Span} \{u\}$ look like? What about $\text{Span} \{u, v\}$? Can you find 2 vectors that span $\mathbb{R}^3$? What about 3 vectors?

Example. At the store, there are two sets of legos. Set $A$ comes with 4 green blocks, 4 blue blocks, and 2 red block. Set $B$ has 3 green, 4 blue, and 5 red. To build a spaceship, you need exactly 19 green blocks, 22 blue blocks, and 20 red blocks. Is it possible to buy some number of sets $A$ and $B$ to have exactly enough to build your spaceship?

Definition 21. A subspace of a vector space $V$ is a subset $W \subseteq V$ such that for any $x, y \in W$, $x + y \in W$, and for any $x \in W, c \in \mathbb{R}, cx \in W$.

Theorem 22. A subspace of a vector space is itself a vector space.

Exercises.

(a) Let $v_1, \ldots, v_n \in V$. Is $\text{Span} \{v_1, \ldots, v_n\}$ a subspace?
(b) Is $0 \in \text{Span} \{v_1, \ldots, v_n\}$?
(c) Come up with two vectors $v, w \in \mathbb{R}^2$ such that $\text{Span} \{v, w\}$ is a line.
(d) Come up with three vectors $u, v, w \in \mathbb{R}^3$ such that $\text{Span} \{u, v, w\}$ is a line.
(e) Let $u = (1, 1)$, and $v = (2, 3)$. What is $\text{Span} \{u, v\}$? Write $(0, 0)$ as a linear combination of $u, v$. Write $(a, b)$ as a linear combination of $u, v$.
(f) Let $u = (1, 2, 3), v = (6, -1, 0)$, and $w = (-1, -1, 2)$. What is $\text{Span} \{u, v, w\}$?
(g) Suppose a lego spaceship requires 5 red blocks, 3 green blocks, and 3 black blocks, and a lego car requires 2 red blocks, 4 green blocks, and 2 black blocks. Lastly, a lego death star requires 1 red block, 2 green blocks, and 8 black blocks. Suppose you have 31 red blocks, 20 green blocks, and 40 black blocks. Is it possible to build a collection of spaceships/cars/death stars and have no blocks left over? If so, how many of each can you build?

Definition 23. If $A$ is an $m \times n$ matrix with columns $a_1, \ldots, a_n$, and $x = (x_1, \ldots, x_n)$ a column vector in $\mathbb{R}^n$, we define the product $Ax$ to be the (column) vector $x_1a_1 + \cdots + x_na_n$.

This only makes sense if the number of columns of $A$ equals the number of entries of $x$. 
Lecture 5. Homework due today, definitions quiz next tuesday.

Is \( \text{Span} \{ (1, 1, 2), (5, 1, 10), (-1, -1, 3) \} = \mathbb{R}^3 \)?

Why can’t 1 vector span \( \mathbb{R}^2 \)? (change one of the vectors coordinates, that won’t be in the span!) Why can’t 2 vectors span \( \mathbb{R}^3 \)? Sure, it makes sense in these two cases, but why can’t say, 10 vectors span all of \( \mathbb{R}^{11} \)? You can see the pattern and while you might be able to generalize to higher dimensions, to really be able to understand it you need to formulate a rigorous argument.

LAST TIME:

Ask for definitions of span and linear combination.

**Definition 24.** If \( A \) is an \( m \times n \) matrix with columns \( a_1, \ldots, a_n \), and \( x = (x_1, \ldots, x_n) \) a column vector in \( \mathbb{R}^n \), we define the product \( Ax \) to be the (column) vector \( x_1a_1 + \cdots + x_na_n \).

This product allows us to write our equations in terms of matrices. Let \( A \) be an \( m \times n \) matrix with columns \( a_1, \ldots, a_n \), and \( x, b \in \mathbb{R}^n \). Compare the two equations:

\[ Ax = b \quad x_1a_1 + \cdots + x_na_n = b \]

By the definition of matrix multiplication, they’re exactly the same equation. Thus, we see that

**Theorem 25.** For a matrix \( A \), the equation \( Ax = b \) has a solution if and only if \( b \) is a linear combination of the columns of \( A \).

Recall:

**Theorem 26.** An augmented matrix \( A \) represents an inconsistent linear system iff the last column is a pivot column.

**Theorem 27.** Let \( A \) be an \( m \times n \) matrix. Then the following are equivalent:

(a) For each \( b \in \mathbb{R}^m \), \( Ax = b \) has a solution.

(b) For each \( b \in \mathbb{R}^m \), the augmented matrix \( [A \ b] \) represents a consistent system.

(c) Each \( b \in \mathbb{R}^m \) is a linear combination of the columns of \( A \).

(d) The columns of \( A \) span \( \mathbb{R}^m \).

(e) \( A \) has a pivot position in every row.

**Proof.** Clearly the first 4 are the same. For (c) \(\implies\) (d), suppose the REF of \( A \) does not have a pivot position in every row, then the row without a pivot must be a zero row and must be on the bottom. Let \( v = (0, \ldots, 0, 1) \). Now suppose \( B \) is the REF of \( A \), then consider the (augmented) matrix \( [B \ v] \). Clearly \( [B \ v] \) is inconsistent. Reverse the row reduction process to obtain an augmented matrix \( [A \ w] \). Since \( [B \ v] \) is inconsistent, so is \( [A \ w] \), and hence \( Ax = w \) does not have a solution, so the columns of \( A \) could not have spanned \( \mathbb{R}^m \). Thus, if they do span \( \mathbb{R}^m \), \( A \) must have a pivot in every row!

**Example:** Consider the matrix

\[
A = \begin{bmatrix}
0 & -6 & -4 \\
1 & 3 & 5 \\
2 & 0 & 6
\end{bmatrix}
\]

Find a vector \( b \) such that \( Ax = b \) is inconsistent.

For (d) \(\implies\) (c), suppose \( A \) has a pivot in every row, then consider the matrix \( [A \ (x_1, \ldots, x_m)] \). Applying the same operations to this matrix as the ones used to row-reduce \( A \) will also give an REF matrix that looks like \( [B \ (y_1, \ldots, y_m)] \) where since \( B \) has a pivot in every row, the last column of \( [B \ (y_1, \ldots, y_m)] \) is not a pivot, so the system is consistent, so the original system was consistent, no matter what the \( x_i \)’s are!
Example. Consider the matrix/vector:

\[
A = \begin{bmatrix}
1 & 5 & 2 & 1 \\
1 & 5 & 3 & 1 \\
2 & 10 & 4 & 3
\end{bmatrix}
\]
\[
b = \begin{bmatrix}
e \\
\pi \\
\sqrt{2}
\end{bmatrix}
\]

Find a solution for \(Ax = b\).

Back to original question. Span \{(1, 1, 2), (5, 1, 10), (−1, −1, 3)\} = \mathbb{R}^3 \text{ iff the three vectors span } \mathbb{R}^3, \text{ ie, the matrix with the vectors as columns has columns who span } \mathbb{R}^3, \text{ ie the matrix has a pivot in every row!}
Lecture 6. QUIZ TODAY. Problem set 3 is online. Thursday office hours moved to 1pm - 2:15pm.

**Theorem 28.** If $A$ is an $m \times n$ matrix, and $u,v \in \mathbb{R}^n$, and $c \in \mathbb{R}$, then

(a) $A(u + v) = Au + Av$
(b) $A(cu) = c(Au)$

**Proof.** Do examples with

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

(don’t compute any sums)

**Definition 29.** A system of linear equations is **homogeneous** if it can be written in the form $Ax = 0$. (ie, if all the constant terms are 0).

**Give examples.** Ask for example of homogeneous system that has no solutions. In fact, all homogeneous systems have at least one solution. (Which solution?) Together with the existence and uniqueness theorem of 1.2, we have:

**Theorem 30.** A homogeneous equation $Ax = 0$ has a nontrivial solution if and only if the equation has at least one free variable (when viewed as a linear system).

**Exercise.** Is the following systems homogeneous? Does it have any nontrivial solutions? (find general solutions)

$$\begin{bmatrix} 3 & 1 & 11 & 0 \\ 3 & 2 & 13 & 0 \\ 1 & 1 & 5 & 0 \end{bmatrix}$$

What about $3x_1 - 6x_2 + 12x_3 = 0$? (write as augmented matrix, row reduce, find general solutions)

What do these general solutions look like? Write general solutions in **parametric form**.

**Theorem 31.** Let $A$ be an $m \times n$ matrix. The solution set to the equation $Ax = 0$ can be written as $\text{Span} \{v_1, \ldots, v_k\}$. In particular, the solution set is a subspace of $\mathbb{R}^n$.

The fact that it’s a subspace also follows from the linearity of matrix multiplication $A(u + v) = Au + Av, A(cu) = c(Au)$.

Now let $b \neq 0$, so that $Ax = b$ is nonhomogeneous. Suppose $p$ is a solution to $Ax = b$, and $q$ is a solution to $Ax = 0$. Must $p + q$ be a solution to $Ax = b$? In fact, all solutions are of this form:

**Theorem 32.** Let $p$ be a solution to the equation $Ax = b$, and let $W$ be the solution set of $Ax = 0$. Then the solution set of $Ax = b$ is the set:

$$p + W = \{p + w : w \in W\}$$

**Exercise.** Write the general solution of $Ax = b$ in parametric vector form.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

In general, to write the solution set of a consistent system in parametric vector form:

(a) Reduce the augmented matrix to reduced echelon form.
(b) Express each basic variable in terms of free variables and/or constants.
(c) Write a typical solution $x$ as a vector whose entries depend on the free variables.
(d) Decompose $x$ into a linear combination of vectors using free variables as parameters.

**Introduce as much of linear independence as possible.**
7. Lecture 7

**Linear Independence!** Look at example 3 in section 1.5 for a good example of finding parametric forms.

Review: go over homogeneous systems...geometric view of solutions sets as being shifted subspaces.

LAST TIME:

**Theorem 33.** Let $A$ be a matrix, $b$ a vector. Let $W$ be the solution set of $Ax = 0$, and let $p$ be a fixed solution of $Ax = b$, then the solution set of $Ax = b$ is exactly $p + W$.

**Question:** Suppose $Ax = 0$ has a nontrivial solution. How many solutions does $Ax = b$ have? Can $Ax = b$ have a different number of solutions than $Ax = 0$? Note that the above theorem says nothing about the case when $Ax = b$ is inconsistent! In general,...

**Theorem 34.** If $Ax = b$ is consistent, then it has exactly the same “number” of solutions (ie, either 1 or $\infty$) as $Ax = 0$.

**Purpose:** Consider two vectors in $\mathbb{R}^2$. What are the possibilities for their span? When is it a point, a line, a plane? What about 3 vectors in $\mathbb{R}^3$? Today, we’ll try to pinpoint exactly what property those three vectors must have.

**Definition 35.** A (finite) set of vectors $S = \{v_1, \ldots, v_p\} \in \mathbb{R}^n$ is said to be linearly independent if the vector equation:

$$x_1v_1 + \cdots + x_pv_p = 0$$

has only the trivial solution. They are said to be linearly dependent if they are not linearly independent (ie, there exists a nontrivial solution).

Or, equivalently, $S$ is linearly independent if the matrix equation $[v_1 \ v_2 \ \ldots \ v_p]x = 0$ has only the trivial solution.

**Example.** Columns of the matrices

\[
\begin{bmatrix}
1 & 3 & 8 & 1 \\
0 & -1 & 3 & 2 \\
3 & 1 & 1 & 0
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 3 & 1 \\
0 & -1 & 1 \\
1 & 2 & 3
\end{bmatrix}
\]

**Special Cases.** Let $S \subset \mathbb{R}^n$ contain only one vector. When is $S$ linearly dependent? What if $|S| = 2$?

**Theorem 36.** A set $S \subset \mathbb{R}^n$ consisting of one vector is linearly dependent if and only if $S = \{0\}$.

If $|S| = 2$ (say $S = \{u, v\}$, $u \neq v$), then $S$ is linearly dependent if and only if one of the vectors is a multiple of the other.

**Theorem 37.** A set $S \subseteq \mathbb{R}^n$ of two or more vectors is linearly dependent if and only if at least one of the vectors in $S$ is a linear combination of the others.

**Proof.** Let $S = \{v_1, \ldots, v_p\}$ be such a set of vectors, then to say that $S$ is linearly dependent is to say that there exist a nontrivial solution $x_1, \ldots, x_n$ to the equation

$$x_1v_1 + \cdots + x_pv_p = 0$$

rearranging, we get:

$$x_1v_1 = -x_2v_2 - x_3v_3 - \cdots - x_pv_p \quad \Rightarrow \quad v_1 = \frac{-x_2}{x_1}v_2 - \cdots - \frac{x_p}{x_1}v_p$$

Where did we use the fact that the solution is nontrivial? Can we always say that every $v \in S$ can be written as a linear combination of the others?

**Theorem 38.** Let $S \subseteq \mathbb{R}^n$, then if $|S| > n$, $S$ is linearly dependent.
8. Lecture 8.

Consider the equation $Ax = b$. Solving the equation amounts to finding a vector such that when multiplied by $A$, gives you $b$. Thus, we see that “multiplication by $A$” is a way of transforming vectors into other vectors. The right terminology for this is a function.

For another simple example, consider the function $f(y) = y^2$.

**Definition 39.** Let $A, B$ be sets, then a function/map/transformation $f$ from $A$ to $B$ is denoted $f : A \rightarrow B$ and is a rule that assigns an element of $B$ to every element of $A$. In a way, we think of $f$ as being a machine that when fed an element of $A$, spits out an element of $B$. Here, $A$ is the domain, and $B$ is the codomain. If $a \in A$, then we say that $T(a)$ is the image of $a$ (under $T$), and we say that the range of $T$ is the set of all images $T(a)$ for $a \in A$.

**Examples.** $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x$, $f(x) = x^2$. Also, $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by $f(x, y) = (x, y, 0)$. (What are the images of these?)

**Definition 40.** Let $V, W$ be vector spaces (for example, $V = \mathbb{R}^n$, $W = \mathbb{R}^m$), then function $T : V \rightarrow W$ is called a linear transformation if:

(a) For all $u, v \in V$, $T(u + v) = T(u) + T(v)$
(b) For all $u \in V, c \in \mathbb{R}$, $T(cu) = cT(u)$

**Theorem 41.** If $T : V \rightarrow W$ is a linear transformation, then for any $v_1, \ldots, v_n \in V$, and $a_1, \ldots, a_n \in \mathbb{R}$, we have

$T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n)$

**Example.** Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $T((x, y)) = (x + y, 2x - y)$. Is this linear? Let $u = (1, 0)$ and $v = (1, 1)$. What is $T(u), T(v)$? What is $T((12, -3))$?

So far we’ve thought of $Ax = b$ as a linear system. But now, we see it in a new light:

Let $A$ be an $m \times n$ matrix, then we define the map: $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $T_A(x) = Ax$. Is $T_A$ a linear transformation?

Here, we essentially view $A$ as a function that maps vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$ via multiplication.

**Notation.** From now on, we write $e_i$ to be the vector with 1 in the $i$th coordinate, and 0’s everywhere else. The set $\{e_1, \ldots, e_n\}$ are called the canonical basis vectors of $\mathbb{R}^n$.

**Examples.** Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad u = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Find $T_A(u)$. Is $b$ in the range of $T_A$? Is $u$ in the range of $T_B$? How many vectors does $T_A$ map to $b$? How many vectors does $T_B$ map to $u$? Recall the theorems:

**Theorem 42.** Let $A$ be an $m \times n$ matrix. Then the following are equivalent:

(a) For each $b \in \mathbb{R}^m$, $Ax = b$ has a solution.
(b) $A$ has a pivot position in every row.

**Theorem 43.** Let $S \subseteq \mathbb{R}^n$, then if $|S| > n$, $S$ is linearly dependent.

**Example.** Consider various matrices. $[\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}], [\begin{bmatrix} 0 \\ 1 \end{bmatrix}], [\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}]$. Compute $T_A(e_i)$. Note that they’re just the columns of the matrices.
9. Lecture 9 Section 1.9 in the book.

New homework online, DUE THURSDAY

LAST TIME:

Definition 44. Let $V, W$ be vector spaces, then a function $T : V \to W$ is called a linear transformation if:

(a) For all $u, v \in V$, $T(u + v) = T(u) + T(v)$
(b) For all $u \in V, c \in \mathbb{R}$, $T(cu) = cT(u)$

In particular, this means that $T(0) = T(0 \cdot u) = 0 \cdot T(u) = 0$, i.e., any linear transformation maps 0 to 0 (so if $T$ is a function with $T(0) \neq 0$, then $T$ is not linear)

Theorem 45. Let $v_1, \ldots, v_n \in V$ (a vector space), $a_1, \ldots, a_n \in \mathbb{R}$, and $T : V \to W$ a linear transformation. Then

$$T(a_1v_1 + \cdots + a_nv_n) = a_1T(v_1) + \cdots + a_nT(v_n)$$

Corollary 46. Suppose $v_1, \ldots, v_p \in \mathbb{R}^n$ span $\mathbb{R}^n$, then any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by how it acts on $v_1, \ldots, v_n$.

In particular, if $v_1, \ldots, v_p$ span $\mathbb{R}^n$ and you know the values $T(v_1), \ldots, T(v_p)$, then you can calculate $T(v)$ for ANY $v \in \mathbb{R}^n$.

Notation. From now on, we’ll refer to $e_i$ as being the vector with a 1 in the $i$-th position and zeroes everywhere else. Note that this vector will depend on context. For example,

$$e_2 \in \mathbb{R}^2 \text{ is } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ but } e_2 \in \mathbb{R}^3 \text{ is } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Corollary 47. Any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its action on the canonical basis vectors $e_1, \ldots, e_n \in \mathbb{R}^n$.

Example. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(v_1) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad T(v_2) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \quad T(v_3) = \begin{bmatrix} -3 \\ 12 \end{bmatrix}$$

What is $T(e_1), T(e_2), T(e_3)$? Since the domain of $T$ is $\mathbb{R}^3$, $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Hint: For each $i = 1, 2, 3$, solving for $[v_1 v_2 v_3]x = e_i$ will show $e_1 = -v_1 + \frac{1}{2}v_2$, $e_2 = \frac{4}{3}v_1 - \frac{1}{6}v_2 - \frac{1}{3}v_3$, and $e_3 = -\frac{1}{3}v_1 + \frac{1}{6}v_2 + \frac{1}{3}v_3$

Now consider $A = \begin{bmatrix} -3 \\ 3 \\ 0 \\ 5 \\ -5 \\ 4 \end{bmatrix}$. Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

What is $T_A(e_1), T_A(e_2), T_A(e_3)$? What is Span $\{e_1, e_2, e_3\}$? (Answer: $T_A(e_i) = T(e_i)$) But the corollary tells us that any linear transformation is completely determined by how it acts on $e_1, e_2, e_3$ (or any other spanning set)! Thus, $T$ must be the same transformation as $T_A$! Indeed, for any $x, y, z$, we have

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = T(xe_1 + ye_2 + ze_3) = xT(e_1) + yT(e_2) + zT(e_3)$$
Theorem 52. In fact, we have the following fact:

\[ \vec{a} \quad A \quad \text{columns of} \quad A \]

Let \( \vec{a} \) be an \( n \times m \) matrix. Then the following are equivalent:

(a) For each \( \vec{b} \in \mathbb{R}^m \), \( A\vec{b} = \vec{c} \) has a solution.

(b) For each \( \vec{b} \in \mathbb{R}^m \), the augmented matrix \([A \ \vec{b}]\) represents a consistent system.

(c) Each \( \vec{b} \in \mathbb{R}^m \) is a linear combination of the columns of \( A \).

(d) The columns of \( A \) span \( \mathbb{R}^m \).

(e) \( A \) has a pivot position in every row. (ie, all rows are pivot rows)

(f) \( T_A : \mathbb{R}^n \to \mathbb{R}^m \) is onto.

In fact, we have the following fact:

Theorem 53. If \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a linear transformation, then the range of \( T \) is the span of the columns of \( A_T \).

Proof. The range of \( T \) is by definition the set of all things that look like \( T(\vec{x}) \) for some \( \vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \). But by definition of \( A_T \), if \( \vec{a}_1, \ldots, \vec{a}_n \) are the columns of \( A_T \), we have

\[ T(\vec{x}) = A_T \vec{x} = x_1 \vec{a}_1 + \cdots + x_n \vec{a}_n \]

where the second equality is just the definition of multiplying a matrix by a vector. Thus, we see that \( T(\vec{x}) \) is a linear combination of the columns of \( A_T \). From this, noting that \( x_1, \ldots, x_n \) are free to range over all possible real numbers, we see that the set of all \( T(\vec{x}) = A_T \vec{x} \) is exactly the set of all linear combinations of \( \vec{a}_1, \ldots, \vec{a}_n \).
10. Lecture 10 Sections 1.9 and 2.1 in the book.

Recall that theorem 51 gave us some characterizations of what it means for a linear transformation to be onto. Today, we discuss what it means for a linear transformation to be 1-1, and then continue on to discuss how to do algebra with matrices.

**Theorem 54.** Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation, then the following are equivalent:

(a) \( T \) is one-to-one.
(b) \( T(x) = 0 \) has only the trivial solution.
(c) The columns of \( A_T \) are linearly independent.
(d) There is a pivot in every column of \( A_T \). (ie, all columns are pivot columns)

**Proof.** First we show (a) \iff (b) If \( T \) is 1-1, then in particular \( T \) can map at most one thing to 0, and since \( T(0) = 0 \), this means that \( T(x) = 0 \) must have only the trivial solution. Conversely, if \( T \) is not one-to-one, then suppose for distinct vectors \( u \neq v \in V \), we have \( T(u) = T(v) \), then

\[
T(u) - T(v) = 0 = T(u - v)
\]

but that since \( u \neq v \), \( u - v \neq 0 \), so \( u - v \) is a nontrivial solution to \( T(x) = 0 \). This shows that (a) is exactly the same as (b).

To see that (b) \iff (c), note that by the definition of \( A_T \), \( T(x) = A_T x \). Thus, the equation \( T(x) = 0 \) is the same as the equation \( A_T x = 0 \), and the latter equation only having a trivial solution is exactly to say that the columns of \( A_T \) are linearly independent.

Lastly to show that (d) is the same as (a),(b),(c), note that:

- \( T(x) = A_T x = 0 \) has a unique solution \iff [there are no free variables] \iff [all variables are basic] \iff [every column corresponding to a variable in the augmented matrix \( [A_T \overline{0}] \) contains a pivot]

Thus (d) is the same as (b), which are the same as (a) and (c), so all of the statements (a),(b),(c),(d) are the same.

**Example.** What is the domain/codomain/range of \( T_A \)? Is \( T_A \) onto? 1-1?

\[
A = \begin{bmatrix}
1 & 3 & -2 \\
0 & 5 & 1 \\
0 & 0 & 8 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

Domain is \( \mathbb{R}^3 \), codomain is \( \mathbb{R}^4 \). The range of \( T_A \) is the span of the columns. Since not every row contains a pivot, by 51, \( T_A \) is not onto. Since every column is a pivot column, by 54, \( T_A \) is 1-1.

**Matrix Operations**

Let \( A \) be a matrix, then we’ll use \( a_{ij} \) to denote the number in the \( i \)th row and \( j \)th column.

**Definition 55.** Let \( A, B \) be matrices of the same size (say \( m \times n \)). Then we define the matrix \( A + B \) to be the matrix \( C \) such that \( c_{ij} = a_{ij} + b_{ij} \) for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \).

For any \( r \in \mathbb{R} \), we define the scalar multiple \( rA \) to be the matrix \( D \) such that \( d_{ij} = ra_{ij} \).

The **zero matrix** is a matrix with all zero entries, which we’ll often denote with 0.

In other words, we add matrices by adding corresponding entries, and multiply matrices by scalars by scaling corresponding entries.

**Example.** I.e,

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix} + \begin{bmatrix}
0 & 1 \\
6 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 3 \\
9 & 5
\end{bmatrix}, \quad \text{and} \quad 2 \begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix} = \begin{bmatrix}
2 & 4 \\
6 & 8
\end{bmatrix}
\]

Addition and scalar multiplication of matrices satisfy the usual rules:
Theorem 56. Let \( A, B, C \) be matrices of the same size, and let \( r, s \in \mathbb{R} \).

(a) \( A + B = B + A \)
(b) \( (A + B) + C = A + (B + C) \)
(c) \( A + 0 = A \)
(d) \( r(A + B) = rA + rB \)
(e) \( (r + s)A = rA + sA \)
(f) \( r(sA) = (rs)A \)

Proof. These all follow directly from the definitions. Do some examples until you believe these properties.

Definition 57. Let \( A \) be an \( m \times n \) matrix, and let \( B \) be an \( n \times p \) matrix with columns \( \vec{b}_1, \ldots, \vec{b}_p \).
Then define the product of \( A \) and \( B \) as:
\[
AB = [A\vec{b}_1 \ A\vec{b}_2 \ \cdots \ A\vec{b}_p]
\]

In light of this definition, note that multiplying a matrix by a vector is a special case of multiplying two matrices. (ie, the vector is just an \( m \times 1 \) matrix.)

Exercise 58. If \( A \) is \( m \times n \), and \( B \) is \( n \times p \), what is the size of \( AB \)?

Example 59. Verify these:
\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 0 \end{bmatrix}
\]

On the other hand,
\[
\begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}
\]
is undefined!!!

The Row-Column rule for matrix multiplication. Suppose \( A, B \) are matrices of appropriate sizes, where say \( B = [\vec{b}_1, \ldots, \vec{b}_p] \), then the \( ij \)-th entry of \( AB \) is the \( i \)-th entry of \( A\vec{b}_j \). To compute that, by the row-column rule for vectors, the \( i \)-th entry of \( A\vec{b}_j \) is just \( \vec{a}_i \cdot \vec{b}_j \), where \( \cdot \) denotes the dot product, and \( \vec{a}_i \) is the \( i \)-th row of \( A \).

Definition 60. Let \( I_m \) be the \( m \times m \) matrix with 1’s along the main diagonal and 0’s everywhere else. We’ll call \( I_m \) the \( m \times m \) identity matrix. If the size is clear from context, we sometimes just use the letter \( I \).

Example 61. \( I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), and \( I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \).

Theorem 62. Let \( r \in \mathbb{R} \), \( A \) be an \( m \times n \) matrix, and let \( B, C \) have sizes for which the expressions below make sense.

(a) \( A(BC) = (AB)C \)
(b) \( A(B + C) = AB + AC \)
(c) \( (B + C)A = BA + CA \)
(d) \( r(AB) = (rA)B = A(rB) \)
(e) \( I_mA = A = AI_n \)

Proof. These all follow directly from the definitions. Do some examples until you believe these properties.
Of these, (a) is especially interesting. While at first it may seem like a technicality, to understand what it means, consider the following example.

**Example 63.** Suppose \( A \) is 4 \( \times \) 3, and \( B \) is 3 \( \times \) 2. Then, we have the associated linear transformations \( T_A : \mathbb{R}^3 \to \mathbb{R}^4 \) and \( T_B : \mathbb{R}^2 \to \mathbb{R}^3 \). (What are the standard matrices of \( T_A, T_B ? \))

Now, consider the map \( T : \mathbb{R}^2 \to \mathbb{R}^4 \) defined by first sending any vector \( x \in \mathbb{R}^2 \) to \( T_B(x) \), and then sending \( T_B(x) \) to \( T_A(T_B(x)) \). Pictorially, we have the following diagram:

\[
\begin{array}{ccc}
T : \mathbb{R}^2 & \rightarrow & \mathbb{R}^3 \\
\downarrow & & \downarrow \\
T_B & \rightarrow & T_A(T_B)
\end{array}
\]

I.e., \( T \) is the composition of the linear transformations \( T_A \) and \( T_B \), denoted \( T = T_A \circ T_B \), where \( T(x) = T_A(T_B(x)) \). In other words, \( T \) is the map obtained by first applying \( T_B \), and then \( T_A \).

Referring to the above diagram, \( T \) is defined by the rule: “for any \( x \in \mathbb{R}^2 \), \( T(x) = A(Bx) \).”

Now, using 62(a) and setting \( C = x \), we find \( A(Bx) = (AB)x \). Thus,

\[
T(x) = A(Bx) = (AB)x
\]

What’s the standard matrix for \( T \)?

Since \( T(x) = (AB)x \), by the definition of the standard matrix, the standard matrix for \( T \) is \( AB! \) (i.e., \( T = T_{AB} \))

In particular, since \( T \) is the map defined by “multiplication by \( AB \),”, \( T \) is a linear transformation. We summarize this in the following theorem.

**Theorem 64.** Let \( T_B : \mathbb{R}^p \to \mathbb{R}^n \), and \( T_A : \mathbb{R}^n \to \mathbb{R}^m \) be linear transformations given by their standard matrices \( A, B \). Then the composition \( T_A \circ T_B \) is a linear transformation from \( \mathbb{R}^p \to \mathbb{R}^m \) defined by

“starting in \( \mathbb{R}^p \), apply \( T_B \), then apply \( T_A \) and land in \( \mathbb{R}^m \)”

More precisely, \( (T_A \circ T_B)(x) = T_A(T_B(x)) \). Furthermore, the standard matrix for \( T_A \circ T_B \) is \( AB \).

Also, \( A_S \) is \( m \times n \), \( A_T \) is \( n \times p \), so \( A_S A_T \) is \( m \times p \).

**Example 65.** Suppose \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). Note that

\[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}
\]

Suppose \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is the map defined by first rotating counterclockwise by 90°, and then reflecting across the line \( x = y \). What is the standard matrix for \( T \)?

The usual method involves computing \( T(e_1) \) and \( T(e_2) \). Then, the standard matrix is just \( [T(e_1) \ T(e_2)] \), which in this case is \( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \).

Here’s another (not necessarily easier) way to think about it. Firstly,

- let \( R : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation “Rotate counterclockwise by 90°”
- let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) the linear transformation “ReFlect across the line \( x = y \)”

Note that the standard matrix for \( R \) is the matrix \( A_R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \), and the standard matrix for \( F \) is \( A_F = A \) (the matrix above).

Then, the entire transformation \( T \) is really just the composition \( F \circ R \) (i.e., first Rotate, then ReFlect). theorem 64 tells us that \( T \) is indeed a linear transformation, and its standard matrix is \( A_F A_R \), i.e.

By theorem 64 the standard matrix of \( T \) is

\[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]
WARNING 66.

(a) In general, $AB \neq BA$.

In the above example, note that $A_F A_R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, but $A_R A_F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Indeed, $A_F A_R$ represents the transformation “first rotate, then reflect”, and $A_R A_F$ represents the transformation “first reflect, then rotate”. Think about what these transformations do to $e_1$ (or $e_2$)

(b) Cancellation laws do not in general hold for matrix multiplication. That is, even if $AB = AC$, that does not mean that $B = C$. (See exercise 2.1.10 in the book)

(c) If $AB = 0$ (ie, the zero matrix), then from this information alone you cannot conclude that either $A = 0$ or $B = 0$. (See exercise 2.1.12 in the book)

MATRIX TRANSPOSE

We probably won’t have time to get to this, but matrix transposes are pretty easy. They’re just a “flipped” version of the matrix. Here’s the precise definition and some properties.

Definition 67. Let $A$ be $m \times n$, then the transpose of $A$ is denoted $A^T$ and has entries $(a^T)_{ij} = a_{ji}$.

Theorem 68. Let $A, B$ denote matrices of appropriate sizes, then

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) For any scalar $r$, $(rA)^T = rA^T$

(d) $(AB)^T = B^T A^T$
11. **Lecture 11.** (Section 2.2) Quiz next Thursday (no definitions/proofs, mostly computations and true/false), covering everything through next Tuesday’s lecture. Also, problem set 5 will be due next Thursday. It’ll be a little shorter than pset4. Use it to study for the quiz!

Recall the warning from last time: Given $AB = AC$, this does not general mean that $B = C$! When/why does this cancellation law fail?

**Example 69.** FREE POINTS!

- Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $P(x,y) = (x,0)$. (think of this as projection onto the $x$-axis)
- Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that scales all vectors by 2.
- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation where $T(1,0) = (2,0)$, and $T(0,1) = (0,3)$.

What are the standard matrices for $P, S, T$?

Now, computing $A_P A_S$ and $A_P A_T$, we see that $A_P A_S = A_P A_T$, and yet $A_S \neq A_T$! What happened here? To analyze this situation, we’ll try to understand these matrices geometrically. Recall that matrices represent linear transformations, and matrix multiplication is really just the composition of the associated linear transformations. In other words,

- the matrix $A_P A_S$ is the standard matrix for the transformation “first apply $S$ (ie multiply by $A_S$), then apply $P$ (ie multiply by $A_P$)”

  or equivalently, “first scale by 2, then project onto the $x$-axis”

- the matrix $A_P A_T$ is the standard matrix for the transformation “first apply $T$ (ie multiply by $A_T$), then apply $P$ (ie multiply by $A_P$)”

  or equivalently, “first scale horizontally by 2 and vertically by 3, and then project onto the $x$-axis”.

Phrased another way, this is saying that even though $S \neq T$ as linear transformations, $P \circ S = P \circ T$! Why did this happen?

First, note that $S$ and $T$ can be defined as: $S(x,y) = (2x,2y)$, and $T(x,y) = (2x,3y)$.

Now, the effect of $P$ is that it makes the $y$-coordinate of the input 0, no matter what it used to be (and leaves the $x$-coordinate alone). Thus, $P$ in a way throws away all the information associated with the $y$-coordinate. Now, how do $S$ and $T$ differ? Well, they do the same thing to the $x$-coordinate, and in fact the only way they differ is in how they treat the $y$-coordinate. $S$ scales it by 2, and $T$ scales it by 3. But in the compositions $P \circ S$ and $P \circ T$, since $P$ is composed on the left, this means that after $S$ or $T$ are applied (to some input vector), $P$ is applied. Thus, no matter how differently $S$ and $T$ acts on the $y$-coordinate, $P$ washes away all the evidence of $S, T$ being distinct by making the $y$ coordinate uniformly 0. Specifically, we can see that on the vector $(1,1) \in \mathbb{R}^2$, we have

$$P(S(1,1)) = P(2,2) = (2,0) \quad \text{and} \quad P(T(1,1)) = P(2,3) = (2,0)$$

Thus, even though $S$ and $T$ act differently on the vector $(1,1)$, the compositions $P \circ S$ and $P \circ T$ act the same on $(1,1)$. Why? Because applying $P$ destroyed the distinction between $S(1,1) = (2,2)$ and $T(1,1) = (2,3)$.

It’s like having two objects, which are exactly the same except that one is red and one is blue. Even though they’re different, after dipping both in green paint, both will end up being indistinguishable! In this analogy, the dipping in green paint corresponds to applying the transformation $P$.

In a sense, the matrix $P$ and the process of dipping in green paint destroys some of the information about the original objects. Specifically, multiplying both $S$ and $T$ on the left by $P$ throws away information about how $S, T$ acted on the $y$-coordinate, and dipping both objects in green paint throws away information about their previous colors.
Now that we’ve seen the kinds of situations in which the cancellation law fails, ie where

$$AB = AC \quad \text{but} \quad B \neq C,$$

it remains to ask: when does it hold? From the example above, it would seem to hold when the matrix $A$ does not throw away any information. Such matrices will be called invertible.

**Definition 70.** Let $A$ be an $n \times n$ matrix, then $A$ is said to be invertible if there exists an $n \times n$ matrix $B$ with $AB = BA = I_n$. We call $B$ the inverse of $A$, or $A^{-1}$.

**Theorem 71.** If $A, B$ are $n \times n$ and $AB = I_n$, then $BA = I_n$. In particular, if $A, B$ are square and $AB = I$, then both $A, B$ are invertible with $A^{-1} = B$ and $B^{-1} = A$.

**Theorem 72.** Let $A, B$ be invertible matrices of the same size, then

(a) $A^{-1}$ is invertible, with $(A^{-1})^{-1} = A$
(b) $AB$ is invertible, with $(AB)^{-1} = B^{-1}A^{-1}$
(c) $A^T$ is invertible, with $(A^T)^{-1} = (A^{-1})^T$

*Proof.* Follows directly from the definitions. \( \square \)

Note that (b) says that the inverse of a product is the product of the inverses, in reverse order. This is because without being about to reorder the matrices in a product, $ABA^{-1}B^{-1}$ is not in general equal to $I$. On the other hand,

$$ABBA^{-1}B^{-1} = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

**Theorem 73.** If $A$ is an $n \times n$ invertible matrix, then the equation $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.

*Proof.* Set $x = A^{-1}b$. Then $Ax = A(A^{-1}b) = Ib = b$, so $x = A^{-1}b$ is indeed a solution!

Conversely, if $x$ is any solution, ie $Ax = b$, then multiplying on the left on both sides by $A^{-1}$ gives

$$A^{-1}Ax = A^{-1}b \quad \text{ie...} \quad Ix = A^{-1}b \quad \text{so} \quad x = A^{-1}b$$

\( \square \)

**Exercise 74.** How many pivots must an $n \times n$ invertible matrix have?

**Definition 75.** An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix.

**Example 76.** Consider

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Compute $E_1A, E_2A, E_3A$. Note that $E_1A$ is just the matrix obtained from $A$ by adding 3 times the first row to the third row. $E_2A$ is the matrix obtained by swapping the first two rows of $A$, and $E_3A$ is the matrix obtained by scaling the third row of $A$. We summarize this in the following theorem.

**Theorem 77.** Let $A$ be $m \times n$. Suppose $E$ is the $m \times m$ matrix obtained by applying some elementary row operation to $I_m$. Then $EA$ is the matrix obtained by applying the same elementary row operation to $A$.

In other words, multiplying on the left by an elementary matrix $E$ is equivalent to applying the elementary row operation that $E$ represents.

In particular, this means that the process of row-reducing a matrix is exactly the same as left-multiplying by a sequence of elementary matrices.
Example 78. Are elementary matrices invertible? What’s the inverse of $E_1, E_2, E_3$?

Theorem 79. An $n \times n$ matrix $A$ is invertible if and only if $A$ is row equivalent to $I_n$, and the same sequence of row operations that reduces $A$ to $I_n$ also reduces $I_n$ into $A^{-1}$.

Proof. Suppose $A$ is invertible, then since $Ax = b$ has a solution for every $b$, every row is a pivot row, but since $A$ is square, the REF of $A$ must be $I_n$ (think about the possible REF’s of a square matrix with a pivot in every row), ie $A$ is row-equivalent to $I_n$.

Conversely, suppose $E_1, \ldots, E_k$ are elementary matrices that reduce $A$ to $I_n$, where the row operation represented by $E_1$ is applied first, then $E_2$, and so on..., then

$$
(E_k \ldots E_2 E_1)A = I_n
$$

ie, $A^{-1} = E_k E_{k-1} \ldots E_2 E_1 = (E_k E_{k-1} \ldots E_2 E_1)I$. In particular, this shows that $A$ is invertible.

This in fact gives us an algorithm for finding the inverse of any matrix $A$. If we can row-reduce $A$ to the identity matrix, then if we keep track of the row-operations used to reduce $A$ to $I_n$, applying the same operations to $I_n$ will give us $A^{-1}$. The following algorithm gives a nice way of keeping track of the row-operations used.

Algorithm for finding $A^{-1}$.

1. Row reduce the matrix $[A \ I]$.
2. If $A$ reduces to $I$, then $[A \ I]$ reduces to $[I \ A^{-1}]$. Otherwise $A$ doesn’t have an inverse.

Another View of Matrix Inversion. Let $A$ be $n \times n$, then $A^{-1}$ is the matrix such that $AA^{-1} = I$. Let $b_1, \ldots, b_n$ be the columns of $A^{-1}$. By the definition of matrix multiplication,

$$
AA^{-1} = I \quad \text{means that} \quad [Ab_1 \ldots Ab_n] = I
$$

which is to say that for each $1 \leq i \leq n$, $Ab_i = e_i$. In other words, the $i$th column of $A^{-1}$ is just the solution to the equation $Ax = e_i$. We sum this up in the following theorem:

Theorem 80. Let $A$ be an $n \times n$ matrix. If $A$ is invertible, then the $i$th column of $A^{-1}$ is the unique solution of the equation $Ax = e_i$.

Theorem 81. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A$ is invertible iff $ad - bc \neq 0$, and its inverse is

$$
\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
$$

Example 82. Consider the matrices

$$
R_{90} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad S = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}
$$

Using theorem 81, compute the inverses of these matrices. Note that $R_{90}$ is just the familiar “rotate counterclockwise by 90°” matrix, $F$ just reflects everything across the line $x = -y$, and $S$ scales everything by 2.

Given these descriptions, it’s easy to see that the inverse of the transformation “rotate counterclockwise by 90°” is “rotate clockwise by 90°”, the inverse of “reflect across the line $x = -y$” is exactly the same transformation (reflecting twice is the identity!), and the inverse of “scaling by 2” is “scaling by 1/2²”. Now, using the standard technique (theorem 81) for determining standard matrices, compute the standard matrices of these inverse transformations, and verify that they coincide with the inverse matrices computed using theorem 81.
1. Let \( \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \). Suppose \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) is a linear transformation defined by \( T(\vec{x}) = A\vec{x} \) such that \( T(\vec{x}) = \vec{b} \) has no solution.

(a) Is it possible for \( A \) to have no pivot positions? If so, construct a matrix \( A \) such that the linear transformation \( T(\vec{x}) = A\vec{x} \) has the property described above. (ie, the property that \( T(\vec{x}) = \vec{b} \) has no solution). If not, explain why not.

\[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Proof. Yes. Let \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). This clearly works as well.

(b) Is it possible for \( A \) to have exactly one pivot position? If so, construct a matrix \( A \) such that the linear transformation \( T(\vec{x}) = A\vec{x} \) has the property described above. If not, explain why not.

\[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Proof. Yes! Consider \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). This works! (check it)

(c) Is it possible for \( A \) to have exactly two pivot positions? If so, construct a matrix \( A \) such that the linear transformation \( T(\vec{x}) = A\vec{x} \) has the property described above. If not, explain why not.

\[ \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 4 & 3 & 1 \end{bmatrix} \]

Proof. No! If \( A \) has three pivot positions, then it must have a pivot in every row, so by thm 51, we know that \( Ax = c \) has a solution for \( c \in \mathbb{R}^3 \). In particular, \( Ax = b \) has a solution.

(d) Is it possible for \( A \) to have three pivot positions? If so, construct a matrix \( A \) such that the linear transformation \( T(\vec{x}) = A\vec{x} \) has the property described above. If not, explain why not.

\[ \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 4 & 3 & 1 \end{bmatrix} \]

Proof. No! If \( A \) has three pivot positions, then it must have a pivot in every row, so by thm 51, we know that \( Ax = c \) has a solution for \( c \in \mathbb{R}^3 \). In particular, \( Ax = b \) has a solution.

Hint: What is the range of a linear transformation? Can you relate the property of \( T(\vec{x}) = \vec{b} \) not having a solution to a statement about the range of \( T \)? Theorems 51 and 52 may be useful.

2. Let \( A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 4 & 3 & 1 \end{bmatrix} \). Using thm 80, find the (4,4)th entry of \( A^{-1} \). (ie, the entry in the bottom right corner)

\[ \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 4 & 3 & 1 \end{bmatrix} \]

Proof. By thm 80 (and the discussion preceding it), you just want to solve for \( Ax = e_4 \), which ends up giving you \( x = (-0.25, 0.25, 0.25, -0.25) \), so the (4,4)th entry of \( A^{-1} \) is just the last entry of \( x \) (which by the theorem is the fourth column of \( A^{-1} \)), which is \(-0.25\)
3. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad T(\vec{e}_2) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

(a) Can $T$ be 1-1? If yes, give an example of a $T$ that is both 1-1 and satisfies the properties described above. If no, explain why.

**Hint:** If you said yes, then the easiest way to define a linear transformation is to come up with a matrix $A$, and define $T(\vec{x}) = A\vec{x}$. Note that here, the matrix will have to be $3 \times 3$.

**Hint:** Recall the definition of 1-1 \((50)\), and linear transformation \((40)\).

**Proof.** Nope! Let $A$ be the standard matrix for $T$. Note that the two conditions above imply that the first two columns of $A$ are $T(\vec{e}_1), T(\vec{e}_2)$. I.e, $A$ must have the form

$$A = \begin{bmatrix} 1 & 2 & a \\ 1 & 2 & b \\ 1 & 2 & c \end{bmatrix}$$

But no matter what the last column contains, the first two columns are linearly dependent, so all three columns must be linearly dependent! (the first column is a linear combination of the second one, and hence also a linear combination of the second and third columns). Thus, by thm\(54\), $T$ cannot be 1-1.

(b) Can $T$ be onto? If yes, give an example. If no, explain why.

**Hint:** As usual, when asked if something can be/is onto, theorem\(51\) will be useful.

**Proof.** From above, we know that $T$ cannot be 1-1, and hence since $T$ is square, by thm\(94\), it can’t be onto either.

(c) Is the set \(\{\vec{e}_1, \vec{e}_2\}\) linearly independent? Is the set \(\{T(\vec{e}_1), T(\vec{e}_2)\}\) linearly independent?

**Proof.** Yes to the first question, no to the second.

(d) Suppose:

$$T \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

Compute $T(\vec{w})$.

**Proof.** First, write $\vec{w}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $e_1$, and $e_2$. In other words, find $x_1, x_2, x_3$ such that

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Then use linearity of $T$ to compute

$$T \left( \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right) = x_1 T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + x_2 T \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) + x_3 T \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

But you know all the values on the right hand side of the above equation, so you can compute this for yourself.
11. **Lecture 11.** (Section 2.3) Exam and review session details on

[http://www.math.psu.edu/dmh/Teaching/Fall2011-220/](http://www.math.psu.edu/dmh/Teaching/Fall2011-220/)

- REVIEW Session on Sunday, 10/9 starting at 6pm in 117 Osmond
- EXAM on Wednesday, 10/12 from 6:30 - 7:45pm in 102 Forum
- CONFLICT Exam on Wednesday, 10/12 from 5:05 - 6:20pm in 108 Tyson
- MAKEUP Exam on Thursday, 10/20 from 6:30 - 7:45pm...somewhere else

So far we’ve discussed what it means for linear transformations to be onto and 1-1, what it means for their standard matrices, and how you can tell. Today, we summarize the results we’ve uncovered, and derive some new implications.

Recall:

**Theorem 83.** *(From [48]*) Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation, then there is a unique matrix \( A_T \) such that \( T(x) = A_Tx \) for all \( x \in \mathbb{R}^n \). In fact, \( A_T = [T(e_1) \ T(e_2) \ldots \ T(e_n)] \). We call this matrix \( A_T \) the (standard) matrix for \( T \).

Firstly, many problems will require you to determine if a particular linear transformation is 1-1/onto. If the transformation is defined in terms of its matrix, then the table below gives you a concrete way to tell if the transformation is 1-1/onto. If it isn’t, then just find its matrix! (and then use the table).

Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a linear transformation, and let \( A \) be its matrix, then

<table>
<thead>
<tr>
<th>( T ) is onto</th>
<th>( T ) is not onto</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T ) is 1-1</td>
<td>Every row and column of ( A ) has a pivot.</td>
</tr>
<tr>
<td>( T ) is not 1-1</td>
<td>Every row of ( A ) has a pivot, but not every column.</td>
</tr>
</tbody>
</table>

The above table tells us what 1-1/onto mean in terms of what the matrix of the transformation must look like, but often it is also necessary to understand the other consequences of a linear transformation being 1-1/onto. The most important ones are summed up here:

**Theorem 84.** *(From [51]*) Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation with matrix \( A \). The following are equivalent:

(a) \( T \) is onto.
(b) For every \( b \in \mathbb{R}^m \), the equation \( Ax = b \) has a solution.
(c) Every row of \( A \) has a pivot.
(d) The columns of \( A \) span \( \mathbb{R}^m \).

**Theorem 85.** *(From [54]*) Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation with standard matrix \( A \). The following are equivalent:

(a) \( T \) is 1-1.
(b) The equation \( Ax = 0 \) has a unique solution.
(c) Every column of \( A \) has a pivot.
(d) The columns of \( A \) are linearly independent.

**Exercise 86.** Suppose \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is a linear transformation given by \( T(x,y) = (2x - y, x + y) \), then is \( T \) 1-1? Is \( T \) onto?
Note that in the above exercise, after we discovered that $T$ is 1-1, we automatically know for free that $T$ is onto! This is because any square matrix with a pivot in every column must also have a pivot in every row (and vice versa).

Before stating the following result, we need a quick discussion on invertible transformations.

**Definition 87.** Define $\text{id}_n : \mathbb{R}^n \to \mathbb{R}^n$ to be the linear transformation defined by:

$$\text{id}_n(x) = x \quad \text{for all } x \in \mathbb{R}^n$$

**Definition 88.** A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be an invertible linear transformation if there exists another linear transformation $S : \mathbb{R}^m \to \mathbb{R}^n$ such that

(a) For all $x \in \mathbb{R}^n$, $S(T(x)) = x$.

(b) For all $x \in \mathbb{R}^m$, $T(S(x)) = x$.

In this case, we call $S$ the **inverse of $T$**, or $T^{-1}$.

Note that the two conditions above say that $S \circ T = \text{id}_n$, and $T \circ S = \text{id}_m$. (Recall that $S \circ T$ is the composition of $S$ and $T$, which is a linear transformation defined by “first apply $T$, then apply $S$“.)

Pictorially, the situation above is as follows:

$$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xleftarrow{S} \mathbb{R}^n$$

where $S, T$ are inverses of each other if $S$ “undoes” everything that $T$ does, and similarly $T$ “undoes” everything that $S$ does.

Note that this definition directly parallels the definition for a matrix inverse, which I’ve reproduced here for convenience:

**Definition 89.** (From [70]) Let $A$ be an $n \times n$ matrix, then $A$ is said to be **invertible** if there exists an $n \times n$ matrix $B$ with $AB = BA = I_n$. We call $B$ the inverse of $A$, or $A^{-1}$.

**Example 90.** Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(x_1, x_2, x_3) = (x_1, x_2)$, and $S : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $S(x_1, x_2) = (x_1, x_2, 0)$. Note that $T(1, 2, 0) = (1, 2)$, and $S(1, 2) = (1, 2, 0)$.

(a) Are $S, T$ inverses of each other? (ie, is $T$ invertible with $T^{-1} = S$?)

The answer is no! Note that $T$ maps $(1, 2, 3)$ to $(1, 2)$, but $S$ maps $(1, 2)$ to $(1, 2, 0)$, so $S$ does not reverse the action of $T$ on all $x$.

(b) Is $T$ invertible?

Again the answer is no! Let $A$ be the standard matrix for $T$, then

$T$ maps a large space into a small space $\quad \Rightarrow \quad A$ have more columns than rows

$\iff$ not every column of $A$ contains a pivot

$\iff \quad T$ is not 1-1

$\iff \quad$ there exist distinct vectors $x, y \in \mathbb{R}^3$ such that $T(x) = T(y)$

Let $z = T(x)$, then the last statement says that: “$T$ maps distinct vectors $x, y$ both to $z$”.

This means that for any function $S$ to reverse the action of $T$ on $x$, it must map $z$ to $x$, and for it to reverse the action of $T$ on $y$, it must map $z$ to $y$, but $S$ can’t map $z$ to both $x$ and $y$ (since $S$ is a function!), so no such $S$ exists, and $T$ is not invertible.

So, **WHY** isn’t $T$ invertible? Fundamentally, what went wrong was the fact that $T$ is not 1-1. Thus, since $T$ maps distinct vectors to the same vector, it “discards information”, and thus, given any vector $z$ in the codomain (here, $\mathbb{R}^2$), the vector in the domain (here, $\mathbb{R}^3$) that maps to $z$ cannot be recovered, since it is not unique (there are multiple vectors in $\mathbb{R}^3$ that maps to $z \in \mathbb{R}^2$)

Thus, in general, if $T$ is not 1-1, then $T$ is not invertible. In particular, if $n > m$, then any map $T : \mathbb{R}^n \to \mathbb{R}^m$ cannot be 1-1, hence is not invertible.
Now we consider the other case, where $T$ maps smaller space to a larger space.

**Example 91.** Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(x_1, x_2) = (x_1, x_2, 0)$, and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $S(x_1, x_2, x_3) = (x_1, x_2)$. Note that for any $(x_1, x_2) \in \mathbb{R}^2$, we have

$$S(T(x_1, x_2)) = S(x_1, x_2, 0) = (x_1, x_2)$$

(a) Are $S, T$ inverses of each other? (ie, is $T$ invertible with $T^{-1} = S$?)

Interestingly enough, even though the above formula shows that $S$ reverses everything that $T$ does, in this case $T$ doesn’t reverse everything that $S$ does! Note that:

$$T(S(1, 2, 3)) = T(1, 2, 0) = (1, 2, 0) \neq (1, 2, 3)$$

(b) Is $T$ invertible?

Again, the answer is no, though this time we can use a simpler argument:

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined as above is invertible, then its inverse must be a map

$$T^{-1} : \mathbb{R}^3 \to \mathbb{R}^2$$

which must be itself invertible with inverse $T$, but this is impossible since we’ve just proven that any map from a large space into a smaller space cannot be invertible!

Thus the original assumption that $T$ is invertible must have been false, ie $T$ is not invertible!

In particular, this shows that if $n \neq m$, then any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is not invertible. Equivalently, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is invertible, then $n = m$.

**Exercise 92.** If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation where $n = m$, then can we conclude that $T$ is invertible?

No! Consider the zero map $T : \mathbb{R}^5 \to \mathbb{R}^5$ given by $T(x) = 0$. This is clearly not 1-1, hence not invertible.

**Example 93.** Now we consider a few situations. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation.

(a) If $T$ is invertible, must $T$ be 1-1?

Yes! From above, if $T$ is not 1-1, then $T$ is not invertible, so if $T$ is invertible, it must be 1-1.

(b) If $T$ is 1-1, must $T$ be onto?

Yes! If $T$ is 1-1, then by thm 54 its matrix has a pivot in every column, but since $T : \mathbb{R}^n \to \mathbb{R}^n$, its matrix is $n \times n$, so if it has a pivot in every column, then it also has a pivot in every row, so by thm 51 $T$ is onto!

(c) If $T$ is invertible, must $T$ be both 1-1 and onto?

Yes. This comes from just putting (a) and (b) together.
(d) If $T$ is invertible, then must its standard matrix be invertible?

Yes! If this comes from the fact that every linear transformation has a standard matrix, and multiplication of matrices corresponds to composition of linear transformations (i.e., chaining linear transformations together). In particular, if $T$ is invertible with inverse $S$, then

$$T \circ S = S \circ T = \text{id}_n$$

Turning this into a statement about matrices, if $A_T$ is the matrix for $T$ and $A_S$ the matrix for $S$, this means that

$$A_T A_S = A_S A_T = I_n$$

In particular, don’t let this following statement confuse you, but:

This says that the inverse of the standard matrix of an invertible transformation $T$ is just the matrix of the inverse of $T$. Equivalently, the inverse of the transformation $T$ is just the transformation defined by the inverse of the standard matrix for $T$.

(e) If the standard matrix for $T$ is invertible, must $T$ be invertible?

Yes! This comes from the last statement in (d). Or equivalently, you can verify that if the map $T : \mathbb{R}^n \to \mathbb{R}^n$ has standard matrix $A$ (this is $n \times n$), and $A$ is invertible with inverse $B$ (also $n \times n$), then the linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ given by $S(x) = Bx$ is indeed the inverse to $T$!

Explicitly, note that for all $x \in \mathbb{R}^n$, we have

$$S(T(x)) = S(Ax) = B(Ax) = (BA)x = I_n x = x$$

and

$$T(S(x)) = T(Bx) = A(Bx) = (AB)x = I_n x = x$$

(f) If the standard matrix for $T$ is invertible, must $T$ be 1-1 and onto?

Yes! If the standard matrix for $T$ is invertible, then from (e), $T$ is also invertible, so from (c), $T$ is 1-1 and onto.

(g) If $T$ is 1-1 and onto, must its standard matrix be invertible?

Yes! Let $A$ be the standard matrix of $T$, then if $T$ is both 1-1 and onto, then since $T$ maps $\mathbb{R}^n$ to $\mathbb{R}^n$, we have

- $A$ is square and has a pivot in every row and column $\implies$ The REF of $A$ is $I_n$
- $\implies A$ is row-equivalent to the identity matrix
- $\implies$ By thm79, $A$ is invertible

In fact, note that here we didn’t even need the fact that $T$ is both 1-1 and onto. If $T$ is 1-1 OR onto, then $A$ (and hence $T$), must be invertible!

At this point, we’ve “discovered” the following theorem:
Theorem 94. Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a linear transformation with matrix \( A \). The following are equivalent:

(a) \( T \) is invertible.
(b) \( A \) is invertible.
(c) \( T \) is 1-1 and onto.
(d) \( T \) is 1-1.
(e) \( T \) is onto.

Furthermore, any linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \) for \( n \neq m \) is not invertible, and similarly any matrix that is not square is not invertible.

Note that this theorem is compatible with theorems 51 and 54.

Example 95. Suppose \( T : \mathbb{R}^n \to \mathbb{R}^n \) is a linear transformation, and suppose \( T(x) = b \) has a solution for every \( b \in \mathbb{R}^n \), then is \( T \) 1-1? onto? invertible?

All three are true! Let \( A \) be its standard matrix, then to say that \( T(x) = b \) has a solution for every \( b \in \mathbb{R}^n \) is to say that \( Ax = b \) has a solution for every \( b \in \mathbb{R}^n \), which by thm 51, we see that this means that \( T \) is onto, and hence (by thm 94) \( T \) is 1-1, onto, and invertible!

Exercise 96. (Seriously, do this!) Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( T(x_1, x_2) = (2x_1 + x_2, 5x_1 + 3x_2) \).

(a) Find the standard matrix for \( T \). Call it \( A \). Note that as usual, for all \( x \in \mathbb{R}^2 \), we have \( T(x) = Ax \) (i.e, this gives an equivalent description of \( T \))

(b) Verify that the matrix \( A \) found in (a) is an invertible matrix, and find its inverse, which we’ll call \( B \).

(c) Define the linear transformation \( S : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( S(x) = Bx \).

(d) Verify that \( S \) and \( T \) are inverses of each other (i.e, \( T \) is invertible with inverse \( S \) and vice versa)

(e) Find the standard matrices of \( S \) and \( T \). (You’ve done most of the work already)

(f) Verify that the standard matrices of \( S \) and \( T \) are inverses of each other.

Other random questions:

– What’s the standard matrix for scale vertically by 2? (as a transformation from \( \mathbb{R}^2 \to \mathbb{R}^2 \))
– What’s the standard matrix for reflect across the origin?
– What’s the standard matrix for “scale vertically by 2, then reflect across the origin”?
– What’s its inverse?
– Suppose \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) is a linear transformation, and suppose there are distinct vectors \( u, v \) such that \( T(u) = T(v) \). What can you say about \( T \)? What can you say about its matrix? What if \( u, v \) were LI? What now?
– Suppose \( T(1,0) = (3,3,8) \), and \( T(0,1) = (1,0,1) \). What is \( T(-1,5) \)?
– Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \), and \( T(1, -2, 3) = (1, 1, 1), T(2, -1, 0) = (1, 6, 2) \), and \( T(5, 2, 1) = (-1, -1, -1) \). What is \( T(10, 10, 10) \)? Is there enough information? What is the range of \( T \)?
– Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be linear. If \( T \) is 1-1, what can you say about \( n, m \)? If \( T \) is onto, what can you say about \( n, m \)? What is \( T(0) \)?
– Let \( T \) be defined by \( T(x, y, z) = (2x + y, x + 2y, y - z) \). Is it 1-1? Is it onto? How do you solve this? What is its matrix?
12. **Lecture 12.** (Sections 2.3 and 2.8)

**HOMEWORK DUE NOW!**

Quiz at end of class. Three problems, 6 points per problem, and 2 bonus points for finishing on time.

**Finishing up invertible matrices.** Now that we’ve talked enough about how matrices can be manipulated, it’s valuable to spend a moment to think about doing algebra with matrices. In fact, you can do algebra much the same way, keeping in mind the warnings from Example 97.

**Example 97.** Suppose $A, B$ are invertible $n \times n$ matrices.

(a) Is $AB$ invertible?

Yes! This was discussed earlier. In fact, the inverse is just $B^{-1}A^{-1}$.

(b) Is $A + B$ invertible?

No! (not in general). For example, consider $I$ and $-I$. Clearly both $I$, $-I$ are invertible, but $I + (-I) = 0$.

(c) Suppose $A(C + D) = B$, then is $C + D$ invertible?

Yes! Note that since $A$ is invertible, we can multiply both sides on the left by $A^{-1}$, giving us: $C + D = A^{-1}B$, but $A^{-1}, B$ are both invertible, so $A^{-1}B$ is invertible, hence $C + D$ is.

(d) Suppose $C, D$ are also $n \times n$ matrices, though not necessarily invertible, and $A(X + C)B^{-1} = D$.

Can we solve for $X$?

Yes! Proceed as you would normally do, keeping in mind that you have to multiply on the same side when manipulating the equation.

$$A(X + C)B^{-1} = D$$
$$A^{-1}(X + C)B^{-1} = A^{-1}D$$
$$X + C = A^{-1}DB$$
$$X = A^{-1}DB - C$$

**Subspaces.**

First, recall the definition of a vector space. Really, you don’t need to remember all the technical details word-for-word, but the idea you should have in your mind is that a vector space is “a set where you can add and scale” where also the set must have a 0 element, as well as “negative” elements.

Until now, the only examples of vector spaces that we’ve discussed in detail are the sets $\mathbb{R}^n$. While the category of vector spaces is far more diverse than just $\mathbb{R}^n$, today we’ll expand our bank of examples of vector spaces to *subspaces of* $\mathbb{R}^n$.

**Definition 98.** Let $V$ be a vector space, then a subset $W \subseteq V$ is a subspace if $W$ itself is a vector space.

**Theorem 99.** A subset $W \subseteq V$ is a subspace if and only if

(a) For all $u, v \in W$, $u + v \in W$

(b) For all $u \in W, a \in \mathbb{R}$, $au \in W$. 

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We won’t prove this, but intuitively, this means that a subset \( W \) of a vector space is a vector space if and only if the sum of any two things in \( W \) is also in \( W \), and any scalar multiple of anything in \( W \) is also in \( W \).

(For the purpose of a definitions quiz, I’ll allow this theorem to be cited as an “alternate” definition of subspace)

**Example 100.**

(a) Let \( V \) be a vector space. Must every subspace of \( V \) contain 0 (the zero vector)?

Yes! Let \( W \) be any subspace of \( V \), and let \( u \) be any vector in \( W \). By property (b) of \ref{99}, since \( u \in W \), we must have \( 0u \in W \), but \( 0u = 0 \), so \( 0 \in W \). This gives a quick check to see if certain subsets are not subspaces (ie, if you’re given a subset of a vector space that doesn’t have 0, it can’t be a subspace!)

(b) Let \( v_1, \ldots, v_p \in \mathbb{R}^n \), then is \( \text{Span} \{v_1, \ldots, v_p\} \) a subspace?

Yes! Any sum of linear combinations of \( v_1, \ldots, v_p \) is a linear combination, and so is any scalar multiple. We discussed this earlier. Explicitly, if \( a_i, b_i, c \in \mathbb{R} \), then

\[
(a_1v_1 + a_1v_2 + \cdots + a_pv_p) + (b_1v_1 + b_2v_2 + \cdots + b_pv_p) = (a_1v_1 + b_1v_1) + \cdots + (a_pv_p + b_pv_p) = (a_1 + b_1)v_1 + \cdots + (a_p + b_p)v_p
\]

and

\[
c(a_1v_1 + a_1v_2 + \cdots + a_pv_p) = (ca_1)v_1 + \cdots + (ca_p)v_p
\]

(c) What about the line \( x = y \) in \( \mathbb{R}^2 \)?

Yes! This is just \( \text{Span} \{(1,1)\} \), or equivalently \( \text{Span} \{(2,2)\}, \text{Span} \{-6, -6\} \), or in general \( \text{Span} \{(a,a)\} \) for any \( a \neq 0 \).

(d) Consider \( \text{Span} \{(1,1,1), (1,2,3)\} \subseteq \mathbb{R}^3 \). Now suppose we add a vector \( v \) into the span. For what \( v \) will \( \text{Span} \{(1,1,1), (1,2,3), v\} \neq \text{Span} \{(1,1,1), (1,2,3)\} \)?

Well, if it doesn’t change the span, that means that \( v \) must have already been in \( \text{Span} \{(1,1,1), (1,2,3)\} \), ie, \( \{(1,1,1), (1,2,3), v\} \) is linearly dependent!

On the other hand, if it does change the span, then in fact the set \( \{(1,1,1), (1,2,3), v\} \) will be linearly independent. You can prove this by showing that none of the vectors can be linear combinations of the others, though by next thursday we’ll have a slicker way to argue this.

Recall that the range of a linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is just the span of the columns of its standard matrix. Now, we’ll give this space a name:

**Definition 101.** The column space of a matrix \( A \) is the span of its columns, denoted \( \text{Col} \ A \).

Note that while “range” is a word that applies to functions, “column space” is a word that applies to a matrix. The connection between these two terms is as follows: “the range of a linear transformation \( T \) is the column space of the standard matrix for \( T \)”

**Example 102.** Let \( A \) be an \( m \times n \) matrix. Is \( \text{Col} A \) a subspace? What is it a subspace of?

Yup! By definition, it’s the span of a bunch of vectors, hence is a subspace! Since the vectors are in \( \mathbb{R}^m \), it’s a subspace of \( \mathbb{R}^m \).

**Definition 103.** The null space of a matrix \( A \) is the solution set of \( Ax = 0 \), denoted \( \text{Nul} A \).

**Example 104.**
(a) Let $A$ be $m \times n$. Is $\text{Nul } A$ a subspace? What is it a subspace of?

Yes again! Firstly, if $u \in \text{Nul } A$, then $Au = 0$, so for this to make sense, we must have $u \in \mathbb{R}^n$, so $\text{Nul } A$ is a subset of $\mathbb{R}^n$.

To see that it’s a subspace, suppose $u, v \in \text{Nul } A$, then that’s to say that $Au = Av = 0$, but then $A(u+v) = Au + Av = 0$, so $u+v \in \text{Nul } A$. Furthermore, if $c \in \mathbb{R}$, then $A(cu) = c(Au) = c0 = 0$, so $cu \in \text{Nul } A$, so by 99, $\text{Nul } A$ is a subspace (of $\mathbb{R}^n$).

(b) Now let $A$ be an $m \times n$ matrix, and $b \in \mathbb{R}^m$ with $b \neq 0$. Let $W$ be the solution set of $Ax = b$. Is $W$ a subspace?

No! Recall that the solution set of $Ax = b$ is either empty, or a shifted copy of the solution set of $Ax = 0$ (which we now can call $\text{Nul } A$). Thus, if $Ax = b$ is consistent and $p$ is any solution to $Ax = b$, then

$$\text{[Solution set of } Ax = b] = \{p + w : w \in \text{Nul } A\}$$

Now, does the right hand side contain 0?
Geometrically speaking, you can imagine that the solution set of $Ax = 0$ (ie, $\text{Nul } A$) is going to be some line, plane, or higher dimensional space through the origin, and the solution set of $Ax = b$ is some shifted copy of that. But since it shifts in a direction that is not “along” the line/plane, the shifted copy won’t contain 0. (Note that shifting the line $x = y$ by the vector $(1,1)$ will not change the line at all!)

However, this is not a proof. To prove this, we have to appeal to the algebra:

Algebraically speaking, if $0 = p + w$ for some $w \in \text{Nul } A$, then $p = -w$, so this expresses $p$ as a scalar multiple of something in $\text{Nul } A$, using the fact that $\text{Nul } A$ is a subspace, we can conclude that $p \in \text{Nul } A$.

However, this is impossible, since by assumption $Ap = b$ and $b \neq 0$, so $p \notin \text{Nul } A$, so we cannot write $0 = p + w$ for any $w \in \text{Nul } A$, ie 0 is not in the solution set of $Ax = b$.

The following definition is kind of the most important idea in all of linear algebra.

**Definition 105.** A basis for a vector space is a linearly independent subset of $W$ that spans $W$. 

13. Lecture 13. EXAM REVIEW

Summary of sections covered and things you need to know:

1.1 Systems of Linear Equations
   * What is a linear equation/system?
   * How many solutions can a linear system have?
   * What are elementary row operations/why do we use them?

1.2 Row Reduction and Echelon Forms
   * What is echelon form/reduced echelon form?
   * What are pivot positions?
   * What are basic/free variables?
   * How do you row-reduce a matrix?

1.3 Vector Equations
   * How do you solve a vector equation? (ie, an equation of the form $x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n = \vec{b}$
   * What is a linear combination of $v_1, \ldots, v_n$? What is $\text{Span}\{v_1, \ldots, v_n\}$?
   * How do you determine if a given vector $v$ is in $\text{Span}\{v_1, \ldots, v_n\}$?

1.4 The Matrix Equation $Ax = b$
   * What is the definition of the product of a matrix with a vector? How large must the vector be w.r.t. the matrix?
   * How do you solve matrix equations of the form $Ax = b$?
   * If $A$ is an $m \times n$ matrix, give as many equivalent statements as you can to “For each $b \in \mathbb{R}^m$, the equation $Ax = b$ has a solution”
   * Does multiplying by a matrix distribute across sums of vectors?

1.5 Solution Sets of Linear Systems
   * What is a homogeneous linear system?
   * When does a homogeneous system have a nontrivial solution? (Given a homogeneous system, how would you tell if it had a nontrivial solution?)
   * How do you find the parametric form of the solution set of a linear system?
   * What do solution sets of linear systems look like?
   * If $Ax = b$ is consistent, then how does the solution set of $Ax = b$ compare to the solution set of $Ax = 0$?

1.7 Linear Independence
   * What is the definition of linear independence?
   * What are all the characterizations of linear independence? (seriously, learn them all).
   * When can you easily spot if a set is linearly dependent?
   * How do you determine if a given set of vectors is linearly independent?

1.8 Introduction to Linear Transformations
   * What is the definition of a function? What is its domain/codomain?
   * What is the definition of the range of a function? What’s the image of $x$ under $f$?
   * How can you use a matrix to define a function from $\mathbb{R}^n \to \mathbb{R}^m$?
   * What is the definition of a linear transformation?
   * Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation, and you’re given vectors $v_1, \ldots, v_4$ that span $\mathbb{R}^4$, and you’re also given the values $T(v_1), \ldots, T(v_4)$. Suppose you’re given another vector $v \in \mathbb{R}^4$. How do you determine $T(v)$?

1.9 The Matrix of a Linear Transformation
   * Can any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ be represented by multiplication by a matrix?
* Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, how do you find the matrix $A$ such that $T(x) = Ax$ for all $x \in \mathbb{R}^n$?

* Come up with some $2 \times 2$ matrices and describe their action on the plane.

* What does it mean for a function to be onto? 1-1?

* Come up with as many equivalent statements as you can to “a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is 1-1”

* Come up with as many equivalent statements as you can to “a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto”

2.1 Matrix Operations

* When is the product of two matrices defined/undefined?

* What’s the definition of the product of two matrices $AB$?

* What’s the row-column rule for computing matrix products?

* What is the significance of the product of two matrices $AB$ w.r.t. the linear transformations they represent?

* Is $AB = BA$ for all matrices $A, B$ (of the appropriate sizes) ?

* If $AB = AC$, then does that mean $B = C$?

* If $AB = 0$, must $A = 0$ or $B = 0$?

* Given an equation involving matrices like $A(I + X)B = D$ how can you solve for $X$? (When is it possible?)

* What is the transpose of a matrix? What are the properties of the transpose?

2.2 The Inverse of a Matrix

* What does it mean for a matrix to be invertible?

* When is a $2 \times 2$ matrix invertible?

* If $A$ is invertible, then what does that say about the equation $Ax = b$?

* If $A, B$ are invertible, what is the inverse of $AB$?

* What is an elementary matrix?

* What must the REF of an invertible matrix look like?

* How can you tell if a matrix is invertible? If it is, how do you find its inverse?

* How can you compute a particular column of the inverse without computing the entire inverse?

2.3 Characterizations of Invertible Matrices

* If $m \neq n$, and $A$ is $m \times n$, can $A$ be invertible?

* If $A$ is square, come up with as many equivalent statements as you can to “$A$ is invertible”

* What is an invertible linear transformation?

* If $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible, how would you find its inverse?

2.8 Subspaces of $\mathbb{R}^n$

* What is a subspace of $\mathbb{R}^n$?

* What is the column space/null space of a matrix?

* What is a basis for a vector space?
14. **Lecture 14.** (Sections 2.8, 2.9) Definitions quiz next Thursday.

Recall the definition of basis:

**Definition 106.** A basis for a vector space $W$ is a linearly independent subset of $W$ that spans $W$.

I’ve cut/pasted these examples from last time:

**Example 107.**

(a) Consider the set $\{e_1, e_2\} \subseteq \mathbb{R}^2$. Is this a basis for $\mathbb{R}^2$?

Yes. To verify this, we have to check that it’s both (a) linearly independent, and (b) spans all of $\mathbb{R}^2$. It’s easy to check this by elementary means, but we’re going to do something else:

Consider the matrix $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since it’s clearly got a pivot in every column, by 54, the columns (which are $e_1, e_2$) are linearly independent. Since it’s clearly got a pivot in every row, by 51, the columns span all of $\mathbb{R}^2$.

We used this method because it is easily generalizable.

(b) Consider the set $\{(−2, 0, 4), (1, 2, 3), (2, 2, 1)\}$. Is this a basis for $\mathbb{R}^3$?

As it turns out, no! Even though you can’t spot any obvious relations between the three, it turns out that the set is in fact linearly dependent, and hence does not span $\mathbb{R}^3$. To see this, consider the matrix with those three vectors as columns. As it turns out...

(c) Do the columns of the above matrix span $\mathbb{R}^2$?

NO! This is a favorite question to ask on the exam. $\mathbb{R}^2$ and $\mathbb{R}^3$ are completely different sets. While we might think of $\mathbb{R}^2$ as being “smaller” than $\mathbb{R}^3$, it is in no way a subset of $\mathbb{R}^3$. Thus, since the columns are vectors in $\mathbb{R}^3$, and the span of vectors in $\mathbb{R}^3$ give subspaces of $\mathbb{R}^3$, the answer is a resounding NO.

(d) Let $u, v, w \in \mathbb{R}^{231}$, and let $W = \text{Span} \{u, v, w\} \subseteq \mathbb{R}^{231}$. Suppose $\{u, v, w\}$ is linearly independent. Is $\{u, v, w\}$ a basis for $W$?

Yes. By definition they span $W$, and by assumption they’re linearly independent, so they’re a basis for $W$!

They however are not a basis for $\mathbb{R}^{231}$.

Now that we’re comfortable checking to see if particular sets of vectors form a basis for $\mathbb{R}^n$, let’s try to find a basis for some subspaces of $\mathbb{R}^n$.

(e) Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

And thus since there isn’t a pivot in the last row/column, the columns are not linearly independent, AND the columns do not span $\mathbb{R}^3$.

To do this, we’ll want to write the solution of $Ax = 0$ in parametric vector form.

$$[A \; 0] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \\ 0 = 0 \end{bmatrix}$$
Writing the basic variables in terms of the free variables, we get:

\[ x_1 = 2x_2 + x_4 - 3x_5 \]
\[ x_3 = -2x_4 + 2x_5 \]

Thus, a general solution will look like:

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{bmatrix} = \begin{bmatrix}
  2x_2 + x_4 - 3x_5 \\
  x_2 \\
  -2x_4 + 2x_5 \\
  x_4 \\
  x_5
\end{bmatrix} = x_2 \begin{bmatrix} 2 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}
\]

Note here that the matrix \([u \ v \ w]\) is already in echelon form, and in fact has a pivot in every column! This tells us that \([u, v, w]\) is linearly independent. On the other hand, this computation shows us that \(\text{Nul } A\) is exactly \(\text{Span } [u, v, w]\).

In example (e) above, in general the matrix \([u, v, w]\) may not be in echelon form. On the other hand, it is \([u, v, w]\) will always be linearly independent. To see this, note that each vector \(u, v, w\) corresponds to a free variable. In this case, \(u\) corresponds to \(x_2\), \(v\) to \(x_4\), and \(w\) to \(x_5\). Note that \(u\) is the only vector of the 3 that has a nonzero entry in the \(x_2\) position, and similarly for \(v\) with \(x_4\), and \(w\) with \(x_5\). Thus, if

\[ x_2u + x_4v + x_5w = 0 \]

this means that in particular the sums of the 2nd, 4th, and 5th coordinates of \((x_2u + x_4v + x_5w)\) are zero, but since the only possible contributor to the 2nd coordinate is \(u\), and \(v\) for the 4th, and \(w\) for the 5th, we see that each coefficient \(x_2, x_4, x_5\) must all be 0.

**Example 108.** Consider the REF matrix

\[
A = \begin{bmatrix}
  1 & 0 & 2 & 0 & -3 \\
  0 & 1 & 6 & 0 & 5 \\
  0 & 0 & 0 & 1 & 6 \\
  0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

How can you describe \(\text{Col } A\)? What is a basis for \(\text{Col } A\)?

Clearly the span of the columns of \(A\) will only contain vectors with a 0 in the 4th coordinate. However, looking at the pivot columns, we see that any vector with a 0 in the 4th coordinate can be expressed as a linear combination of the columns. Furthermore, the pivot columns are obviously linearly independent, so in this case we see that they form a basis for \(\text{Col } A\).

Now consider

\[
B = \begin{bmatrix}
  2 & 0 & 4 & 0 & -6 \\
  1 & 1 & 8 & 1 & 2 \\
  0 & 1 & 6 & 0 & 5 \\
  1 & 0 & 2 & 1 & 3
\end{bmatrix}
\]

It can be shown that \(B \sim A\), ie \(A\) is the REF of \(B\). But the column space of \(B\) is clearly different from that of \(A\), since it contains stuff with nonzero 4th coordinate. Indeed, \(\text{Col } B\) is much more difficult to understand and coming up with a basis for \(\text{Col } B\) is also not easy.

Can anyone think of a way to find a basis for \(\text{Col } B\)?

In our efforts to understand \(\text{Col } B\), we’ll need to discuss some further properties of linear transformations.
Lemma 109. Suppose \( v \) is a linear combination of \( \{v_1, \ldots, v_p\} \), and \( T : V \to W \) a linear transformation, then \( T(v) \) is a linear combination of \( T(v_1), \ldots, T(v_p) \).

Proof. Suppose \( v = a_1v_1 + \cdots + a_pv_p \), then \( T(v) = T(a_1v_1 + \cdots + a_pv_p) = a_1T(v_1) + \cdots + a_pT(v_p) \).

Lemma 110. Suppose \( \{v_1, \ldots, v_p\} \) is a linearly independent set, and \( T : V \to W \) is a 1-1 linear transformation, then \( \{T(v_1), \ldots, T(v_p)\} \) is also linearly independent.

Proof. Suppose \( \{T(v_1), \ldots, T(v_p)\} \) were linearly dependent, then there would be scalars \( a_1, \ldots, a_p \in \mathbb{R} \), not all 0, such that
\[
a_1T(v_1) + \cdots + a_pT(v_p) = 0
\]
But that means \( T(a_1v_1 + \cdots + a_pv_p) = 0 \). Since \( T \) is 1-1, this means that \( a_1v_1 + \cdots + a_pv_p = 0 \), which is to say that \( v_1, \ldots, v_p \) were linearly dependent (since not all the \( a_i \)'s are 0). Since \( v_1, \ldots, v_p \) were not linearly dependent, \( T(v_1), \ldots, T(v_p) \) couldn't have been linearly dependent either!

Now we can prove the following theorem, which tells us exactly how to find a basis for \( \text{Nul} \ A \) and \( \text{Col} \ A \) of any matrix \( A \).

Theorem 111. Given an \( m \times n \) matrix \( A \)

(a) The vectors obtained in the parametric form of the solution to \( Ax = 0 \) form a basis for \( \text{Nul} \ A \).

(b) The pivot columns of \( A \) form a basis for \( \text{Col} \ A \).

Proof. Only the second part remains to be proven.

To see that the pivot columns of \( A \) form a basis, we must prove two things:

To see that they are linearly independent, let \( B \) be the REF of \( A \). Then for some elementary matrices \( E_1, \ldots, E_k \), we have \( (E_k \cdots E_1)A = B \). Let \( E = E_k \cdots E_1 \).

Clearly the pivot columns of \( B \) are linearly independent (think about what the pivot columns of a matrix in REF must look like), but also, since \( E_1, \ldots, E_k \) are invertible, so is \( E \), so we have
\[
EA = B \quad A = E^{-1}B
\]

But letting \( b_1, \ldots, b_n \) be the columns of \( B \), we have \( E^{-1}B = [E^{-1}b_1 \ldots E^{-1}b_n] \), so the pivot columns of \( A \) are just \( E^{-1} \) times the pivot columns of \( B \), so by lemma \( \ref{lem:linearly-independent} \) the pivot columns of \( A \) are also linearly independent.

Furthermore, they span the column space because:

(a) The non-pivot columns of \( B \) are linear combinations of the pivot columns of \( B \).

(b) The non-pivot columns of \( A \) are just \( E^{-1} \) times the non-pivot columns of \( B \), and thus by \( \ref{lem:linearly-independent} \) are linear combinations of the pivot columns of \( A \).

In other words. Suppose \( b_1, \ldots, b_r \) are the pivot columns of \( B \), and \( b_{r+1}, \ldots, b_n \) are the non-pivot columns, and similarly \( a_1, \ldots, a_r \) are the pivot columns of \( A \), and \( a_{r+1}, \ldots, a_n \) the non-pivot columns of \( A \). (in general the pivot columns might not come first, but it doesn’t matter for this argument)

Then, we know that for each \( i = 1, \ldots, n \), we have \( a_i = E^{-1}b_i \). Remark (a) tells us that for each \( i = r + 1, \ldots, n \), we have \( b_i \) is a linear combination of \( b_1, \ldots, b_r \). Since multiplication by \( E^{-1} \) is a linear transformation, so by \( \ref{lem:linearly-independent} \) we also have that each \( a_i = E^{-1}b_i \) is a linear combination of \( a_1 = E^{-1}b_1, \ldots, a_r = E^{-1}b_r \).

\( \square \)
Coordinate Systems

**Theorem 112.** Suppose $B = \{b_1, \ldots, b_p\}$ is a basis for a vector space $V$, then for any $x \in V$, there exist uniquely determined $a_1, \ldots, a_p \in \mathbb{R}$ such that $x = a_1 b_1 + \cdots + a_p b_p$.

**Proof.** Since a basis must span $V$, clearly such $a_1, \ldots, a_p$ exist. Now suppose there $c_1, \ldots, c_p \in \mathbb{R}$ also has the property that $x = c_1 b_1 + \cdots + c_p b_p$, then subtracting, we get:

$$0 = x - x = (a_1 - c_1)b_1 + \cdots + (a_p - c_p)b_p$$

but since the $b_1, \ldots, b_p$ are linearly independent, all the coefficients on the right hand side of the above equation are zero, ie $a_i = c_i$ for each $i$. \qed

**Definition 113.** Two vector spaces $V, W$ are called isomorphic if there is an invertible linear transformation $T : V \to W$. We write $V \cong W$ to mean $V$ is isomorphic to $W$.

**Theorem 114.** Suppose $B = \{b_1, \ldots, b_n\}$ is a basis for $V \subseteq \mathbb{R}^m$, then $V \cong \mathbb{R}^n$.

**Proof.** Define a linear transformation $T : \mathbb{R}^n \to V$ by sending $e_i \mapsto b_i$, ie

$$T(a_1 e_1 + \cdots + a_n e_n) = a_1 b_1 + \cdots + a_n b_n \quad \text{for all } a_i \in \mathbb{R}$$

Now, this is 1-1, since the standard matrix of $T$ (viewed as a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$) is just $A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$, but since $b_1, \ldots, b_n$ are a basis, they’re linearly independent, so the standard matrix must represent a 1-1 transformation.

Furthermore, it’s onto, since $V = \text{Span} \{b_1, \ldots, b_n\}$, but clearly $T$ will map to any linear combination of the $b_i$’s, so $T$ is onto.

Note that we also have a map $T^{-1} : V \to \mathbb{R}^n$ given by

$$T^{-1}(a_1 b_1 + \cdots + a_n b_n) = a_1 e_1 + \cdots + a_n e_n$$

This map essentially gives us a way of writing vectors of $V$ as vectors of $\mathbb{R}^n$. \qed

**Note.** In the proof above, note that if we think of $T$ as a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$, it’s not onto, but if we restrict the codomain to $V \subseteq \mathbb{R}^m$, it is onto!

Indeed, this fact is reflected in the following question: Must the matrix $A$ defined above be invertible?

 Nope! Note that $A$ is $m \times n$, so if $m \neq n$, $A$ isn’t invertible!

In a way, the above theorem says that we can work with $V$ in essentially the same way as working with $\mathbb{R}^n$. Concretely speaking, we need a good way of representing vectors in $V$:

**Definition 115.** Suppose the set $B = \{b_1, \ldots, b_p\}$ is a basis for a vector space $V$, then for each $x \in V$, the coordinates of $x$ relative to the basis $B$ are the coefficients $c_1, \ldots, c_p$ such that

$$x = c_1 b_1 + \cdots + c_p b_p$$

The vector:

$$[x]_B = \begin{bmatrix} c_1 \\
\vdots \\
c_p \end{bmatrix}$$

is called the coordinate vector of $x$ relative to $B$, or the $B$-coordinate vector of $x$. 

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Example 116. Consider the vectors:

\[
\begin{align*}
v &= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, & b_1 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, & b_2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{align*}
\]

(a) Are \(b_1, b_2\) linearly independent? For what vector space \(V\) is \(B = \{b_1, b_2\}\) a basis?

Since \(B\) is linearly independent obviously \(B\) spans \(\text{Span} B\), it’s a basis for \(\text{Span} B\)!

Now, theorem 114 tells us that \(V\) must behave exactly like \(\mathbb{R}^2\). How exactly does it behave like \(\mathbb{R}^2\)?

(b) Is \(v \in \text{Span} B\)? What is \([v]_B\)?

Yes, \(v = b_1 + 2b_2\). By the definition, we have \([v]_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\). Indeed, the map:

\[
[\cdot]_B : \text{Span} B \rightarrow \mathbb{R}^2
\]

that takes any \(v \in \text{Span} B\) to \([v]_B \in \mathbb{R}^2\) is really just the map \(T^{-1}\) defined in thm 114. This allows us to identify \(\mathbb{R}^2\) with a subspace of \(\mathbb{R}^3\).

In this way we can identify any vector space as \(\mathbb{R}^n\) for some \(n\).

(c) What I’m about to say is a little philosophical:

Consider all the points on the (infinite) blackboard. I’ll draw a special point called \(O\), and I’ll define the scalar multiple \(aP (a \in \mathbb{R})\) of a point \(P\) as just being the point \(Q\) that lies on the same line as \(P\) and \(O\), but is \(a\)-times as far away from \(O\) as \(P\).

Furthermore, I’ll define the sum of points \(P + Q\) as the fourth point of the parallelogram with vertices at \(P, Q, O\) and edges \(PO, QO\). Using these definitions, we can similarly define linear combinations, span, linear independence, and so on.

Now, it can be shown geometrically that this is a vector space! But, clearly

\[
\mathbb{B} = \{\text{all points on the infinite blackboard}\} \neq \{\text{all lists of real numbers } (x, y) \text{ with } x, y \in \mathbb{R}\} = \mathbb{R}^2
\]

They’re just completely different sets, in the sense that the set of all apples is different from the set of all oranges. However, they are similar in a certain sense, in that both apples and oranges are fruits, and so any knowledge we have of apples that depend only on their “fruitiness” carries over to the land of oranges. On the other hand, oranges are tasty and apples are gross, so they’re not all the same.

In the same sense, even though you can “erase” points on the blackboard, but you can’t erase lists of real numbers (since they’re in your head), these two sets are similar as vector spaces.

Consider the following questions, and ask them of \(\mathbb{B}\) and \(\mathbb{R}^2\):

i. Is there a nonzero element?

ii. Does there exist an element \(x\) such that every element is a scalar multiple of \(x\)?

iii. Do there exist elements \(x, y\) such that every element is a linear combination of \(x, y\)?

We see that each question has the same answer for \(\mathbb{B}\) and \(\mathbb{R}^2\)!

Thus, in this sense, they are similar. On the other hand, for us it’s easier to add vectors in \(\mathbb{R}^2\) than it is to add points by drawing a parallelogram and finding the 4th vertex. Thus, we want a way of connecting \(\mathbb{B}\) with \(\mathbb{R}^2\) in a way that allows easy transition between the two worlds.

This ends up just amounting to putting a coordinate system on \(\mathbb{B}\). Pick any two linearly independent points \(P, Q \in \mathbb{B}\), and define the linear transformation

\[
T : \mathbb{B} \rightarrow \mathbb{R}^2 \quad \text{by} \quad T(aP + bQ) = \begin{bmatrix} a \\ b \end{bmatrix}
\]
It can be shown (completely geometrically!) that $P, Q$ are a basis for $B$, and so by thm[114] we see that $T$ is both 1-1 and onto, and thus we can identify $B$ with $\mathbb{R}^2$ via $T$.

The key point here is that you can put different coordinate systems on $B$ by choosing different points for $P, Q$.

(d) The fact that you can put different coordinate systems on any vector space is essentially why we talk about the standard matrix for a linear transformation. You can represent the same linear transformation via different matrices! In fact, given any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ and a choice of basis $B$ for $\mathbb{R}^n$ and $C$ for $\mathbb{R}^m$, you can define a matrix $A$ such that $A[v]_B = [T(v)]_C$.

**Theorem 117.** Let $V$ be a vector space and let $B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_m\}$ be two bases for $V$. Then $n = m$.

**Proof.** Suppose not, that, say, $n < m$, then we have invertible linear transformations $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : V \to \mathbb{R}^m$, so $T \circ S$ is an invertible linear transformation $\mathbb{R}^n \to \mathbb{R}^m$, which is impossible from our discussions on section 2.3.

**Lemma 118.** Suppose $V$ is a vector space, and let $\{v_1, \ldots, v_n\} \subseteq V$ be linearly independent. Let $w \in V$ with $w \notin \text{Span} \{v_1, \ldots, v_n\}$, then $\{w, v_1, \ldots, v_n\}$ is also linearly independent.

**Proof.** Suppose there exist $a_0, \ldots, a_n \in \mathbb{R}$ with $a_0w + a_1v_1 + \cdots + a_nv_n = 0$. To show that $\{w, v_1, \ldots, v_n\}$ is linearly independent, we’d like to show that $a_0 = a_1 = \cdots = a_n = 0$.

Now, first consider the case where $a_0 = 0$. But that would mean that $a_1v_1 + \cdots + a_nv_n = 0$, so by linear independence of $\{v_1, \ldots, v_n\}$, we must have $a_1, \ldots, a_n = 0$, so it proves it in this case.

Now suppose $a_0 \neq 0$. Then, we can write

$$a_0w = -a_1v_1 - \cdots - a_nv_n$$

so...

$$w = \frac{a_1}{a_0}v_1 - \cdots - \frac{a_n}{a_0}v_n$$

But this shows that $w$ is a linear combination of $v_1, \ldots, v_n$, which can’t happen because by assumption $w \notin \text{Span} \{v_1, \ldots, v_n\}$.

**Theorem 119.** Every vector space $V$ has a basis.

**Proof.** The idea is to “grow” a basis starting with the set $\{v_1\}$ for $u \in V$ any nonzero vector. Let $v_1 \in V$ be any nonzero vector. If $\{v_1\}$ doesn’t form a basis, since it’s clearly linearly independent, it must be that $\{v_1\}$ doesn’t span $V$, so you can find something (call it $v_2$) that lies outside $\text{Span} \{v_1\}$, and throw it in.

Now by the above lemma, we have that $\{v_1, v_2\}$ is also linearly independent, so if $\{v_1, v_2\}$ isn’t a basis, you can find $v_3 \notin \text{Span} \{v_1, v_2\}$, and consider $\{v_1, v_2, v_3\}$.

Continue in this way until your set spans $V$. (since at each step the set we have is linearly independent, the final set will be a basis for $V$)

**Example 120.** In this example we’ll try to come up with a good definition of dimension.

Now, it’s natural to think of $\mathbb{R}^1$ as being a line, $\mathbb{R}^2$ as a plane, and $\mathbb{R}^3$ as being 3-space, and hence in a way, these spaces have dimensions 1, 2, and 3, respectively. However, $\mathbb{R}^n$ isn’t the only example of a space we might think of as being n-dimensional. For example, consider

$$V = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad U = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Geometrically, $V$ is a line in 3-space, and $W, U$ are planes in $\mathbb{R}^3$.

How can we come up with a nice definition of dimension that will agree with the fact that $\mathbb{R}^n$ has dimension $n$, $\dim V = 1$, and $\dim W = \dim U = 2$?
Definition 121. The dimension of a nonzero vector space $V$ is the size of any basis for $V$. The dimension of the vector space $\{0\}$ is defined to be 0.

Definition 122. The rank of a matrix $A$ is the dimension of its column space. The kernel of a linear transformation, denoted $\text{Ker } T$ is the null space of its standard matrix. Ie, if $A$ is the standard matrix of $T$, then $\text{Ker } T = \text{Nul } A$ and $\text{Ran } T = \text{Col } A$.

Example 123. Let $A$ be row-equivalent to the following matrix:

\[
\begin{bmatrix}
1 & 0 & 2 & -1 & 0 \\
0 & 1 & 1 & 3 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(a) What is the dimension of $\text{Col } A$?

By thm 111 we see that the pivot columns form a basis for $\text{Col } A$. Since $A$ clearly has three pivot columns, $\dim \text{Col } A = 3$.

(b) What is the dimension of $\text{Nul } A$?

By thm 111 we see that the vectors obtained in the parametric form of the solution set of $Ax = 0$ form a basis for $\text{Nul } A$. By reviewing the example of how one finds the parametric solution set, we see that there is one vector for each free variable, and since the matrix $[A \ 0]$ has clearly 3 pivots, and hence 3 basic variables, it has 2 free variables, so $\dim \text{Nul } A = 2$.

(c) Suppose now that $B$ is a $4 \times 7$ matrix, and suppose that $\{v_1, v_2, v_3\}$ is a basis for $\text{Nul } B$. What is $\dim \text{Col } B$?

Since $\text{Nul } B$ has a basis of 3 elements, $\dim \text{Nul } B = 3$, i.e. there are 3 free variables in the equation $Bx = 0$. On the other hand, there are 7 variables total in that equation, hence there are 4 basic variables, which means there must be 4 pivots in $B$, corresponding to 4 pivot columns, so by thm 111 $\dim \text{Col } B = 4$.

Example (c) illustrates a useful line of reasoning that we’ll summarize in the following theorem:

Theorem 124. (The Rank-Nullity Theorem) Let $A$ be an $m \times n$ matrix, then

\[
\dim \text{Nul } A + \dim \text{Col } A = n
\]

Proof. The dimension of $\text{Nul } A$ is the number of free variables in the equation $Ax = 0$, corresponding to the number of columns of $A$ that are not pivot columns. Since the total number of columns if $n$, the rest must be pivot columns, the number of which is the dimension of $\text{Col } A$. Thus, we have $\dim \text{Nul } A + \dim \text{Col } A = n$.

Example 125. Suppose $T : \mathbb{R}^5 \to \mathbb{R}^3$ is a linear transformation, and $\{u, v\} \subseteq \mathbb{R}^5$ is linearly independent. Suppose $T(u) = T(v) = 0$. What can you say about $\dim \text{Ran } T$? Can it be 0? 1? 2? 3? 4? 5?

Let $A$ be the standard matrix of $T$, then $A$ is $3 \times 5$, and to say that $T(u) = T(v) = 0$ is to say that $u, v \in \text{Nul } A$. Since $u, v$ are linearly independent, this means that $\dim \text{Nul } A \geq 2$ (with equality if and only if $\text{Nul } A = \text{Span } \{u, v\}$). Since $\dim \text{Nul } A \geq 2$, and $\dim \text{Nul } A + \dim \text{Col } A = 4$, this means that $\dim \text{Col } A = \dim \text{Ran } T \leq 3$. 

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This example in a way illustrates how to best think of the rank-nullity theorem. Any $m \times n$ matrix $A$ represents a linear transformation that maps from $T_A : \mathbb{R}^n \to \mathbb{R}^m$. In this case, the domain is what we want to look at. Now, $T$ will map some of the domain to 0, and some of it to nonzero vectors in $\mathbb{R}^m$. In a way, the vectors it maps to zero form the null space and do not contribute to the dimension of the range, whereas the vectors that don’t get mapped to zero contribute everything to the dimension of the range. Thus, since $\mathbb{R}^n$ has $n$ dimensions for $T$ to work on, intuitively speaking, the dimensions of $\mathbb{R}^n$ that $T$ maps to zero correspond to the dimension of the kernel, and the other ones correspond to the the dimension of the range.

**Example 126.** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Can you always find a basis for ker $T$ consisting of nothing but the vectors $\{e_1, e_2, e_3\}$? (not necessarily including them all)

Nope. Let’s think about what kinds of subspaces $\{e_1, e_2, e_3\}$ can span. They can span only three 1-dimensional subspaces (namely the $x, y, z$ axes), and can span only three 2-dimensional subspaces (namely the $xy$-plane, $xz$-plane, and the $yz$-plane). They can also span the 0-space $\{0\}$, and all of $\mathbb{R}^3$.

On the other hand, ker $T = \text{Nul } A$ (where $A$ is the standard matrix of $T$) doesn’t have to be so well-behaved. We’ve seen many examples where Nul $A$ is a line or a plane that doesn’t fit squarely on any collections of axes. Indeed, it’s possible to come up with a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose null space is Span $\{(1, 1, 1)\}$.

**PROBLEM SET 6 PART TWO** (one problem).
Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that

$$\text{ker } T = \text{Span } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

1. What is dim Ran $T$? (Hint: Use the rank nullity theorem [124] on the standard matrix of $T$)

*Proof. Recall that Ran $T$ denotes the range of $T$, which is exactly the same as the column space of the standard matrix of $T$. Let $A$ be the standard matrix of $T$. Now, clearly ker $T$ is onto dimensional, since $(1, 1, 1)$ is certainly a basis for it. Thus, since ker $T = \text{Nul } A$, we have dim Nul $A = 1$, so by the rank-nullity theorem, dim Nul $A + \text{dim Col } A = 3$, so dim Col $A = 2$. 

2. Given an example of such a linear transformation.
(Hint: Come up with an appropriate matrix $A$ such that Nul $A = \text{Span } \{(1, 1, 1)\}$, the define $T$ by $T(x) = Ax$. What must such a matrix look like?)

*Proof. From part (a), we see that since dim Col $A = 2$, $A$ must have two pivots. Thus, you simply have to come up with a matrix $A$ with two pivots such that $A \cdot (1, 1, 1) = (0, 0, 0)$. There are many possibilities - here’s one:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 127. Let $H$ be a $p$-dimensional vector space.

(a) Suppose $\{v_1, \ldots, v_p\}$ is a linearly independent set of vectors in $H$. Must they form a basis?

If they weren’t a basis, then that’d be to say that they did not span $H$, so we can find additional vectors $v_{p+1}, \ldots$ such that $\{v_1, \ldots, v_p, v_{p+1}, \ldots\}$ is also linearly independent, which would mean that $H$ had dimension $\geq p + 1$, which is a contradiction.
(b) Suppose instead that \( \{v_1, \ldots, v_p\} \) spans \( H \). Must they form a basis?

If they weren’t a basis, then that’d be to say that the set is linearly dependent, which is to say that at least one of the vectors is a linear combination of the others, so we can remove at least one vector and have the resulting set still span \( H \). Since we can repeat this until we arrive at a basis for \( H \), this would show that \( H \) had dimension \( < p \), which is also a contradiction.

**Example 128.** Let \( A \) be \( n \times n \).

(a) If the columns of \( A \) are linearly independent, must they form a basis for \( \mathbb{R}^n \)?

Yes! Since \( A \) is square, the columns must also span \( \mathbb{R}^n \), hence they’re a basis for \( \mathbb{R}^n \).

(b) In this case, what is rank \( A \)? \( \dim \text{Col } A \)? \( \text{Nul } A \)? \( \dim \text{Nul } A \)?
Determinants

Recall that for $2 \times 2$ matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we saw that they were invertible if and only if $ad - bc \neq 0$, where $ad - bc$ was called the determinant of the matrix.

Notation. If $A$ is a matrix, then we use $A_{ij}$ to denote the submatrix obtained by removing the $i$th row and $j$th column.

**Definition 129.** For $n \geq 2$, the determinant of an $n \times n$ matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is:

$$
\det A = |A| = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}
$$

We’ll define the determinant of a $1 \times 1$ matrix $[a]$ to be $a$.

**Example 130.** (a) Find the determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

From the definition, we see that $\det A = |A| = (-1)^2 a \cdot \det[d] + (-1)^3 b \cdot \det[c] = ad - bc$.

Thus, the quantity $ad - bc$ is indeed the determinant of $A$ as it was defined.

(b) Find the determinant of the matrix

$$
B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}
$$

Again by applying the definition, we have

$$
\det B = (-1)^2 a \cdot \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} + (-1)^3 b \cdot \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + (-1)^4 c \cdot \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}
$$

Thus, we see that in general, the determinant is a complicated polynomial function in the entries of your matrix, and in fact, for general matrices, this determinant is rather difficult to compute. However, in some cases it’s easy.

(c) Find the determinant of the matrix

$$
C = \begin{bmatrix} 0 & 0 & 0 \\ d & e & f \\ g & h & i \end{bmatrix}
$$

The above computation shows that $|C| = 0$!

Indeed, this shows that the presence of $0$’s in the right places can make determinants significantly easier to compute. However, to fully take advantage of this technique, we need some more technology.
Definition 131. Given a square matrix $A$, then $(i, j)$-cofactor of $A$ is the number $C_{ij}$ given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Theorem 132. For any $n \times n$ matrix $A$, we have for any $i$ or $j$,

$$\det A = a_{i1}C_{i1} + \cdots + a_{in}C_{in} = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj}$$

Here, the first is called the cofactor expansion along the $i$th row, and the second is the cofactor expansion along the $j$th column.

Proof. Very messy and not very enlightening. (do it for homework! (actually, don’t)) \hfill \Box

Example 133.

(a) Use cofactor expansion across the second row to compute $\det A$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\det A = a_{21}(-1)^3 \det A_{21} + a_{22}(-1)^4 \det A_{22} + a_{23}(-1)^5 \det A_{23}$$

$$= 0(-1)^3 \det A_{21} + a_{22}(-1)^4 \det A_{22} + 0(-1)^5 \det A_{23}$$

$$= \det A_{22}$$

$$= \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= 1 \cdot 1 - 3 \cdot 2$$

$$= -5$$

(b) Use cofactor expansion to compute $\det B$, where

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 4 & 1 \\ 6 & 1 & 0 & 4 \\ 3 & 2 & 0 & 1 \end{bmatrix}$$

First cofactor expand across the third column, giving us

$$\det B = 0 \cdot (\ldots) + 4 \cdot (-1)^5 \begin{vmatrix} 0 & 1 & 0 \\ 6 & 1 & 4 \\ 3 & 2 & 1 \end{vmatrix} + 0 \cdot (\ldots) + 0 \cdot (\ldots)$$

$$= -4 \cdot \begin{vmatrix} 0 & 1 & 0 \\ 6 & 1 & 4 \\ 3 & 2 & 1 \end{vmatrix}$$

Now to compute the determinant of this matrix, we can cofactor expand across the first row:

$$\ldots = -4 \cdot \left(1 \cdot (-1)^3, \begin{vmatrix} 6 & 4 \\ 3 & 1 \end{vmatrix} \right)$$

$$= 4 \cdot (6 \cdot 1 - 4 \cdot 3)$$

$$= -24$$
(c) Use cofactor expansion to compute the determinant of the following matrix

\[
C = \begin{bmatrix}
  a & b & c & d \\
  0 & e & f & g \\
  0 & 0 & h & i \\
  0 & 0 & 0 & j
\end{bmatrix}
\]

First cofactor-expand along the first column:

\[
\det C = a \begin{bmatrix}
  e & f & g \\
  0 & h & i \\
  0 & 0 & j
\end{bmatrix}
\]

then again expand along the first column of the 3 \times 3 matrix:

\[
\cdots = a \cdot (e \begin{bmatrix}
  h & i \\
  0 & j
\end{bmatrix})
\]

and then again...

\[
\cdots = a \cdot (e(hj)) = aehj
\]

Indeed, this illustrates the following fact:

**Definition 134.** A **triangular** matrix is a square matrix such that all entries either above or below the main diagonal are zero.

In the case where everything below the diagonal are zeros, we call the matrix **upper triangular**. If everything above the diagonal are zero, we call it **lower triangular**.

**Theorem 135.** If \( A \) is a triangular matrix, then \( \det A \) is the product of all the entries on the main diagonal.

**Proof.** We’ve already seen that if \( A \) is upper-triangular, repeated cofactor expansion along the first column proves the claim. For the case where \( A \) is lower-triangular, simply cofactor-expand across the first row.

**Corollary 136.** If \( A \) is a triangular matrix with a zero on the main diagonal, then \( \det A = 0 \).

**Properties of Determinants**

Now that we know how to compute determinants, what are they good for?

Before proceeding, we’ll take this theorem for granted:

**Theorem 137.** If \( A, B \) are square matrices of the same size, then \( |AB| = |A||B| \).

**Proof.** Kind of difficult. Assuming this however makes proving other stuff a lot simpler.

**Theorem 138.** \( |A| = |A^T| \).

**Proof.** This comes from the fact that you can either cofactor-expand along rows or columns, without changing the value of the determinant. Proving this is also unnecessarily messy and unenlightening.
Example 139. Compute the determinant of the following matrices:

\[
E_1 = \begin{bmatrix}
k & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \quad E_2 = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \quad E_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & k \\
0 & 0 & 1 \\
\end{bmatrix}
\]

The first and third are triangular, and it's easily seen that \(|E_1| = k|E_3| = 1|.

On the other hand, cofactor-expanding along either the third row or column will show that \(|E_2| = -1|.

In fact, this illustrates a more general fact, which we also won't prove in full rigor:

**Theorem 140.** Let \( E \) be an elementary matrix, and let \( I \) be the identity matrix of the same size.

(a) If \( E \) was obtained by scaling a row of \( I \) by \( k \), then \(|E| = k|I|\).

(b) If \( E \) was obtained by interchanging two rows of \( I \), then \(|E| = -1|I|\).

(c) If \( E \) was obtained by a row replacement operation, then \(|E| = 1|I|\).

In particular, the determinant of any elementary matrix is always nonzero.

**Corollary 141.** Let \( A \) be a square matrix.

(a) If \( B \) is obtained by scaling a row of \( A \) by \( k \), then \(|B| = k|A|\).

(b) If \( B \) is obtained by interchanging two rows of \( A \), then \(|B| = -|A|\).

(c) If \( B \) is obtained by applying a row-replacement operation to \( A \), then \(|B| = |A|\).

**Proof.** This corollary just comes from the fact that applying an elementary row operation corresponds to multiplying by the appropriate elementary matrix. I.e., in (a), \( B = EA \), where \( E \) is the elementary matrix representing the “scale a row \( k \)” operation, so applying the multiplicativity of the determinant, we have

\[|B| = |EA| = |E||A| = k|A|\]

The same technique also proves (b) and (c).

**IN FACT!** Since \(|A| = |A^T|\), the previous corollary is true when “row” is replaced by “column”.

**Example 142.** Now suppose \( A \) is row-equivalent to the matrix

\[
B = \begin{bmatrix}
1 & 2 & 3 \\
0 & -1 & 2 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

What is det \( A \)?

First note that \(|B| = 0\) (it’s triangular with a 0 on the main diagonal).

Now, to say that \( A \sim B \) is to say that there exist elementary matrices \( E_1, \ldots, E_n \) such that

\[
B = E_n E_{n-1} \cdots E_1 A \quad \text{so} \quad |B| = |E_n E_{n-1} \cdots E_1 A| = |E_n||E_{n-1}|\cdots|E_1||A|
\]

But since none of the \( E_i \)'s have determinant zero, so since \(|B| = 0\), we must also have \(|A| = 0\).

**Warning.** Note that in general, if \( B \sim A \), it is NOT true that \(|B| = |A|\). However, we do know that if one of them is zero, then the other is zero, since multiplying by elementary matrices cannot make nonzero determinants zero.

This example demonstrates the following fact:
Corollary 143. Suppose $A, B$ are square matrices and $A \sim B$, then $|A| = 0$ if and only if $|B| = 0$.

Example 144. Suppose $A$ is a square matrix. Let $B$ be any echelon form of $A$.

(a) Is $B$ triangular?

Yes, in fact it’s upper-triangular.

(b) If $A$ is invertible, what can you tell about $\det B$?

Since it’s invertible, it must have a pivot in every row/column. Since $B$ is in echelon form, every diagonal entry must be a row-leader, and hence is nonzero. Thus, $\det B \neq 0$.

(b) If $A$ is not invertible, what can you say about $\det B$?

If $A$ is not invertible, then it won’t have a pivot in every row/column. Just thinking about the possible echelon forms of a square matrix that don’t have a pivot in every row/column, we see that at least one of the diagonal entries must be zero, hence $\det B = 0$.

Combined with the previous corollary, we get the following theorem:

Theorem 145. A square matrix $A$ is invertible if and only if $\det A \neq 0$.

Notation. For any $n \times n$ matrix $A$ and $b \in \mathbb{R}^n$, let $A_i(b)$ be the matrix obtained from $A$ by replacing column $i$ by the vector $b$.

Theorem 146. Let $A$ be an invertible $n \times n$ matrix. For any $b \in \mathbb{R}^n$, the unique solution $x$ of $Ax = b$ has entries given by

$$x_i = \frac{\det A_i(b)}{\det A} \quad i = 1, 2, \ldots, n$$
17. **Lecture 17.** Eigenvectors/Eigenvalues/Eigenspaces. Sections 5.1 and 5.2. Pset 6 due Thursday.

**Example 147.** Consider the matrix

\[
A = \begin{bmatrix}
4 & -1 \\
2 & 1
\end{bmatrix},
\]

we can compute: \(A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}\), \(A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\)

So at first the action of \(A\) on the plane is difficult to describe. On the other hand, we can ask:

Does there exist a vector \(x \in \mathbb{R}^2\) such that \(Ax = \lambda x\) for some \(\lambda \in \mathbb{R}\)? (in other words, do there exist vectors \(x \in \mathbb{R}^2\) that are scaled by \(A\)?)

**Definition 148.** An eigenvector of a square matrix \(A\) is a nonzero vector \(x\) such that \(Ax = \lambda x\) for some \(\lambda \in \mathbb{R}\). An eigenvalue of a matrix \(A\) is a number \(\lambda\) such that for some nonzero vector \(x\), \(Ax = \lambda x\).

Here, \(\lambda\) is the eigenvalue corresponding to the eigenvector \(x\), and \(x\) is an eigenvector with eigenvalue \(\lambda\).

**Example 149.** Let \(A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}\) as above.

(a) Is \(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\) an eigenvector of \(A\)?

We can compute \(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}\) for any \(\lambda \in \mathbb{R}\), and hence is not an eigenvector.

(b) Is \(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\) an eigenvector of \(A\)?

We can compute \(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}\), so \(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\) is an eigenvector of \(A\) with eigenvalue \(\lambda = 2\).

Thus, we see that indeed there do exist vectors \(x \in \mathbb{R}^2\) that are scaled by \(A\). Now that we have one, we can ask:

(c) How many eigenvectors of \(A\) are there?

Infinitely many! This is because if \(Ax = \lambda x\), then for any \(c \in \mathbb{R}\), \(A(cx) = c(Ax) = c\lambda x = \lambda(cx)\), so \(cx\) is also an eigenvector (as long as \(c \neq 0\)).

In fact, note that if \(x, y\) are eigenvectors of \(A\) with eigenvalue \(\lambda\), then \(Ax = \lambda x\) and \(Ay = \lambda y\), so

\[A(x + y) = Ax + Ay = \lambda x + \lambda y = \lambda(x + y),\]

which shows that \((x + y)\) is also an eigenvector of \(A\) with the same eigenvalue!

(d) Let \(\lambda \in \mathbb{R}\), then is the set of all eigenvectors of \(A\) with eigenvalue \(\lambda\) a subspace of \(\mathbb{R}^2\)?

Almost! (but not quite). Note that technically, the zero vector \(\vec{0}\) is not an eigenvector. However, the above computations show that

The set of all eigenvectors of \(A\) with eigenvalue \(\lambda\), together with \(\vec{0}\), is a subspace of \(\mathbb{R}^2\)

We'll see this in another light later on.

Now, we've found that \(A\) has infinitely many eigenvectors. However, all the examples we've got are all scalar multiples of \((1, 2)\), so now we may ask: are these the only eigenvectors of \(A\)? To answer this question, we'll first consider another:
(e) Is 5 an eigenvalue of $A$?

To see if it’s an eigenvalue, we want to ask: do there exist nonzero $x \in \mathbb{R}^2$ such that $Ax = 5x$?

but the above equation is the same as: (let $I$ be the $2 \times 2$ identity matrix)

$$\cdots Ax = 5Ix$$ or equivalently $Ax - 5Ix = 0$ or.. $(A - 5I)x = 0$

Thus, the question can be rephrased as:

do there exist nontrivial solutions to the homogeneous equation $(A - 5I)x = 0$?

In this case, we see that $A - 5I = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix}$. However, the determinant of this matrix is $4 + 2 = 6 \neq 0$, so this matrix is invertible, and hence there are no nontrivial solutions to $(A - 5I)x = 0$.

Thus, we can conclude that 5 is not an eigenvalue.

(f) What are the eigenvalues of $A$?

The nice part of the above computation, is that it’s easily generalizable. Let $\lambda \in \mathbb{R}$ be any number. What does it mean for $\lambda$ to be a eigenvalue of $A$? From above, we see that

$$[\lambda \text{ is an eigenvalue of } A] \iff [(A - \lambda I)x = 0 \text{ has a nontrivial solution}]$$

but by the characterizations of invertible square matrices, $(A - \lambda I)x = 0$ having a nontrivial solution is the same as..

$$\cdots \iff [(A - \lambda I) \text{ is not invertible}] \iff [\det(A - \lambda I) = 0]$$

Thus, $\lambda$ is an eigenvalue of $A$ if and only if $\det(A - \lambda I) = 0$.

How can we use this to figure out the eigenvalues of $A$?

Viewing the above statement in another light, we see that the eigenvalues of $A$ are exactly the values $\lambda$ for which $\det(A - \lambda I) = 0$. So, if we treat $\lambda$ as a variable, the eigenvalues of $A$ are exactly the zeroes/roots of the function $\det(A - \lambda I)$. In fact, in this case, we see that

$$\det(A - \lambda I) = \det \begin{bmatrix} 4 - \lambda & -1 \\ 2 & 1 - \lambda \end{bmatrix} = (4 - \lambda)(1 - \lambda) - (-1 \cdot 2) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$$

thus, we see that the eigenvalues of $A$ are exactly 2 and 3.

(g) How can you describe the set of all eigenvectors of $A$?

The only eigenvalues of $A$ are 2 and 3, so the eigenvectors of $A$ fall into two categories - those with eigenvalue 2, and those with eigenvalue 3. From the definitions, the eigenvectors of $A$ with eigenvalue 2 are exactly the nonzero solutions to the equation

$$(A - 2I)x = 0$$

Similarly, the eigenvectors of $A$ with eigenvalue 3 are exactly the nonzero solutions to the equation

$$(A - 3I)x = 0$$

Even though this was all computed for the specific $2 \times 2$ matrix $A$, the ideas hold in general. We summarize all this in the following theorems.
Theorem 150. A number $\lambda$ is an eigenvalue for a square matrix $A$ if and only if there is a nontrivial solution to the equation

$$(A - \lambda I)x = 0$$

Definition 151. Let $A$ be an $n \times n$ square matrix, then $\det(A - \lambda I)$ as a function in the variable $\lambda$ is a polynomial of degree $n$. We call this the characteristic polynomial of $A$, denoted $\chi_A(\lambda) = \det(A - \lambda I)$

Corollary 152. The eigenvalues of any square matrix $A$ are the roots of the polynomial $\chi_A(\lambda)$.

Definition 153. For any square matrix $A$ and $\lambda \in \mathbb{R}$, the eigenspace of $A$ corresponding to $\lambda$ (or the $\lambda$-eigenspace) is the solution set of the equation

$$(A - \lambda I)x = 0$$

Note that $\lambda$ is an eigenvalue iff its eigenspace is nontrivial (ie, has dimension $> 0$)

Note. Here, we’re certainly justified in using the name “eigenspace” in the above definition. Indeed, if $x \in \mathbb{R}^n$ is an eigenvector of an $n \times n$ matrix $A$ with eigenvalue $\lambda$, then $Ax = \lambda x$, so $(A - \lambda I)x = 0$, so $x$ is indeed in the $\lambda$-eigenspace of $A$. Thus, the $\lambda$-eigenspace of $A$ certainly contains all eigenvectors of $A$ with eigenvalue $\lambda$.

Furthermore, everything (except $\vec{0}$) in the $\lambda$-eigenspace is also an eigenvector, since

$$(A - \lambda I)x = 0 \quad \text{is to say} \quad Ax = \lambda x$$

Thus, we see that the $\lambda$-eigenspace is exactly the set of all eigenvectors of $A$ with eigenvalue $\lambda$, together with the vector $\vec{0}$.

Corollary 154. An $n \times n$ matrix $A$ is invertible if and only if $0$ is not an eigenvalue for $A$.

Proof. If it’s noninvertible, then there must be a nonzero vector $x \in \mathbb{R}^n$ such that $Ax = 0$, but that’s to say that $0$ is an eigenvalue of $A$.

Conversely, if $A$ is invertible, then $(A - 0I)x = Ax = 0$ has no nontrivial solutions, so the 0-eigenspace is trivial, so 0 is not an eigenvalue of $A$.

Example 155. Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. Find a basis for the 2-eigenspace.

By definition, the 2-eigenspace is exactly the null space of the matrix

$$A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since this matrix has two non-pivot columns, the equation $(A - 2I)x = 0$ has a null space of dimension 2. To find a basis for the null space, simply find the parametric form of the solution set of $(A - 2I)x = 0$ and apply theorem [111].

Example 156. Now consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & -5 \end{bmatrix}$$

What are the eigenvalues of this matrix?
Applying corollary 152, we see that the eigenvalues are just the roots of $\chi_A(\lambda)$, which in this case is

$$
\chi_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix}
1 - \lambda & 2 & -4 \\
0 & 2 - \lambda & 2 \\
0 & 0 & -5 - \lambda
\end{bmatrix}
$$

Since this matrix is triangular, we can easily calculate the determinant to be the product of the entries on the main diagonal, which are

$$
\cdots = (1 - \lambda)(2 - \lambda)(-5 - \lambda)
$$

Thus, we see that the eigenvalues of the triangular matrix $A$ are exactly the roots of $\chi_A(\lambda)$, which for any triangular matrix are exactly the entries on the main diagonal of $A$.

**Theorem 157.** Suppose $A$ is a triangular (square) matrix. Then the eigenvalues of $A$ are exactly the entries on the main diagonal.
18. **Lecture 18.** Sections 5.1 and 5.2, Problem Set 6 due NOW! Problem set 7 due next next tuesday.

Firstly, recall the theorem that described how applying row operations to your matrix affects the determinant (thm 141). To summarize, the theorem said:

- scaling a row by $k$ also scales the determinant by $k$.
- interchanging two rows flips the sign of the determinant.
- applying a replacement operation does not change the determinant.

Now, even though $A \sim B$ doesn’t mean $|A| = |B|$, in some situations we can profitably use the above theorem to compute the determinant of various matrices.

**Example 158.** Compute the determinant of the following matrices:

(a) $A = \begin{bmatrix} 0 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix}$

There are a number of ways. Using cofactor expansion, you can either expand along the bottom row, or the leftmost column. If you expand along the bottom row, you get

$$|A| = 6 \cdot \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 6 \cdot (-4 \cdot 1) = -24$$

If you cofactor expand along the leftmost column, you get:

$$|A| = 1 \cdot \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = -1 \cdot 4 \cdot 6 = -24$$

Or, you could note that swapping the first two rows gives you a triangular matrix, whose diagonal entries are $1, 4, 6$, so the triangular matrix you get after the interchange has determinant 24, so the original matrix has determinant -24.

(b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 0 & 3 & -1 \end{bmatrix}$

This matrix is a little lacking in zeroes, so cofactor expansion will be a little difficult. However, note that if you subtract the first row from the second and then interchange the last two rows, we get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

which has determinant 3. Since we applied only a row replacement and an interchange operation to get to this triangular matrix, the original matrix $B$ must have had determinant -3.

Alright, back to eigenvectors. Last time, we learned how to find the eigenvectors of any matrix.

**Example 159.**

(a) What are the eigenvalues of the matrix

$$A = \begin{bmatrix} -1 & 4 & 8 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Recall that for any matrix $A$, the eigenvalues of the matrix are exactly the roots of the polynomial $\det(A - \lambda I)$. In this case, this polynomial is:

$$\chi_A(\lambda) = \begin{vmatrix} -1 - \lambda & 4 & 8 \\ 0 & 3 - \lambda & 0 \\ 0 & 0 & 3 - \lambda \end{vmatrix} = (-1-\lambda)(3-\lambda)(3-\lambda)$$

(since this matrix is triangular)
Thus, the eigenvalues are exactly -1, 3. In this case, 3 is a double root of $\chi_A$, so we say that
the eigenvalue 3 has **algebraic multiplicity** 2.

(b) As before, we see that the eigenvectors of $A$ come in two flavors: those with eigenvalue -1, and those with eigenvalue 3. Let’s consider these two flavors separately.

**How many linearly independent eigenvectors of $A$ can you find with eigenvalue -1?**

Recall that the $\lambda$-eigenspace of a matrix $A$ is defined to be the null space of the matrix $A - \lambda I$.

(This is also the set of all eigenvectors of $A$ with eigenvalue $\lambda$, together with $\vec{0}$)

Thus, in this case, we’re looking for the dimension of $\text{Nul} (A + I)$. Note that

$$A + I = \begin{bmatrix} 0 & 4 & 8 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Note that this matrix is not in REF.

There are a variety of ways to calculate the null space of this matrix. You could either row reduce the matrix, and note that it has two pivots, or you can note that the dimension of the column space is clearly 2, since columns 2 and 3 certainly are linearly independent and hence give a basis for the column space. Thus, by the rank-nullity theorem, we see that $\dim \text{Nul} (A + I) = 1$.

(c) **How many linearly independent eigenvectors of $A$ are there with eigenvalue 3?**

We know the eigenvectors of $A$ with eigenvalue $-1$ are exactly the nonzero vectors in $\text{Nul} (A - (-1)I) = \text{Nul} (A + I)$. In this case, we see that

$$[(A + I) \ 0] \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{so...} \quad \begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_1 = \text{free} \end{cases}$$

Thus, $(1, 0, 0)$ is in fact a basis for the $(-1)$-eigenspace.

(c) **How many linearly independent eigenvectors of $A$ are there with eigenvalue 3?**

(ie, what is the dimension of the 3-eigenspace of $A$?)

In this case the 3-eigenspace of $A$ is the null space of the matrix $A - 3I$. In this case, we have

$$A - 3I = \begin{bmatrix} -4 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix clearly has one pivot, so $\dim \text{Nul} (A - 3I) = 2$.

(d) **Since the dimension of the 3-eigenspace is 2, that means that you can find 2 linearly independent eigenvectors of $A$ with eigenvalue 3. How could you find such vectors?**

Well, as usual, it would suffice to find a basis for $\text{Nul} (A - 3I)$. In this case, we have

$$[(A - 3I) \ 0] \sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{so...} \quad \begin{cases} x_1 = x_2 + 2x_3 \\ x_2 = \text{free} \\ x_3 = \text{free} \end{cases}$$

The parametric form of the solution set is:

$$\begin{bmatrix} x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

so $(1, 1, 0)$ and $(2, 0, 1)$ are a basis for $\text{Nul} (A - 3I)$, and hence for the 3-eigenspace of $A$. 

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To summarize, given a matrix $A$, we’ve found bases for both the $(-1)$-eigenspace and the $3$-eigenspace of $A$:

basis for the $(-1)$-eigenspace :  
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}  

basis for the $3$-eigenspace:  
\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}  

Note in particular that if we put these vectors together, the set:

\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},  
\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},  
\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}  

form a basis for $\mathbb{R}^3$ (why is this?)

Thus, in the case of $A$, we’ve found a basis for $\mathbb{R}^3$ consisting of eigenvectors. Geometrically, this means that we can describe the effect that $A$ has on $\mathbb{R}^3$ in terms of vectors that are scaled by $A$. This gives us a nice way of understanding how the matrix $A$ acts on $\mathbb{R}^3$.

In fact... let $v = (1, 1, 0)$. Note that $v$ is an eigenvector of $A$ with eigenvalue $3$. Then, let's compute $A^2v$:

$$A^2v = A(Av) = A(3v) = 3Av = 3(3v) = 9v$$

A good way to think about it is as follows: If $x$ is an eigenvector for a $3 \times 3$ matrix $B$, then you want to think of Span $\{x\}$ as being a “track” in $3$-space such that applying $B$ to any vector on this subspace just pushes the vector further along, and the speed at which the vector is pushed along the track is just the eigenvalue. In the case of the matrix $A$ above, it has three linearly independent tracks, one with “speed” $-1$, and two with “speed” $3$.

In the case of the above computation, we saw that applying $A$ two times to $v$ essentially just pushed it along the $v$-track with speed $3$, two times (of course, speed is probably not the right word here, since we ended up with $A^2v = 9v$ (as opposed to $6v$), but it’s okay as an analogy).

In general, $n \times n$ matrices $B$ for which it is possible to find a basis of $\mathbb{R}^n$ consisting of eigenvectors of $B$ are called diagonalizable. We’ll see why this is in a moment, but first, some setup:

**Definition 160.** Two matrices $A, B$ are similar if there exists an invertible matrix $P$ such that $A = PBP^{-1}$.

**Lemma 161.** If $P$ is an invertible matrix, then $|P^{-1}| = 1/|P|$.

**Proof.** Note that $I = PP^{-1}$, so $1 = |I| = |PP^{-1}| = |P||P^{-1}|$, so $|P^{-1}| = 1/|P|$.

**Theorem 162.** If two matrices $A, B$ are similar, then $\chi_A = \chi_B$.

**Proof.** Suppose $A, B$ are similar, ie, $A = PBP^{-1}$ for some invertible matrix $P$, then

$$A - \lambda I = PBP^{-1} - \lambda PIP^{-1} = P(B - \lambda I)P^{-1}$$

taking determinants, we get:

$$\chi_A = |A - \lambda I| = |P(B - \lambda I)P^{-1}| = |P||B - \lambda I||P^{-1}| = |B - \lambda I| = \chi_B$$

**Definition 163.** A diagonal matrix is a square matrix that is both upper and lower triangular. (ie, the only nonzero entries are on the main diagonal)

**Example 164.** Let $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Clearly $D$ is diagonal.
(a) What are the eigenvalues of \( D \)?

Since \( D \) is triangular, the eigenvalues of \( D \) are exactly the entries on the main diagonal, in this case -1 and 3 (with multiplicity 2).

(b) Can you find a basis for \( \mathbb{R}^3 \) consisting of eigenvectors of \( D \)?

It’s easy to see (either just by inspection, or by using the methods described in the last example)

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
\]

are eigenvectors with eigenvalues -1, 3, and 3 respectively. They obviously form a basis for \( \mathbb{R}^3 \).

**Definition 165.** A matrix \( A \) is said to be diagonalizable if \( A \) is similar to a diagonal matrix. If \( A \) is similar to a diagonal matrix \( D \), we may call \( D \) a diagonalization of \( A \).

We often want to talk about diagonal matrices because they’re especially easy to understand. The action of any \( n \times n \) diagonal matrix with diagonal entries \( a_1, \ldots, a_n \) is to scale the first coordinate by \( a_1 \), the second by \( a_2 \), etc.

**Example 166.** Now we’ll investigate the relationship between \( A \) and \( D \). Recall:

\[
A = \begin{bmatrix}
-1 & 4 & 8 \\
0 & 3 & 0 \\
0 & 0 & 3
\end{bmatrix}, \quad D = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 3
\end{bmatrix}
\]

Note that \( A,D \) have exactly the same eigenvalues. Furthermore, the dimensions of the corresponding eigenspaces for both \( A,D \) are the same. Recall that we found the following bases for the eigenspaces of \( A \):

basis for the \((-1)\)-eigenspace : \( \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ v_1 \end{bmatrix} \right\} \) 

basis for the \(3\)-eigenspace: \( \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \)

I claim that \( A \) is diagonal, and \( D \) is a diagonalization of \( A \). To see this, we want to find an invertible matrix \( P \) such that \( A = P^{-1}DP \).

Idea: Start with the vector \( e_1 \in \mathbb{R}^3 \). Use \( P \) to move \( e_1 \) onto the track defined by an eigenvector basis element of \( A \). Then, if we apply \( A \), and then apply \( P^{-1} \) to move it back to where it should be, the net effect should be the same as applying \( D \) (if the eigenvalue of the eigenvector we chose agrees with the eigenvalue of \( e_1 \) for \( D \))

Plan: Find a matrix \( P \) that maps \( e_1 \) to \( v_1 \), \( e_2 \) to \( v_2 \), and \( e_3 \) to \( v_3 \).

But this is easy, since we’re just looking for a matrix with columns \( v_1, v_2, v_3 \), ie, set:

\[
P = \begin{bmatrix}
1 & 1 & 2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Then, we have

\[
(P^{-1}AP)e_1 = P^{-1}(A(PE_1)) = P^{-1}(Av_1) = P^{-1}(-v_1) = -P^{-1}v_1 = -e_1 = D e_1
\]

\[
(P^{-1}AP)e_2 = P^{-1}(A(PE_2)) = P^{-1}(Av_2) = P^{-1}(3v_2) = 3P^{-1}v_2 = 3e_2 = De_2
\]

\[
(P^{-1}AP)e_3 = P^{-1}(A(PE_3)) = P^{-1}(Av_3) = P^{-1}(3e_3) = 3P^{-1}v_3 = 3e_3 = De_3
\]

Thus, we see that \( P^{-1}AP \) acts on \( e_1, e_2, e_3 \) exactly as \( D \) would, so since \( e_1, e_2, e_3 \) are a basis for \( \mathbb{R}^3 \), it must be that \( D = P^{-1}AP \). (If you’re unconvinced, you can multiply out \( P^{-1}AP \) and verify that it’s equal to \( D \)).
Let $A = \begin{bmatrix} -1 & 4 & 8 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ be as in the examples above.

In the examples above, we found that $A$ had two eigenvalues, namely -1 and 3. Of these, the eigenvalue 3 had algebraic multiplicity 2. By this, we meant that the characteristic polynomial $\chi_A(\lambda) = (-1 - \lambda)(3 - \lambda)(3 - \lambda) = -(\lambda + 1)(\lambda - 3)(\lambda - 3)$ has 3 as a double root.

However, in the case of $A$, we saw that the 3-eigenspace also had dimension 2. (The dimension of the $\lambda$-eigenspace is called the geometric multiplicity of the eigenvalue $\lambda$.) In general, this is not the case.

1. Come up with a square matrix $B$ such that $B$ has an eigenvalue $\lambda$ with algebraic multiplicity 2, but geometric multiplicity 1.

In other words, find a square matrix $B$ and a number $\lambda \in \mathbb{R}$ such that $\lambda$ is an eigenvalue of $B$, $\chi_B(x)$ as a polynomial is divisible by $(x - \lambda)^2$, and the dimension of the $\lambda$-eigenspace is 1.

(Hint: One possible solution can be obtained by tweaking the matrix $A$ so that 3 is still an eigenvalue with algebraic multiplicity 2, but with geometric multiplicity 1).

Proof. There are multiple answers to this, one of which is in fact given in the lecture notes for lecture 19, as the matrices $A$ and $B$.

Another example would be the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$. It has a single eigenvalue, namely 3 with algebraic multiplicity 2, but the dimension of the 3-eigenspace (which is also the null space of $A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$), is 1.

2. Suppose...

(a) there is a robot that when fed a banana, will poop out 2 perfectly clean and tasty bananas.
(b) this robot, when fed an apple, will poop out 3 perfectly clean and tasty apples.
(c) this robot, when fed an orange, will poop out 4 perfectly clean and tasty oranges.
(d) There exist traders that will trade you any fruit for any fruit (one for one).
(e) You have a rambutan, a kumquat, and a mangosteen.
(f) You have an uncontrollable urge to acquire 2 rambutans, 3 kumquats, and 4 mangosteens.

If this is not done, you and your true love will never meet again on the bonnie bonnie banks of loch lomond.

Here’s the question: Without robbing anyone (or in general doing anything that you’re not explicitly allowed to do), how do you avoid never meeting your true love on the bonnie bonnie banks of loch lomond?

(Hint: If $A$ is a square diagonal matrix, and $D$ a diagonal matrix with the same eigenvalues, then this problem is essentially the same as finding the invertible matrix $P$ such that

$$D = P^{-1}AP$$

)
Lecture 19. Homework 7 online. Yes, it’s all extra credit! (whatever that means)

**Notation.** From now on, the act of “diagonalizing a matrix \( A \)” will refer to the process of finding a diagonal matrix \( D \) such that \( A = PDP^{-1} \) for some invertible matrix \( P \). (Note this is equivalent to finding \( D, P \) such that \( D = PAP^{-1} \), or \( D = P^{-1}AP \), or \( A = P^{-1}DP \)).

Last time we saw how to diagonalize the matrix 
\[
\begin{bmatrix}
-1 & 4 & 8 \\
0 & 3 & 0 \\
0 & 0 & 3
\end{bmatrix}
\]

Now, let’s try to diagonalize another matrix.

**Example 167.** Consider the matrix 
\[
\begin{bmatrix}
-1 & 4 & 8 \\
0 & 3 & 1 \\
0 & 0 & 3
\end{bmatrix}
\]

(a) What are the eigenvalues of \( B \)?

Recall that the eigenvalues of a matrix \( B \) are exactly the roots of its characteristic polynomial. In this case, we have
\[
\chi_B(\lambda) = \left| \begin{array}{ccc}
-1 - \lambda & 4 & 8 \\
0 & 3 - \lambda & 1 \\
0 & 0 & 3 - \lambda
\end{array} \right| = (-1 - \lambda)(3 - \lambda)(3 - \lambda)
\]

Thus, the eigenvalues are again \(-1\) and \(3\), with the eigenvalue \(3\) having algebraic multiplicity 2.

(b) Find a basis for the \(-1\)-eigenspace of \( B \).

Again, the \(-1\)-eigenspace of \( B \) is just the solution set of \((B - (-I))x = (B + I)x = 0\). In this case, we have
\[
B + I = \begin{bmatrix}
0 & 4 & 8 \\
0 & 4 & 1 \\
0 & 0 & 4
\end{bmatrix}
\]

So, we get:
\[
[(B + I) \ 0] \sim \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
x_1 = \text{free} \quad x_2 = 0 \quad \text{so parametric solution is: } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

And hence \((1, 0, 0)\) is a basis for the \(-1\)-eigenspace, just as in the case with \( A \).

(c) Find a basis for the 3-eigenspace of \( B \).

As usual, this is the same as finding a basis for \( \text{Nul} \ (B - 3I) \). In this case, we have
\[
B - 3I = \begin{bmatrix}
-4 & 4 & 8 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

so...
\[
[(B - 3I) \ 0] \sim \begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
x_1 = x_2 \\
x_2 = \text{free} \\
x_3 = 0
\]

The parametric form of the solution set is:
\[
\begin{bmatrix}
x_2 \\
x_2 \\
0
\end{bmatrix} = x_2 \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix}
\]

so \((1, 1, 0)\) is a basis for the 3-eigenspace of \( B \). This is the crucial difference between \( A \) and \( B \). While the 3-eigenspace of \( A \) has dimension 2, the 3-eigenspace of \( B \) has dimension 1.
(d) Is $B$ diagonalizable?

What happens when you try to diagonalize it? Well, your best guess for a diagonalization of $B$ would probably be something like

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

just like the diagonalization of $A$. Now, we have to find a suitable invertible matrix $P$ such that $D = P^{-1}BP$. Let’s try our old idea, ie

(a) We want the process of first multiplying by $P$, then $B$, then $P^{-1}$ to be the same as the effect of multiplying by $D$. To show this, it suffices to show that $P^{-1}BPe_1 = De_1$, and similarly with $e_2, e_3$.

(b) To do this, we’ll want to let $B$ do most of the work, and use $P$ to trade the input vectors $e_1, e_2, e_3$ into vectors we can use $B$ to scale. Thus, we’ll want to map $e_1, e_2, e_3$ to eigenvectors of $B$.

But at this point, we see we’re stuck! Why? Well, first, let’s consider the following lemma:

**Lemma 168.** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let $\{v_1, \ldots, v_p\} \subseteq \mathbb{R}^n$ be linearly independent. Suppose now that $\{T(v_1), \ldots, T(v_p)\}$ is linearly dependent. Then $T$ isn’t invertible!

**Proof.** Firstly, note that since $\{T(v_1), \ldots, T(v_p)\}$ is linearly dependent, the dimension of their span is strictly lower than $p$. Now, what’s essentially happening here is that $T$ is mapping vectors that span a $p$-dimensional subspace, and squeezing them into a subspace of lower dimension. In other words, you can think of $T$ as mapping a big space to a small space, so naturally $T$ discards information, and hence can’t be invertible!

More formally, suppose $a_1 T(v_1) + \cdots + a_p T(v_p) = 0$ is a nontrivial linear dependence relation (ie, not all the $a_i$’s are 0). On the other hand, we have (by linearity of $T$):

$$0 = a_1 T(v_1) + \cdots + a_p T(v_p) = T(a_1 v_1 + \cdots + a_p v_p)$$

Since the $a_i$’s are not all 0, and $v_1, \ldots, v_p$ are linearly independent, this means that $a_1 v_1 + \cdots + a_p v_p \neq 0$, so $T$ maps a nonzero vector to 0, so since it also maps 0 to 0, it’s not 1-1, and hence not invertible. \qed

Now let’s return to the question of finding a suitable invertible matrix $P$ such that $D = P^{-1}BP$. Now, via the lemma, we see that for $P$ to be invertible, we must define $P$ by sending $e_1, e_2, e_3$ to linearly independent eigenvectors of $B$. **Why is this impossible?**

It’s impossible because the eigenvectors of $D$ all come in two flavors, either with eigenvalue -1 or with eigenvalue 3. However, since the (-1)-eigenspace has dimension 1, and the 3-eigenspace has dimension 1, we see that we can find at most two linearly independent eigenvectors of $B$!

Thus, this shows that our method, given our initial guess for $D$, is doomed to failure! As it turns out, for this particular matrix $B$, there does not exist a diagonal matrix that is similar to $B$. Ie, $B$ is not diagonalizable! In fact, we have a more general theorem:
Theorem 169. An \( n \times n \) matrix \( A \) is diagonalizable if and only if \( A \) has \( n \) linearly independent eigenvectors.

Proof. First we’ll show that if \( A \) is diagonalizable, then it must have \( n \) linearly independent eigenvectors. To see this, suppose \( A = PDP^{-1} \) for some invertible matrix \( P \) and diagonal matrix \( D \). Suppose

\[
D = \begin{bmatrix}
d_1 & 0 & 0 \\
0 & d_2 & 0 \\
0 & 0 & d_3
\end{bmatrix}
\]

Then, what are the vectors that \( P^{-1} \) maps to \( e_1, e_2, e_3 \)?

Well, clearly \( Pe_1, Pe_2, Pe_3 \) are the columns of \( P \), so if \( P = [\vec{p}_1 \ \vec{p}_2 \ \vec{p}_3] \), then \( P^{-1} \) must map \( \vec{p}_1, \vec{p}_2, \vec{p}_3 \) to \( e_1, e_2, e_3 \) respectively (since inverses reverse the action of the original map).

Now, note that for \( i = 1, 2, 3 \), we have:

\[
Ap_i = PDP^{-1}p_i = PDe_i = P(d_i e_i) = d_i(Pe_i) = d_i \vec{p}_i
\]

Thus we see that \( p_1, p_2, p_3 \) are eigenvectors of \( A \) with eigenvalues \( d_1, d_2, d_3 \). On the other hand, we know that since \( \vec{p}_1, \vec{p}_2, \vec{p}_3 \) are the columns of \( P \), and \( P \) is invertible, they must be linearly independent, so indeed we’ve found 3 linearly independent eigenvectors of \( A \)!

Now suppose that \( v_1, v_2, v_3 \) are three linearly independent eigenvectors of \( A \) (with eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \)). Then, we want to find an invertible matrix \( P \) such that \( D = P^{-1}AP \) for some diagonal matrix \( D \). As described earlier, we’d like \( P \) to map \( e_1, e_2, e_3 \) to \( v_1, v_2, v_3 \). Thus, if you let \( P = [v_1 \ v_2 \ v_3] \), we’d have that for \( i = 1, 2, 3 \)

\[
P^{-1}APe_i = P^{-1}Av_i = P^{-1}\lambda_i v_i = \lambda_i(P^{-1}v_i) = \lambda_i e_i
\]

But this tells you exactly that the columns of \( P^{-1}AP \) are \( \lambda_1 e_1, \lambda_2 e_2, \) and \( \lambda_3 e_3 \). In other words,

\[
P^{-1}AP = [\lambda_1 e_1 \ \lambda_2 e_2 \ \lambda_3 e_3]
\]

which is indeed diagonal!

\[\square\]

Important! Note that here, all we needed was for \( P \) to map the set \( \{e_1, e_2, e_3\} \) onto the set \( \{v_1, v_2, v_3\} \). It didn’t necessarily need to map \( e_i \) to \( v_i \). Ie, it could mix them up a bit. For example, if \( P = [v_2 \ v_3 \ v_1] \), this would still work! (Though you’d end up with another diagonal matrix \( D \))

In fact, I would urge you to try this on the homework. Ie, when diagonalizing, try mixing up the order a bit and see what happens.

Example 170. Consider the matrix \( A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \). Is it diagonalizable?

So, by the above theorem [169] we see that \( A \) is diagonalizable if and only if it has 3 linearly independent eigenvectors. To see this, we’ll first ask, what are the eigenvalues of \( A \)?

As usual, since \( A \) is triangular, the eigenvalues are just the entries down the main diagonal. In this case, 1, 2, and 3. Thus, this shows that the eigenvectors of \( A \) come in 3 flavors - those with eigenvalue 1, those with eigenvalue 2, and those with eigenvalue 3. Thus, each of those eigenvalues will come with its own eigenspace, each of which in this case will be 1-dimensional. Let \( v_1, v_2, v_3 \) be eigenvectors corresponding to the eigenvalues 1,2,3. Are these eigenvectors linearly independent? In fact, they are:
Theorem 171. If $A$ is an $n \times n$ matrix with eigenvectors $v_1, \ldots, v_p$ with eigenvalues $\lambda_1, \ldots, \lambda_p$. If the eigenvalues are all distinct, then $\{v_1, \ldots, v_p\}$ are linearly independent.

Proof. Suppose $\{v_1, \ldots, v_p\}$ were linearly dependent, then we $a_1, \ldots, a_p$, not all zero, such that $a_1 v_1 + \cdots + a_p v_p = 0$. Now, there may be many choices for the $a_i$’s, so choose the $a_i$’s so that as many $a_i$’s are zero as possible. Ie, find a linear dependence relation that involves as few vectors as possible. Possibly reordering/renaming the vectors, suppose the vectors involved in this minimal relation are $v_1, \ldots, v_r$, so we have:

$$a_1 v_1 + \cdots + a_r v_r = 0$$

where $a_1, \ldots, a_r$ are all nonzero, and there is no smaller nontrivial linear combination of $v_1, \ldots, v_r$ that equals 0. Now, if we apply $T$, we get:

$$T(a_1 v_1 + \cdots + a_r v_r) = a_1 T(v_1) + \cdots + a_r T(v_r) = a_1 \lambda_1 v_1 + \cdots + a_r \lambda_r v_r = 0$$

On the other hand, multiplying the entire relation by $\lambda_r$, we get

$$a_1 \lambda_r v_1 + a_2 \lambda_r v_1 + \cdots + a_r \lambda_r v_r = 0$$

Thus, subtracting the two equations, we get:

$$a_1 (\lambda_1 - \lambda_r) v_1 + a_2 (\lambda_2 - \lambda_r) v_2 + \cdots + a_r (\lambda_r - \lambda_r) v_r = 0$$

However, since the lambda’s are all distinct, the coefficients above are all nonzero, except for the last one $a_r (\lambda_r - \lambda_r) = 0$. But, this certainly gives us a smaller linear dependent relation of the set $v_1, \ldots, v_r$, contradicting the minimality of the relation we picked in the first place.

Thus, we conclude that our assumption (that $\{v_1, \ldots, v_p\}$ were linearly dependent), must have been false. Ie, they’re linearly independent.

Corollary 172. Let $A$ be an $n \times n$ matrix.

(a) For any eigenvalue $\lambda$ of $A$, the dimension of the $\lambda$-eigenspace is less than or equal to the algebraic multiplicity of $\lambda$.

(b) If $A$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.

(c) $A$ is diagonalizable if and only if the sums of the dimensions of the eigenspaces equals $n$.

(d) If $A$ is diagonalizable whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$, then if $B_k$ is a basis for the eigenspace corresponding to $\lambda_k$, the total collection of the vectors in $B_1, \ldots, B_p$ form a linearly independent set of eigenvectors. If (c) is true, then they form an eigenvector basis for $\mathbb{R}^n$.

Proof. We haven’t proven (a), but it can be deduced by examining the matrix $A - \lambda I$.

Everything else is more or less straightforward.

Example 173.

(a) Diagonalize the matrix $A = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, if possible.

Since this is triangular, we can easily see that the eigenvalues are 4, 0, and 2. Since these give 3 distinct eigenvalues, we see that $A$ is diagonalizable.

Then, by thm 169, $P^{-1} A P$ is diagonal, where the columns of $P$ are 3 linearly independent eigenvectors of $A$. 


(b) Diagonalize the matrix \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \) if possible.

The characteristic polynomial of \( A \) is:

\[
\chi_A(\lambda) = \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} = (1 - \lambda)(\lambda^2 + 1)
\]

Unfortunately this polynomial only has one real root, and hence only 1 real eigenvalue, so this matrix is not diagonalizable (over \( \mathbb{R} \)). Note that it IS diagonalizable if we allow our matrices to have complex entries.

(c) Diagonalize the matrix \( A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \) if possible.

Read the book.

**Example 174.**

(a) Suppose \( D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \). Calculate \( D^4 \).

So, you could either do this manually, and after a while you might notice a pattern, or you might not, but it’s easier and more enlightening to notice that \( De_1 = 2e_1, De_2 = 3e_2, \) and \( De_3 = 4e_3 \). From this, you can easily see:

\[
D^4 e_1 = DDDD e_1 = DDD(2e_1) = 2DDD e_1 = 2DD(2e_1) = 4DD e_1 = 4D(2e_1) = 8De_1 = 16e_1
\]

And similarly, we have

\[
D^4 e_2 = 81e_2 \quad \text{and} \quad D^4 e_3 = 256e_3
\]

But how does \( D^4 e_1, D^4 e_2, D^4 e_3 \) relate to the entries of \( D^4 \)? Indeed, they are just the columns! From this we see immediately that

\[
D^4 = \begin{bmatrix} 2^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 4^4 \end{bmatrix}
\]

Using this, we can simplify the exponentiation of diagonalizable matrices significantly:

(b) Suppose \( A = PDP^{-1} \) for some diagonal matrix \( D \) and invertible \( P \). Write \( A^3 \) in terms of \( P, D, P^{-1} \) and simplify:

\[
A^3 \quad = \quad (PDP^{-1})(PDP^{-1})(PDP^{-1}) = PDIPIDP^{-1} = PD^3P^{-1}
\]

(c) Are diagonalizable matrices always invertible?

Not always. They’re invertible if and only if all of their eigenvalues are nonzero.
Lecture 20. Problem Set 7 due Tuesday.

Definition 175. For two column vectors \( u, v \in \mathbb{R}^n \), we define the dot product of \( u \) and \( v \), denoted \( u \cdot v \) as \( u_1 v_1 + \cdots + u_n v_n \). I.e., \( u \cdot v = u^T v \).

Example 176.

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} \cdot \begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix} = \begin{bmatrix}
1 & 2 & 3
\end{bmatrix} \begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32
\]

In particular,

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} \cdot \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} = 1^2 + 2^2 + 3^2 = 14
\]

Proposition 177. Let \( u, v, w \in \mathbb{R}^n \), and \( c \in \mathbb{R} \), then

(a) \( u \cdot v = v \cdot u \).
(b) \( (u + v) \cdot w = u \cdot w + v \cdot w \).
(c) \( (cu) \cdot v = c(u \cdot v) = u \cdot (cv) \).
(d) \( u \cdot u \geq 0 \), and \( u \cdot u = 0 \) if and only if \( u = 0 \).

Proof. Easy to verify.

Definition 178. For \( v \in \mathbb{R}^n \), the norm (or length) of \( v \) is the value \( \sqrt{v \cdot v} \), denoted \( \|v\| \).

Theorem 179. For any \( c \in \mathbb{R} \), and \( v \in \mathbb{R}^n \), we have

\[\|cv\| = |c|\|v\|\]

Proof. \( \|cv\| = \sqrt{(cv) \cdot (cv)} = \sqrt{c(v \cdot v)} = \sqrt{c^2(v \cdot v)} = |c|=\sqrt{v \cdot v} = |c|\|v\|\]

Definition 180. A unit vector is a vector of norm 1. Given \( v \in \mathbb{R}^n \), we define the normalization of \( v \) to be the unit vector \( u = \frac{1}{\|v\|} v \).

Example 181. The normalization of \( v = (1, -2, 2) \) is...

\[\cdot = \frac{1}{\|v\|} \cdot v = \frac{1}{\sqrt{1 + 4 + 4}} \cdot v = \begin{bmatrix}
1/3 \\
-2/3 \\
2/3
\end{bmatrix} \]

Definition 182. For \( u, v \in \mathbb{R}^n \), the distance between \( u \) and \( v \), is defined to be \( \|u - v\| \), sometimes written \( \text{dist}(u, v) \).

Note that this definition incorporates the distance formula that we’ve learned for \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), and generalizes this to higher dimensions as well.

Example 183. What is the distance between the vectors \( (1, 1) \) and \( (4, 5) \)?

By the formula, the distance is \( \|(1, 1) - (4, 5)\| = \|(-3, -4)\| = \sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5 \).

Now that we’ve defined some notion of distance, a natural next step is to somehow define angle. As it turns out, for our purposes, we won’t actually need a precise definition of angle (though such a definition is possible). Instead, we’ll only need the notion of orthogonality.

Naturally, we have some notion of orthogonality of vectors in the plane, or 3-space, but it’s clear that the power of linear algebra comes from the fact that it allows us to easily generalize into higher
dimensions, so naturally we'll want to come up with a satisfactory definition of orthogonality for $\mathbb{R}^n$.

How do you define orthogonality in the plane? In $\mathbb{R}^3$?

For $\mathbb{R}^2$, we can describe the phenomenon of orthogonality using the mathematical framework we've developed so far as follows:

Consider two lines through the origin in $\mathbb{R}^2$ determined by the vectors $u,v$. In $\mathbb{R}^2$, these two lines are perpendicular if and only if $\|u - v\| = \|u - (-v)\| = \|u + v\|$, which the same as requiring that $\|u + v\|^2 = \|u - v\|^2$. But also, we have:

$$\|u + v\|^2 = (u + v) \cdot (u + v) = u \cdot u + 2u \cdot v + v \cdot v = \|u\|^2 + \|v\|^2 + 2u \cdot v$$

similarly,

$$\|u - v\|^2 = (u - v) \cdot (u - v) = u \cdot u - 2u \cdot v + v \cdot v = \|u\|^2 + \|v\|^2 - 2u \cdot v$$

and $\|u - v\|^2 = \|u + v\|^2$ if and only if

$$\|u\|^2 + \|v\|^2 + 2u \cdot v = \|u\|^2 + \|v\|^2 - 2u \cdot v$$

which is exactly to say that $u \cdot v = 0$, so we see that $u,v \in \mathbb{R}^2$ are orthogonal if and only if $u \cdot v = 0$.

This connection allows us to generalize the idea of orthogonality to $\mathbb{R}^n$.

**Definition 184.** Given $u,v \in \mathbb{R}^n$, we say that $u,v$ are orthogonal (to each other) if $u \cdot v = 0$

The computation above also gives us this nice little fact:

**Theorem 185.** Two vectors $u,v$ are orthogonal if and only if $\|u + v\|^2 = \|u\|^2 + \|v\|^2$. 
21. **Lecture 21.** Homework due NOW!

**Definition 186.** Given a subspace $W$ of $\mathbb{R}^n$, we define the orthogonal complement of $W$ inside $\mathbb{R}^n$, denoted $W^\perp$, to be the set:

$$W^\perp = \{ x \in \mathbb{R}^n : x \text{ is orthogonal to every } w \in W \}$$

or equivalently,

$$W^\perp = \{ x \in \mathbb{R}^n : \text{for all } w \in W, x \cdot w = 0 \}$$

**Example 187.**

(a) Find a nonzero vector that is orthogonal to both $(1, 0, 0)$, and $(0, 1, 0)$.

We’re trying to find a vector $(a, b, c)$ such that $(a, b, c) \cdot (1, 0, 0) = 0$, and $(a, b, c) \cdot (0, 1, 0) = 0$. However,

$$(a, b, c) \cdot (1, 0, 0) = a \quad \text{and} \quad (a, b, c) \cdot (0, 1, 0) = b$$

So from this, we must have $a = b = 0$. Thus, since our vector has to be nonzero, we can choose $(0, 0, 1)$.

(b) Is $(0, 0, 1)$ orthogonal to every vector of the form $(x, y, 0)$?

Yes! Note that

$$(x, y, 0) \cdot (0, 0, 1) = x \cdot 0 + y \cdot 0 + 0 \cdot 1 = 0$$

(c) What is the orthogonal complement of the $xy$-plane in $\mathbb{R}^3$?

From the examples above, it’s just $\text{Span}\{ (0, 0, 1) \}$. In other words, the $z$-axis.

(d) Suppose $W = \text{Span}\{ u, v \}$ is a subspace of $\mathbb{R}^n$. Suppose for some $x \in \mathbb{R}^n$, $x \cdot u = x \cdot v = 0$. Is $x \in W^\perp$?

By definition, the vectors in $W^\perp$ are the vectors of $\mathbb{R}^n$ that are orthogonal to every vector of $W$. Since $W = \text{Span}\{ u, v \}$, for any $w \in W$ we can write $w = au + bv$ for some $a, b \in \mathbb{R}$. Then, we see that

$$x \cdot w = x \cdot (au + bv) = x \cdot (au) + x \cdot (bv) = a(x \cdot u) + b(x \cdot v) = a \cdot 0 + b \cdot 0 = 0$$

**Theorem 188.** Suppose $W$ is a subspace of $\mathbb{R}^n$, and $W$ is spanned by $v_1, \ldots, v_k$, then $W^\perp$ is also a subspace of $\mathbb{R}^n$, and can be characterized as:

$$W^\perp = \{ x \in \mathbb{R}^n : x \cdot v_i = 0 \text{ for each } i \}$$

At this point, it becomes useful to introduce a new term:

**Definition 189.** The row space of a matrix $A$ is the span of its rows.

**Example 190.** Suppose two matrices $A, B$ are row-equivalent.

(a) Must $\text{Col } A = \text{Col } B$?

No! Note that even though

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

the column space of the first is $\text{Span}\{ (1, 0) \}$, whereas the column space of the second is $\text{Span}\{ (1, 1) \}$.
(b) Must Row $A = \text{Row } B$?

As it turns out, yes! To see this, we analyze what happens when you perform a single elementary row operation:

Clearly interchanging two rows doesn’t change the span, and scaling any row by a nonzero number doesn’t change the span. On the other hand, suppose the rows of your matrix are called $v_1, v_2, v_3$, and suppose we add $a$ times $v_1$ to $v_3$. Then, consider the two spans:

\[
W = \text{Span } \{v_1, v_2, v_3\} \quad V = \text{Span } \{v_1, v_2, av_1 + v_3\}
\]

Clearly $av_1 + v_3 \in W$, so certainly $W \supset V$. On the other hand, $-av_1 \in V$, so $(-av_1) + (av_1 + v_3) = v_3 \in V$, so $V \supset W$, so we must have $V = W$.

**Theorem 191.** If $A$ and $B$ are row equivalent, then $\text{Row } A = \text{Row } B$.

Now let $A$ be a matrix with $v_1, \ldots, v_n$ as rows, ie $A = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$. Then given a vector $x$, applying the row-column rule, we see that

\[
Ax = \begin{bmatrix} v_1 \cdot x \\ \vdots \\ v_n \cdot x \end{bmatrix}
\]

From this, we see that if $Ax = 0$, then $x$ must be orthogonal to every row of $A$! (and conversely, if $x \cdot v_i = 0$ for each $i$, then $Ax = 0$).

**Theorem 192.** Let $A$ be any matrix, then

\[
\left(\text{Row } A\right)^\perp = \text{Nul } A \quad \text{and} \quad \left(\text{Col } A\right)^\perp = \text{Nul } A^T
\]

**Example 193.** Let $W = \text{Span } \{v_1, v_2, v_3\}$, where

\[
v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \\ -1 \end{bmatrix}
\]

Find a basis for $W^\perp$.

Consider the matrix $A$ with $v_1, v_2, v_3$ as rows, namely

\[
A = \begin{bmatrix} 1 & 0 & 2 & 3 & -1 \\ 3 & 1 & 0 & -1 & 0 \\ -2 & 1 & 5 & -1 & 1 \end{bmatrix}
\]

then $W = \text{Row } A$, so by the theorem above, we have $W^\perp = \left(\text{Row } A\right)^\perp = \text{Nul } A$, so we only need to find a basis for Nul $A$, which is something we already know how to do.
Definition 194. A set of vectors \( \{u_1, \ldots, u_p\} \subseteq \mathbb{R}^n \) is an orthogonal set if the vectors are pairwise orthogonal. That is, \( u_i \cdot u_j = 0 \) for all \( i \neq j \).

Theorem 195. Any orthogonal set of nonzero vectors is linearly independent.

Proof. Suppose \( \{u_1, \ldots, u_n\} \) is orthogonal, then if \( c_1u_1 + \cdots + c_nu_n = 0 \), then
\[
u_1 \cdot (c_1u_1 + \cdots + c_nu_n) = u_1 \cdot 0 = 0
\]
but the left side is just \( c_1u_1 \), and so since \( u_1 \neq 0 \), it must be that \( c_1 = 0 \). Repeating this for each \( c_i \) shows that \( u_1, \ldots, u_n \) is linearly independent.

Example 196. Consider the following vectors in \( \mathbb{R}^3 \).

\[
v_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}
\]

(a) Is this an orthogonal set?

To see if it’s an orthogonal set, you have to check to see if \( v_1 \cdot v_2 = v_1 \cdot v_3 = v_2 \cdot v_3 = 0 \). In this case, all these are zero, so indeed \( \{v_1, v_2, v_3\} \) is an orthogonal set.

(b) Do \( \{v_1, v_2, v_3\} \) form a basis for \( \mathbb{R}^3 \)?

Yes. By the preceding theorem, this set is linearly independent. Since it’s a linearly independent set of 3 vectors in the 3-dimensional space \( \mathbb{R}^3 \), it’s a basis for \( \mathbb{R}^3 \).

(c) What are the coordinates of \( w = (1, 1, 1) \) relative to this basis?

This is a problem we already know how to do. Ie, we’re just solving the vector equation \( w = x_1v_1 + x_2v_2 + x_3v_3 \). Another way to do this involves taking advantage of the properties of orthogonal sets.

Since \( v_1, v_2, v_3 \) form a basis for \( \mathbb{R}^3 \), we know that there must exist \( c_1, c_2, c_3 \) such that
\[
w = c_1v_1 + c_2v_2 + c_3v_3
\]

So now, let’s start by trying to figure out what \( c_1 \) is. Now, consider dotting both sides by \( v_1 \). Then, we get
\[
v_1 \cdot w = v_1 \cdot (c_1v_1 + c_2v_2 + c_3v_3)
\]
\[
= v_1 \cdot (c_1v_1) + v_2 \cdot (c_2v_2) + v_3 \cdot (c_3v_3)
\]
\[
= c_1(v_1 \cdot v_1) + c_2(v_1 \cdot v_2) + c_3(v_1 \cdot v_3)
\]
\[
= c_1\|v_1\|^2
\]

Thus, we see that
\[
c_1 = \frac{v_1 \cdot w}{\|v_1\|^2} = \left( \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) / \left( \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right) = 5/11
\]

Repeating this process with \( v_2, v_3 \), we see that
\[
c_2 = \frac{v_2 \cdot w}{\|v_2\|^2} \quad \text{and} \quad c_3 = \frac{v_3 \cdot w}{\|v_3\|^2}
\]
Theorem 197. Let \( \{u_1, \ldots, u_p\} \) be an orthogonal basis for a subspace \( W \) of \( \mathbb{R}^n \), then for any \( w \in W \), we have
\[
w = c_1 u_1 + \cdots + c_p u_p
\]
where \( c_i = \frac{w \cdot u_i}{\|u_i\|^2} \).

Proof. Since \( \{u_1, \ldots, u_p\} \) span \( W \), these coefficients must exist. Thus, dotting by \( u_i \), we get:
\[
w \cdot u_i = (c_1 u_1 + \cdots + c_p u_p) \cdot u_i = c_i (u_i \cdot u_i)
\]
so \( c_i = \frac{w \cdot u_i}{\|u_i\|^2} \).

This theorem is a special case of a more general idea.

Example 198. Suppose \( W \) is a 1-dimensional subspace of \( \mathbb{R}^n \), generated by \( w \) (ie, \( W = \text{Span} \{w\} \)), and \( y \in \mathbb{R}^n \).

(a) How could we break \( y \) down into two pieces, one that lies in \( W \), and one that is orthogonal to \( W \)?

Ie, we want to write \( y = \hat{y} + z \), where \( \hat{y} \in W \) and \( z \in W^\perp \).

Since \( z = y - \hat{y} \), the question is, for what \( \hat{y} \in W \) will \( y - \hat{y} \in W^\perp \)?

Well, since \( y - \hat{y} \in W^\perp \) if and only if \( y - \hat{y} \) is orthogonal to \( w \), so this is equivalent to solving the equation
\[
w \cdot (y - \hat{y}) = 0
\]
or equivalently, since \( \hat{y} \in W \), we can write \( \hat{y} = \alpha w \), so we’re really asking, “for what \( \alpha \) is \( w \cdot (y - \alpha w) = 0 \)?”

Thus, it must be that \( \alpha = \frac{w \cdot y}{\|w\|^2} \).

(b) Suppose \( W = \text{Span} \{(1,2)\} \subseteq \mathbb{R}^2 \). Let \( y = (2,1) \). Decompose \( y = \hat{y} + z \), where \( \hat{y} \in W \) and \( z \in W^\perp \).

Well, as usual, we want to find \( \hat{y} \in W \) such that \( y - \hat{y} \in W^\perp \). In other words, we want to find \( \alpha \in \mathbb{R} \) such that \( y - \alpha (1,2) \in W^\perp \), ie we want to find \( \alpha \) such that
\[
\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \cdot \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = 0
\]

Thus, as usual, solving for \( \alpha \), we get:
\[
0 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \alpha \left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\|^2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 5\alpha = 3 - 5\alpha
\]
so \( \alpha = 3/5 \).

Alternatively, directly applying the formula from (a), we get
\[
\alpha = \frac{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\|^2} = \frac{3}{5}
\]

From this, we see that \( y = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{3}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{3}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \).
Interlude on Orthonormal Sets

**Definition 199.** A set of vectors \( \{u_1, \ldots, u_p\} \) is an orthonormal set if it is an orthogonal set of unit vectors. An orthonormal basis for a vector space \( V \) is an orthonormal set that is also a basis for \( V \).

**Example 200.** Consider the set

\[
\begin{align*}
v_1 &= \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, & v_2 &= \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, & v_3 &= \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}
\end{align*}
\]

(a) Is it an orthonormal set?

You can compute that \( v_1 \cdot v_2 = v_1 \cdot v_3 = v_2 \cdot v_3 = 0 \). On the other hand, they are not all unit vectors (in fact, none of them are). Thus, they are not orthonormal.

(b) How could you modify \( v_1, v_2, v_3 \) to form an orthonormal set?

Note that scaling orthogonal sets preserves the property of orthogonality. Thus, we simply need to scale each of the vectors by their norms, ie, the set

\[
\left\{ \frac{1}{\|v_1\|} v_1, \frac{1}{\|v_2\|} v_2, \frac{1}{\|v_3\|} v_3 \right\}
\]

is orthonormal.

**Theorem 201.** An \( m \times n \) matrix \( U \) has orthonormal columns if and only if \( U^T U = I \).

**Proof.** For simplicity, suppose \( n = 3 \), then let \( U = [u_1 \ u_2 \ u_3] \). Consider the product

\[
U^T U = \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix} [u_1 \ u_2 \ u_3]
\]

**Corollary 202.** An \( m \times n \) matrix \( U \) has orthonormal rows if and only if (ask class) \( U U^T = I \).

**Example 203.**

(a) Suppose \( U \) is 5 \( \times \) 4 and has orthonormal columns. Can it have orthonormal rows? Must it?

No, because then it would be invertible. Also, if it did, then \( U^T \) would be a 4 \( \times \) 5 matrix with orthonormal(LI) columns.

(b) Suppose \( U \) is 5 \( \times \) 5. Suppose it has orthonormal columns. Must it have orthonormal rows?

Yes. Since it has orthonormal columns, its columns are linearly independent, and hence since it’s square, it must be invertible. On the other hand, since \( U^T U = I \), its inverse is exactly \( U^T \), so we also know that \( UU^T = I \), ie \( U \) has orthonormal rows.

**Theorem 204.** Let \( U \) be an \( m \times n \) matrix with orthonormal columns, and let \( x \) and \( y \) be in \( \mathbb{R}^n \). Then, \( (Ux) \cdot (Uy) = x \cdot y \).
Proof. Let \( x_i, y_i \) be the coordinates of \( x, y \), and \( u_i \) be the columns of \( U \), then
\[
(Ux) \cdot (Uy) = \left( \sum x_i u_i \right) \cdot \left( \sum y_i u_i \right) = \sum x_i y_i \| u_i \|^2 = \sum x_i y_i = x \cdot y
\]
\[\square\]

Corollary 205. Let \( U \) be an \( m \times n \) matrix with orthonormal columns, and let \( x, y \in \mathbb{R}^n \). Then,
(a) \( \| Ux \| = \| x \| \)
(b) \( (Ux) \cdot (Uy) = 0 \) if and only if \( x \cdot y = 0 \)

Proof. For (a), recall that \( \| Ux \| = (Ux) \cdot (Ux) \), but by the previous theorem, we have
\[
(Ux) \cdot (Ux) = x \cdot x = \| x \|
\]
This proves (a). Part (b) also follows immediately from the previous theorem.
\[\square\]

Orthogonal Projections

Last time, we considered problems like this:

Example 206. Suppose \( W = \text{Span} \{ (1, 0, 0) \} \subseteq \mathbb{R}^3 \). Let \( y = (1, 1, 1) \). Decompose \( y = \hat{y} + z \), where \( \hat{y} \in W \) and \( z \in W^\perp \).

In this case, we want to find \( \alpha \) such that
\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\end{bmatrix}

\begin{bmatrix}
\begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\end{bmatrix}
- \alpha
\begin{bmatrix}
\begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\end{bmatrix}

\begin{bmatrix}
\begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\end{bmatrix}

\begin{bmatrix}
\begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\end{bmatrix}

= 0
\]

ie,
\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
- \alpha
\begin{bmatrix}
\begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\end{bmatrix}

\begin{bmatrix}
\begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\end{bmatrix}

\begin{bmatrix}
\begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\end{bmatrix}

\begin{bmatrix}
\begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\end{bmatrix}

= 0
\]

simplifying... \[ 1 - \alpha = 0 \]

So, \( \alpha = 1 \). Indeed, we can write
\[
y = \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} + \left( \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix} - \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \right) = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
\]

Here, we say that the vector \( \hat{y} \) is called the orthogonal projection of \( y \) onto the 1-dimensional subspace \( W \), and is denoted \( \text{proj}_W y \).

Definition 207. Given a subspace \( W \subseteq \mathbb{R}^n \) and a vector \( y \in \mathbb{R}^n \), the orthogonal projection of \( y \) onto \( W \) is denoted \( \text{proj}_W y \), and is the unique vector \( \hat{y} \in W \) such that \( y - \hat{y} \in W^\perp \).

What would we do if instead \( W \) was 2-dimensional?

Example 208. Suppose \( W = \text{Span} \{ (1, 0, 0), (1, 1, 0) \} \subseteq \mathbb{R}^3 \). Let \( y = (2, 2, 2) \). Find \( \text{proj}_W y \).

As per the definition, we’re trying to find a vector \( \hat{y} \in W \) such that \( y - \hat{y} \in W^\perp \). Let
\[
v_1 = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad v_2 = \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix}
\]
So we see that $W = \text{Span} \{v_1, v_2\}$. Now, everything in $W$ can be written as $a_1v_1 + a_2v_2$, so we’re really just trying to find the coefficients $a_1, a_2$ such that

$$
\begin{bmatrix}
  1 & 2 \\
  1 & 2 \\
  1 & 2 \\
\end{bmatrix}
\begin{bmatrix}
  y \\
  v_1 \\
  v_2 \\
\end{bmatrix}
\in W^\perp
$$

Once we have $a_1, a_2$, we can declare $\hat{y} = a_1v_1 + a_2v_2$, and say $z = y - \hat{y}$. By thm188 since $W$ is spanned by $v_1, v_2$, it suffices to find $a_1, a_2$ such that:

$$
\begin{pmatrix}
  2 \\
  2 \\
  2 \\
\end{pmatrix}
\begin{bmatrix}
  a_1 \\
  0 \\
  a_2 \\
\end{bmatrix}
\begin{pmatrix}
  1 \\
  0 \\
  1 \\
\end{pmatrix}
= 0
$$

and

$$
\begin{pmatrix}
  2 \\
  2 \\
  2 \\
\end{pmatrix}
\begin{bmatrix}
  a_1 \\
  0 \\
  a_2 \\
\end{bmatrix}
\begin{pmatrix}
  1 \\
  0 \\
  1 \\
\end{pmatrix}
= 0
$$

Now, if you simplify the above two equations as much as possible, you can see that each equation is in fact a linear equation in the variables $a_1, a_2$, and so solving the two equations above is really just the same as solving a linear system.

This method will in general work. However, we’re going to consider another method.

**Example 209.** Consider the problem in the previous example. Note that $\{(1, 0, 0), (0, 1, 0)\}$ = $\{e_1, e_2\}$ is an orthogonal basis for $W$. Now, suppose there exists $\hat{y} \in W$ such that $y - \hat{y} \in W^\perp$, then we can write $\hat{y} = b_1e_1 + b_2e_2$. Thus, figuring out what $\hat{y}$ is is the same as determining $b_1, b_2$. So, what are $b_1, b_2$?

Let’s start with $b_1$. To find $b_1$, we’ll first ask, what is $(y - \hat{y}) \cdot e_1$?

By assumption, since $(y - \hat{y}) \in W^\perp$, it must be the case that $(y - \hat{y}) \cdot e_1 = 0$. Thus, we have

$$
0 = (y - \hat{y}) \cdot e_1
= y \cdot e_1 - \hat{y} \cdot e_1
= y \cdot e_1 - (b_1e_1 + b_2e_2) \cdot e_1
= y \cdot e_1 - b_1e_1 \cdot e_1 - b_2e_2 \cdot e_1
= y \cdot e_1 - b_1\|e_1\|^2 - b_2(e_2 \cdot e_1)
$$

however, since $\{e_1, e_2\}$ form an orthogonal basis for $W$, we see that $e_2 \cdot e_1 = 0$, so:

$$
0 = \cdots = y \cdot e_1 - b_1\|e_1\|^2
$$

Thus, doing this for $b_2$ as well will show that

$$
b_1 = \frac{y \cdot e_1}{\|e_1\|^2}, \quad b_2 = \frac{y \cdot e_2}{\|e_2\|^2}
$$

In fact, you can verify that

$$
y - \frac{y \cdot e_1}{\|e_1\|^2}e_1 - \frac{y \cdot e_2}{\|e_2\|^2}e_2 \in W^\perp
$$

Note that here, we used the crucial fact that $\{e_1, e_2\}$ formed an orthogonal basis for $W$. In fact, in general, we have:
Theorem 210. Suppose \( W \subseteq \mathbb{R}^n \) is a subspace, and \( \{u_1, \ldots, u_p\} \) is an orthogonal basis for \( W \). Then for any \( y \in \mathbb{R}^n \), we can find \( \hat{y}, z \) such that \( \hat{y} \in W, z \in W^\perp \), and \( y = \hat{y} + z \). Here, by definition \( \hat{y} = \text{proj}_W y \). In fact, we have the following formula for \( \hat{y} = \text{proj}_W y \):

\[
\hat{y} = \text{proj}_W y = \frac{y \cdot u_1}{\|u_1\|^2} u_1 + \cdots + \frac{y \cdot u_p}{\|u_p\|^2} u_p
\]

With notation as in the theorem above, let \( W_i = \text{Span} \{u_i\} \). Note that these vectors \( \frac{y \cdot u_i}{\|u_i\|^2} u_i \) are in fact the vectors \( \text{proj}_{W_i} y \). Essentially, the way we construct \( \hat{y} \), is by projecting \( y \) onto each “one-dimensional axis” of \( W \), and then adding all the components together.

Note that the above theorem is only useful if we can find an orthogonal basis for \( W \). We’ll discuss next week how to find orthogonal bases.

Example 211. Let \( W \subseteq \mathbb{R}^n \) be a subspace, and \( y \in W \). What is \( \text{proj}_W y \)?

Recall, we’re just trying to find the vector \( \hat{y} \in W \) such that \( y - \hat{y} \in W^\perp \). However, since \( y \in W \), we can pick \( y = \hat{y} \), in which case we have \( y - \hat{y} = 0 \), which is trivially in \( W^\perp \).

Thus, we see that if \( y \in W \), we have \( \text{proj}_W y = y \).

Theorem 212. (Best Approximation Theorem) Let \( W \) be a subspace of \( \mathbb{R}^n \), and let \( y \in \mathbb{R}^n \). Let \( \hat{y} = \text{proj}_W y \), then for all \( v \in W \) with \( v \neq \hat{y} \), we have

\[
\|y - \hat{y}\| < \|y - v\|
\]

Proof. Clearly \( \hat{y} - v \in W \), and by definition of orthogonal projection, \( y - \hat{y} \in W^\perp \) and hence is orthogonal to \( \hat{y} - v \). Thus, by the pythagorean theorem (if \( x \cdot y = 0 \), then \( \|x + y\|^2 = \|x\|^2 + \|y\|^2 = \|x - y\|^2 \)), we see that.

\[
\|y - v\|^2 = \|(y - \hat{y}) + (\hat{y} - v)\|^2 = \|y - \hat{y}\|^2 + \|\hat{y} - v\|^2
\]

Thus \( \|y - v\|^2 - \|y - \hat{y}\|^2 = \|\hat{y} - v\|^2 > 0 \), which shows that \( \|y - v\| > \|y - \hat{y}\| \).

Note that this ensures that \( \text{proj}_W y \) is in fact unique and well-defined.

Lastly, we have this following fact. The proof is in the book (thm 10, page 351), but it’s really a special case that isn’t very interesting imo.

Theorem 213. If \( \{u_1, \ldots, u_p\} \) is an orthonormal basis for a subspace \( W \) of \( \mathbb{R}^n \), then

\[
\text{proj}_W y = (y \cdot u_1)u_1 + (y \cdot u_2)u_2 + \cdots + (y \cdot u_p)u_p
\]

If \( U = [u_1 \ u_2 \ \cdots \ u_p] \), then

\[
\text{proj}_W y = UU^T y
\]

for all \( y \in \mathbb{R}^n \).
23. Lecture 23. Gram-Schmidt (Section 6.4)

Example 214. We’ll develop this procedure inductively.

(a) We’ll start with the base case: Suppose $W$ is a 1-dimensional subspace of $\mathbb{R}^n$, say $W = \text{Span} \{x_1\}$. What is an orthogonal basis for $W$?

In this case, $\{x_1\}$ is itself an orthogonal basis for $W$! Note that since $W$ is 1-dimensional, $x_1 \neq 0$, so $\{x_1\}$ is a linearly independent (and orthogonal) set, that in this case spans $W$, hence it’s an orthogonal basis for $W$.

(b) Now suppose $W$ is a $k$-dimensional space, and $v_1, \ldots, v_k$ is an orthogonal basis for $W$. Further suppose $V$ is a $(k + 1)$-dimensional space that contains $W$, and suppose $v$ is a vector such that $v \in V$ but $v \not\in W$.

How would you find an orthogonal basis for $V$?

**Hint:** Note that we’ve already got an orthogonal basis for $W$, namely $\{v_1, \ldots, v_k\}$. How could we find a $v_{k+1}$ such that $\{v_1, \ldots, v_k, v_{k+1}\}$ is an orthogonal basis for $V$?

Indeed, we’re looking for a nonzero vector $v_{k+1} \in V$ such that $v_{k+1}$ is orthogonal to $v_1, \ldots, v_k$. How could we find such a vector?

**Hint:** How can you “extract” such a vector $v_{k+1}$ from the vector $v$?

We know we can write $v = \text{proj}_W v + z$, where $z \in W^\perp$. Since $\text{proj}_W v \in W$, and $v \not\in W$, it must be the case that $z \neq 0$. Thus, $z$ is a nonzero vector that is orthogonal to all of $W$, and in particular to $v_1, \ldots, v_k$. Thus, the set

$$\{v_1, \ldots, v_k, z\}$$

is an orthogonal set of nonzero vectors in $V$, and hence they are linearly independent. On the other hand, there are $k + 1$ vectors there, and $V$ is $(k + 1)$-dimensional, so they form an orthogonal basis of $V$ that extends the orthogonal basis for $W$.

The general process can be summarized as follows:

Algorithm 215. (Gram-Schmidt)

(a) Given one-dimensional space $W$, let $v_1 \in W$ be any nonzero vector. Then $\{v_1\}$ forms an orthogonal basis of $W$.

(b) Suppose $W$ is a $k$-dimensional space with orthogonal basis $\{v_1, \ldots, v_k\}$, and suppose $V \supseteq W$ is a $(k + 1)$-dimensional space containing $W$, then we can find a vector $v_{k+1} \in V$ such that $\{v_1, \ldots, v_k, v_{k+1}\}$ is an orthogonal basis for $V$. This is accomplished as follows:

(i) Since $V$ is strictly larger than $W$, it’s possible to find some vector $v \in V$ with $v \not\in W$. Find such a $v$.

(ii) Given such a $v$ as above, we know that we can write

$$v = \text{proj}_W v + z$$

where by definition $\text{proj}_W v \in W$ and $z \in W^\perp$. Since $\{v_1, \ldots, v_k\}$ is an orthogonal basis for $W$, we can apply thm210 to compute $\text{proj}_W v$ as follows:

$$\text{proj}_W v = \frac{v \cdot v_1}{\|v_1\|^2} v_1 + \frac{v \cdot v_2}{\|v_2\|^2} v_2 + \cdots + \frac{v \cdot v_k}{\|v_k\|^2} v_k$$

Now we may compute $z = v - \text{proj}_W v \in W^\perp$. Since $v \not\in W$ and $\text{proj}_W v \in W$, it must be the case that $z \neq 0$. Thus, setting $v_{k+1} := z$, we see that

$$\{v_1, \ldots, v_k, v_{k+1}\}$$

indeed gives us an orthogonal set of $(k + 1)$ nonzero vectors in $V$. Since they are orthogonal and nonzero, the set is linearly independent, so since $\dim V = k + 1$, they form an orthogonal basis for $V$. 

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Consider the vectors (Example 216).

Now we’ll apply this method to a few explicit examples.

**Example 216.** Consider the vectors

\[
x_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}
\]

(a) Is \( \{x_1, x_2, x_3\} \) a basis for \( \mathbb{R}^3 \)?

Yes. The matrix \( [x_1 \ x_2 \ x_3] \) is a \( 3 \times 3 \) matrix with 3 pivots, so its columns are a basis for \( \mathbb{R}^3 \).

(b) Find an orthogonal basis for \( W_1 = \text{Span} \{x_1\} \).

Let \( v_1 := x_1 \), then previous example shows that

\[
\{v_1\} = \{x_1\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}
\]

is itself an orthogonal basis for \( W_1 \).

(c) Find an orthogonal basis for \( W_2 = \text{Span} \{x_1, x_2\} \).

Now we’re in case (b) of the previous example. Here, \( W_1 \) is our \( k \)-dimensional space (in this case \( k = 1 \)), with orthogonal basis \( \{v_1\} \).

In this case, since \( \{x_1, x_2\} \) are linearly independent, we see that \( x_2 \notin W_1 = \text{Span} \{v_1\} \), and hence \( x_2 \) plays the role of the vector \( v \) in case the previous example. Thus, we want to write \( x_2 = \text{proj}_{W_1} x_2 + z \).

To do this, we must first compute \( \text{proj}_{W_1} x_2 \). To do this, we refer to thm 210. The theorem tells us that since \( \{v_1\} \) is an orthogonal basis for \( W_1 \),

\[
\text{proj}_{W_1} x_2 = \frac{x_2 \cdot v_1}{\|v_1\|^2} v_1 = \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}
\]

Thus, solving for \( z \), we get:

\[
z = x_2 - \text{proj}_{W_1} x_2 = x_2 - \frac{x_2 \cdot v_1}{\|v_1\|^2} v_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 7/3 \\ 8/3 \end{bmatrix}
\]

We know from the definition of \( \text{proj}_{W_1} x_2 \) that \( z \) is orthogonal to \( v_1 \), and hence setting

\[
v_2 = z = \begin{bmatrix} 1/3 \\ 7/3 \\ 8/3 \end{bmatrix},
\]

we see that \( \{v_1, v_2\} \) is now an orthogonal basis for \( W_2 \).

(d) Find an orthogonal basis for \( W_3 = \text{Span} \{x_1, x_2, x_3\} = \mathbb{R}^3 \).

Again we’re in case (b) of the previous example, where \( k = 2 \). Here, \( W_2 = \text{Span} \{x_1, x_2\} \) is our \( k \)-dimensional space, with orthogonal basis \( \{v_1, v_2\} \).

Now, again since \( \{x_1, x_2, x_3\} \) are linearly independent, \( x_3 \notin W_2 \), and hence we want to extract the perpendicular component from it to form \( v_3 \). To do this, again we use thm 210 to calculate

\[
\text{proj}_{W_2} x_3 = \frac{x_3 \cdot v_1}{\|v_1\|^2} v_1 + \frac{x_3 \cdot v_2}{\|v_2\|^2} v_2 = \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{3}{38/3} \begin{bmatrix} 1/3 \\ 7/3 \\ 8/3 \end{bmatrix} = \frac{9}{38} \begin{bmatrix} 1/3 \\ 7/3 \\ 8/3 \end{bmatrix}
\]

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Thus, by the definition of \( \text{proj}_{W_2} x_3 \), we know that the vector

\[
z = x_3 - \text{proj}_{W_2} x_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \left( \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{9}{38} \begin{bmatrix} 1/3 \\ 7/3 \\ 8/3 \end{bmatrix} \right)
\]

is orthogonal to \( W_2 \), and hence in particular it is orthogonal to \( v_1, v_2 \). Furthermore, \( z \neq 0 \), so if we set

\[
v_3 = z = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \left( \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{9}{38} \begin{bmatrix} 1/3 \\ 7/3 \\ 8/3 \end{bmatrix} \right)
\]

we see that \( \{v_1, v_2, v_3\} \) is an orthogonal set of 3 nonzero vectors in the 3-dimensional space \( W_3 = \mathbb{R}^3 \), and hence they form an orthogonal basis for \( W_3 \).

**Corollary 217.** In general, given a basis \( \{x_1, \ldots, x_p\} \) for a vector space \( W \), define

\[
v_1 = x_1 \\
v_2 = x_2 - \frac{x_2 \cdot v_1}{\|v_1\|^2} v_1 \\
v_3 = x_3 - \frac{x_3 \cdot v_1}{\|v_1\|^2} v_1 - \frac{x_3 \cdot v_2}{\|v_2\|^2} v_2 \\
\vdots \\
v_p = x_p - \frac{x_p \cdot v_1}{\|v_1\|^2} v_1 - \frac{x_p \cdot v_2}{\|v_2\|^2} v_2 - \cdots - \frac{x_p \cdot v_{p-1}}{\|v_{p-1}\|^2} v_{p-1}
\]

then \( \{v_1, \ldots, v_p\} \) is an orthogonal basis for \( W \).

**Colors**

So far, we’ve only thought of vector spaces in two contexts - either as solution sets of linear systems, or geometric “spaces”, like the line, the plane, or 3-space. In fact, the kinds of vector spaces that appear in science and mathematics form a far more varied lot than the paltry few we’ve considered. Today, we’ll consider one of these weird/crazy/awesome vector spaces.

We know that there are 3 primary colors. What does that mean? It means, that by combining suitable quantities of each of the three primary colors, we can achieve any color that we want. Indeed, this is beginning to sound exactly like the definition of what it means for a vector space to be 3-dimensional. In this sense, then, the space of colors is 3-dimensional. Indeed, you can scale colors (in the sense that you can increase/decrease the intensity of the colors), and you can add colors (by combining them), so it makes sense to call the space of colors, under the two operations of scaling and combining (read: multiplication by scalars and addition), a vector space!

Since it’s 3-dimensional, as it turns out it’s isomorphic to \( \mathbb{R}^3 \), in the following sense:

If we view the vector

\[
\begin{bmatrix} r \\ g \\ b \end{bmatrix} \in \mathbb{R}^3
\]

as representing the color obtained by combining \( r \)-amounts of red, \( g \)-amounts of green, and \( b \)-amounts of blue, where \( r, g, b \in \mathbb{R} \), we can view points in \( \mathbb{R}^3 \) as colors!

On the other hand, what are colors really? Well, fundamentally speaking, a “color” is just a form of visible light, and in fact, if we want to distinguish, say, light red from dark red, we should really identify color with all the data associated to a light wave (not just its frequency).
Now as you’ve probably learned in any intro physics/engineering class, when two waves coincide, we say that they are in “superposition”, and the equation that defines the superposition is just the sum of the two equations for each individual wave.

In this sense, we can think of color as being identified with all the possible wave patterns that you can imagine (even jagged/square waves!). Thus, the operation of combining two colors can be thought of as adding the equations of the two waves that each represents, and scaling a color can be thought of as simply scaling the wave (ie, multiplying the equation by some real number).

Now let’s think. Is the space of all possible waves, 3-dimensional? No! In fact, it’s wildly infinite dimensional (in fact, its dimensionality is what mathematicians call uncountable), and yet, there still exist 3 primary colors!

Why is this?

As it turns out, while the dimensionality of the space of all waves is way way infinite, we humans can’t distinguish between all the different ways in which light waves may differ. In fact, the data that our brain receives comes in three flavors: “red”, “green”, and “blue”. Indeed, the retinas in our eyes have three types of receptors, each of which can either send either a strong or weak signal to our brains at any time. The three types are:

(a) S-cones: these react most strongly to light of wavelengths near 420nm (blue)
(b) M-cones: these react most strongly to light of wavelengths near 534nm (green)
(c) L-cones: these react most strongly to light of wavelengths near 564nm (red)

Thus, at any instant in time, when focused on any one object, our brains are receiving varying responses from each of the three cone-types, which together form a vector in 3 coordinates, which our brains then interpret as color.

**Example 218.** Anyhow, this went a little off topic. Now, given that to us, we can think of light as being 3-dimensional, it makes sense that in all modern TV’s and monitors, every pixel is capable of displaying red, green, and blue, all to different intensities. However, this was not always the case. In the early 1900’s, the first color movies were created, using an early version of technicolor, which for whatever reason could only depict shades of red and green (no blue).

Now, suppose we represent a certain shade of red, green, and blue as the vectors:

\[
R = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

respectively, and suppose we wanted to create the color

\[
C = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}
\]

which is presumably is something bluish-green with a hint of red. However, it being the early 1900’s, we can only use the two colors \(R,G\), so we want to approximate \(C\) as best as we can, using only \(R,G\). In other words, we want to find a vector \(C' \in \text{Span} \{R,G\}\) such that \(\|C - C'\|\) is minimum among all vectors in \(V := \text{Span} \{R,G\}\).

How do we do this?

Recall the Best Approximation Theorem (thm212), which said...

(Best Approximation Theorem) Let \(W\) be a subspace of \(\mathbb{R}^n\), and let \(y \in \mathbb{R}^n\). Let \(\hat{y} = \text{proj}_W y\), then for all \(v \in W\) with \(v \neq \hat{y}\), we have

\[
\|y - \hat{y}\| < \|y - v\|
\]
Indeed, this tells us that the closest vector in $\text{Span } \{R, G\}$ to $C$ is just $\text{proj}_V C$. In this case, this is easy to compute. Since $R, G$ are an orthogonal basis for $V$, we can use thm210 to compute $\text{proj}_V C$ as follows:

$$\text{proj}_V C = \frac{C \cdot R}{\|R\|^2} R + \frac{C \cdot G}{\|G\|^2} G = \frac{1}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{5}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$$

Not surprisingly, the closest vector in the “xy-plane” to $(1, 5, 5)$ is just $(1, 5, 0)$.

Of course, this was just a toy example. The full power in the best approximation theorem, and in fact in everything that we’ve learned so far, is their ability to generalize, build upon each other, and provide context and understanding for the specific cases we’re already familiar with, and in doing so shed light on the mysterious and unfamiliar territory where all the booty is buried.