Chapter 16 - Waves

Objectives (Ch 16)
The Basics of Waves
Energy of Waves
Interference of Waves
Standing Waves

“I’m surfing the giant life wave.”
-William Shatner

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PHYS 213
Objectives for Chapter 16

(a) Determine the wavelength and wave speed from time-and/or position-graphs of a wave or when provided similar information about the wave; in particular, relate frequency, wavelength and wave speed.

(b) Use the principle of linear superposition of waves to analyze the result of the interference of multiple waves.

(c) Determine the effect on wave speed and frequency of oscillation of a standing wave when changing one or more of these variables: length of the standing wave, mass/length of the medium, tension of the medium.
A wave is traveling along a rope to the right and is shown at a particular instant below. Two segments are labeled. Which of the following statements correctly describes the motion of the particles that compose the rope in these segments?

(a) Segment A: downward, segment B: upward.
(b) Segment A: upward, segment B: upward.
(c) Segment A: downward, segment B: downward.
(d) Segment A: upward, segment B: downward.
(e) Segment A: toward the left, segment B: toward the right.
There are three main types of waves in physics:

(a) Mechanical waves: described by Newton’s laws and propagate through matter, such as water, sound and seismic waves.
The Basics of Waves

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(c) Matter waves: described by quantum mechanics, these waves explain the wave nature of fundamental particles (electrons, protons, etc).
The Basics of Waves

Mechanical waves come in two flavors:

transverse
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- **Transverse**
- **Longitudinal**
Mechanical waves come in two flavors:

transverse

longitudinal

The oscillations of matter are perpendicular or parallel to the motion of the wave.
All waves satisfy the so-called “wave equation.”

\[ \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \]
The Basics of Waves

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\[
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- \( y \) is the transverse or longitudinal displacement
- \( x \) is the direction of travel
- \( v \), a constant, is the speed of the wave
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We are looking for \( y(x, t) \).
Mechanical waves are described by a traveling sine wave.

\[ y(x, t) = y_m \sin(kx - \omega t) \]
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\[ y(x, t) = y_m \sin(kx - \omega t) \]

- \( y_m \): amplitude (maximum displacement)
- \( k = \frac{2\pi}{\lambda} \): wavenumber
- \( \lambda \): wavelength
- \( \omega = 2\pi f \): angular frequency
**Reminder:** frequency, period and angular frequency are all related:

\[
\omega = 2\pi f \\
f = \frac{1}{T} \\
\omega = \frac{2\pi}{T}
\]
The Basics of Waves

Reminder: frequency, period and angular frequency are all related:

\[ \omega = 2\pi f \]
\[ f = \frac{1}{T} \]
\[ \omega = \frac{2\pi}{T} \]

The wave number \( k = \frac{2\pi}{\lambda} \) is a “spatial frequency”:

\[ y(x, t = 0) = y_m \sin(kx + 0). \]
Each part of the rope is confined to its x position and travels up and down like a harmonic oscillator:

\[ y(x = 0, t) = y_m \sin(-\omega t) \]
The Basics of Waves

Each part of the rope is confined to its x position and travels up and down like a harmonic oscillator:

\[ y(x = 0, t) = y_m \sin(-\omega t) \]

The speed of this part of the rope is just

\[ u = \frac{dy(t)}{dt} = -\omega y_m \cos(\omega t) \]
We can find the **speed of the wave** from the argument of the sine function \( \sin(kx - \omega t) \), known as the **phase of the wave**.
The Basics of Waves

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\[
kx - \omega t = \text{constant}
\]

\[
\frac{d}{dt}(kx - \omega t) = 0
\]
The Basics of Waves

*We can find the speed of the wave from the argument of the sine function* \( \sin(kx - \omega t) \), known as the phase of the wave.

\[
\begin{align*}
\quad kx - \omega t &= \text{constant} \\
\frac{d}{dt}(kx - \omega t) &= 0 \\
\quad k \frac{dx}{dt} - \omega &= 0 \\
\quad \frac{dx}{dt} &= v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f
\end{align*}
\]
Alice and Bob are floating on a quiet river. At one point, they are 5.0 m apart when a speed boat passes. After the boat passes, they begin bobbing up and down at a frequency of 0.25 Hz. Just as Alice reaches her highest level, Bob is at his lowest level. As it happens, they are always within one wavelength. What is the speed of these waves?

(a) 1.3 m/s
(b) 2.5 m/s
(c) 3.8 m/s
(d) 5.0 m/s
(e) 7.5 m/s
We can derive the wave speed for a stretched rope with tension $\tau$ and mass density $\mu$ kg/m.
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\[
F = ma \\
2\tau \sin(\theta) = m \frac{v^2}{R}
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We can derive the wave speed for a stretched rope with tension $\tau$ and mass density $\mu \text{ kg/m}$. 

\[
\begin{align*}
F &= ma \\
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\tau(2\theta) &\approx \frac{\mu \Delta l v^2}{R}
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\tau \frac{\Delta l}{R} \approx \frac{\mu \Delta l v^2}{R} \\
v = \sqrt{\frac{\tau}{\mu}}
\]
Example 1: A wave traveling along a string is described by

\[ y(x, t) = 3.27 \sin(72.1x - 2.72t) \]

with 3.27 in mm, 72.1 in rad/m and 2.72 in rad/s.

(a) What is the amplitude?

(b) What are the wavelength, period and frequency?

(c) What is the wave velocity?

(d) What is the displacement of the string at 22.5 cm and 18.9 s?
Example 2: For the same wave,

\[ y(x, t) = 3.27 \sin(72.1x - 2.72t), \]

(a) what is the transverse velocity?

(b) what is the transverse acceleration?
Energy of Waves

Waves transfer energy in the direction of travel. The rate of energy transfer is the power.
Energy of Waves

Waves transfer energy in the direction of travel.
The rate of energy transfer is the power.

- The mass in region $b$ has $K$ but no $U$.
- The mass in region $a$ has $U$ but no $K$. 
Energy of Waves

\[ P_{\text{avg}} = \left( \frac{dK}{dt} \right)_{\text{avg}} + \left( \frac{dU}{dt} \right)_{\text{avg}} \]
Energy of Waves

\[ P_{avg} = \left( \frac{dK}{dt} \right)_{avg} + \left( \frac{dU}{dt} \right)_{avg} \]

\[ = 2 \left( \frac{dK}{dt} \right)_{avg} \]
Energy of Waves

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\[ = 2 \left( \frac{1}{2} (\mu dx) u^2 \right) \]

The wave generates power in proportion to its mass, velocity and the square of the frequency and amplitude.
Energy of Waves

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\[ = 2 \left( \frac{1}{2} (\mu \, dx) \, u^2 \right) \]

\[ = \left( \mu v \left[ -\omega y_m \cos(kx - \omega t) \right]^2 \right)_{avg} \]
Energy of Waves

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\[ = \mu v \omega^2 y_m^2 (\cos^2(kx - \omega t))_{avg} \]
Energy of Waves

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\[ = 2 \left( \frac{1}{2} (\mu \ dx) u^2 \right) \]

\[ = \mu \nu \omega^2 y_m^2 \left( \cos^2(kx - \omega t) \right)_{avg} \]

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Energy of Waves

\[ P_{\text{avg}} = \left( \frac{dK}{dt} \right)_{\text{avg}} + \left( \frac{dU}{dt} \right)_{\text{avg}} \]

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\[ = (\mu v [-\omega y_m \cos(kx - \omega t)]^2)_{\text{avg}} \]

\[ = \mu v \omega^2 y_m^2 (\cos^2(kx - \omega t))_{\text{avg}} \]

\[ P_{\text{avg}} = \frac{1}{2} \mu v \omega^2 y_m^2 \]

The wave generates power in proportion to its mass, velocity and the square of the frequency and amplitude.
Example 3: A string of linear density 525 g/m is under a tension of 45 N. If a sinusoidal wave of 120 Hz with amplitude 8.5 mm is sent down its length, how much power does this wave transmit?
When two waves on a string overlap, their displacements add algebraically resulting in interference:

\[ y'(x, t) = y_1(x, t) + y_2(x, t) \]
Interference of Waves

*When two waves on a string overlap, their displacements add algebraically resulting in interference:*

\[ y'(x, t) = y_1(x, t) + y_2(x, t) \]

Note: the waves emerge without alteration.
Interference of Waves

Special case: two waves of equal magnitude, wavelength and velocity travel along the same string.

\[ y'(x, t) = y_1(x, t) + y_2(x, t) \]
\[ = y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi) \]
Interference of Waves

*Special case: two waves of equal magnitude, wavelength and velocity travel along the same string.*

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\[ = y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi) \]
\[ = y_m \left[ \sin(kx - \omega t + \phi/2 - \phi/2) + \sin(kx - \omega t + \phi/2 + \phi/2) \right] \]
Interference of Waves

**Special case:** two waves of equal magnitude, wavelength and velocity travel along the same string.

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y'(x, t) = y_1(x, t) + y_2(x, t)
\]
\[
= y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi)
\]
\[
= y_m \left[ \sin(kx - \omega t + \phi/2 - \phi/2) + \sin(kx - \omega t + \phi/2 + \phi/2) \right]
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\]

\[
= y_m \left[ \sin(kx - \omega t + \phi/2 - \phi/2) + \sin(kx - \omega t + \phi/2 + \phi/2) \right]
\]

\[
= [2y_m \cos(\phi/2)] \sin(kx - \omega t + \phi/2)
\]

amplitude \quad oscillation
Interference of Waves

Two waves are traveling along a string. The left wave is traveling to the right at 0.5 cm/s and the right wave is traveling to the left at 2.0 cm/s. At what elapsed time will the two waves completely overlap and what will the maximum amplitude be at that time?

(a) 2.0 s, 1.5 cm
(b) 1.6 s, 2.5 cm
(c) 1.0 s, 1.5 cm
(d) 1.0 s, 2.5 cm
(e) 1.3 s, 0.0 cm
Interference of Waves

\[ y'(x, t) = [2y_m \cos(\phi/2)] \sin(kx - \omega t + \phi/2) \]
### Interference of Waves

<table>
<thead>
<tr>
<th>Phase Difference, in</th>
<th>Amplitude of Resultant Wave</th>
<th>Type of Interference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degrees</td>
<td>Radians</td>
<td>Wavelengths</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>120</td>
<td>$\frac{2}{3}\pi$</td>
<td>0.33</td>
</tr>
<tr>
<td>180</td>
<td>$\pi$</td>
<td>0.50</td>
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<tr>
<td>240</td>
<td>$\frac{4}{3}\pi$</td>
<td>0.67</td>
</tr>
<tr>
<td>360</td>
<td>$2\pi$</td>
<td>1.00</td>
</tr>
<tr>
<td>865</td>
<td>15.1</td>
<td>2.40</td>
</tr>
</tbody>
</table>

*The phase difference is between two otherwise identical waves, with amplitude $y_m$, moving in the same direction.*
Example 4: Two identical sinusoidal waves travel along the same direction on a stretched rope. The amplitude of each wave is 9.8 mm and the phase difference between them is 100°.

(a) What is the amplitude of the resultant wave due to interference?

(b) What phase difference is required to have an amplitude of 4.9 mm?
We can represent a wave with a **phasor**, a vector of length $y_m$ that rotates about the origin at a frequency of $\omega$. 

![Diagram of phasor representing wave motion](image.png)
Interference of Waves

We can represent a wave with a phasor, a vector of length $y_m$ that rotates about the origin at a frequency of $\omega$.

The vertical projection is the displacement of the wave at a particular point.
For two waves, you add the phasors to get the resulting displacement.
**Example 5:** Two waves $y_1(x, t)$ and $y_2(x, t)$ have the same wavelength and travel in the same direction along a string. Their amplitudes are $y_{m1} = 4.0 \text{ mm}$ and $y_{m2} = 3.0 \text{ mm}$ and their phase difference is $\pi/3$.

(a) Draw the phasor diagram for these two waves.

(b) Find the resulting amplitude and the phase constant.

(c) Write out the equation of the wave.
When two waves travel on the same string in opposite directions, the result is a standing wave.
Standing Waves

*When two waves travel on the same string in opposite directions, the result is a standing wave.*

Nodes are spots where the displacement is always zero, and anti-nodes are the spots of maximum amplitude.
Algebraically, a standing wave looks like:

\[ y'(x, t) = y_1(x, t) + y_2(x, t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t) \]
Standing Waves

Algebraically, a standing wave looks like:

\[ y'(x, t) = y_1(x, t) + y_2(x, t) \]

\[ = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t) \]

\[ = y_m \left[ \sin(kx) \cos(\omega t) - \cos(kx) \sin(\omega t) + \sin(kx) \cos(\omega t) + \cos(kx) \sin(\omega t) \right] \]
Algebraically, a standing wave looks like:

$$y'(x, t) = y_1(x, t) + y_2(x, t)$$

$$= y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t)$$

$$= y_m [\sin(kx) \cos(\omega t) - \cos(kx) \sin(\omega t) +$$

$$\sin(kx) \cos(\omega t) + \cos(kx) \sin(\omega t)]$$

$$= 2y_m \sin(kx) \cos(\omega t)$$

The amplitude term gives the nodes and anti-nodes.
Standing Waves

Algebraically, a standing wave looks like:

\[ y'(x, t) = y_1(x, t) + y_2(x, t) \]
\[ = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t) \]
\[ = y_m \left[ \sin(kx) \cos(\omega t) - \cos(kx) \sin(\omega t) + \right. \]
\[ \left. \sin(kx) \cos(\omega t) + \cos(kx) \sin(\omega t) \right] \]
\[ = 2y_m \sin(kx) \cos(\omega t) \]

The amplitude term gives the nodes and anti-nodes.
Standing Waves

A **node** is when the amplitude is always zero:

\[
\sin(kx) = 0
\]

\[
kx = n\pi \text{ for } n = 0, 1, 2, \ldots
\]

\[
x = n\pi/k = n\lambda/2
\]
A **node** is when the amplitude is always zero:

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kx = n\pi \text{ for } n = 0, 1, 2, \ldots \\
x = n\pi/k = n\lambda/2
\]

An **anti-node** is when the amplitude is maximum:

\[
\sin(kx) = 1 \\
kx = (n + 1/2)\pi \text{ for } n = 0, 1, 2, \ldots \\
x = (n + 1/2)\pi/k = (n + 1/2)\lambda/2
\]
When a rope reflects at a barrier, the result depends on the nature of the barrier.
When a rope oscillates with two fixed points, certain frequencies called harmonics result in standing waves with nodes and large anti-nodes.
Standing Waves

When a rope oscillates with two fixed points, certain frequencies called **harmonics** result in standing waves with nodes and large anti-nodes.

\[ \lambda = \frac{2L}{n} \text{ for } n = 1, 2, 3 \ldots \]

\[ f = \frac{v}{\lambda} = n \frac{v}{2L} \]
Standing Waves

When a rope oscillates with two fixed points, certain frequencies called **harmonics** result in standing waves with nodes and large anti-nodes.

\[
\lambda = \frac{2L}{n} \quad \text{for} \quad n = 1, 2, 3 \ldots
\]

\[
f = \frac{v}{\lambda} = n \frac{v}{2L}
\]

If we know the wave velocity (string tension and density), we can predict the harmonic frequencies.
Standing Waves
Example 6: In the figure below, the string has mass 2.5 g and length 0.80 m. If the tension force is 325 N,

(a) What is the wavelength of the standing wave?
(b) Which harmonic $n$ is this?
(c) What is the frequency of the wave?
(d) What is the maximum transverse velocity at $x = 0.180$ m?
(e) At what time is this velocity maximum?