Annuities, Sinking Funds, and Amortization
Math Analysis and Discrete Math – Sections 5.3 and 5.4

I. Warm-Up Problem

Previously, we have computed the future value of an investment when a fixed amount of money is deposited in an account that pays interest compounded periodically. Often, however, people do not deposit money and then sit back and watch it grow. Rather, money is invested in small amounts at periodic intervals.

Consider these problems:

1. Chrissy deposits $200 each year into a savings account that has an annual interest rate of 8% compounded annually. How much money will Chrissy have in her account after three years?

   *Hint:* Make up a table of how much she has in her account by year.

2. Ben saves $50 per month in a credit union that has an annual interest rate of 6% compounded monthly. How much money will Ben have in his account after he has made six deposits?
II. Generalization

Let's generalize the situation. Suppose we deposit \( P \) dollars each payment period for \( n \) payment periods at rate of interest \( i \) per payment period.

a. Consider the first deposit only.

During how many payment periods does interest get applied to this investment? ____________

Using the compound interest formula, how much is this part of the investment worth? Call this quantity \( A_1 \).

__________________________

b. Consider the second deposit only.

During how many payment periods does interest get applied to this investment? ____________

Using the compound interest formula, how much is this part of the investment worth? Call this quantity \( A_2 \).

__________________________

c. Generalize. Consider the \( k \)th deposit only.

During how many payment periods does interest get applied to this investment? ____________

Using the compound interest formula, how much is this part of the investment worth? Call this quantity \( A_1 \).

__________________________

d. So, in general, how much is the investment worth after \( n \) payment periods?

Note that there is a geometric series in the result from (d). We apply the geometric series formula and do some algebraic manipulation to come up with the formula below. For reference, the geometric series formula is

\[
\sum_{r=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}
\]

So, in summary…

**Definition:** An annuity is a sequence of equal periodic deposits. When the deposits are made at the same time the interest is credited, the annuity is termed ordinary. The **amount of the annuity** is the sum of all deposits made plus all interest accumulated.

<table>
<thead>
<tr>
<th>Amount of an Annuity</th>
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<tbody>
<tr>
<td>If ( P ) represents the deposit in dollars made at each payment period for an annuity at ( i ) percent interest per payment period, the amount ( A ) of the annuity after ( n ) payment periods is given by:</td>
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| \[
A = P \frac{(1 + i)^n - 1}{i}
\] |

**Definition:** A person with a debt may decide to accumulate sufficient funds to pay off the debt by agreeing to set aside enough money each month (or quarter or year) so that when the debt becomes payable, the money set aside each month plus the interest earned will equal the debt. The fund created by such a plan is called a **sinking fund**.
III. Practice Problems for Annuities and Sinking Funds

For Problems #1-3, find the amount of each annuity.

1. The deposit is $1000 monthly for 1 year at 5% compounded monthly.

2. The deposit is $500 quarterly for 2 years at 4% compounded quarterly.

3. The deposit is $1000 annually for 20 years at 6% compounded annually.

For Problems #4-5, find the payment required for each sinking fund.

4. The amount required is $50,000 after 10 years at 7% compounded semiannually. What is the semiannual payment?

5. The amount required is $2500 after 2 years at 5% compounded quarterly. What is the quarterly payment?

IV. Applications

Problem 1: Saving for a Car

Emily wants to invest an amount every month so that she will have $5,000 in 3 years to make a down payment on a new car. Her account pays 8% compounded monthly. How much should she deposit each month?
Problem 2: Saving for a House
In 6 years, you would like to have $20,000 for a down payment on a beach house. How much should you deposit each quarter into a savings account paying 3% interest compounded quarterly?

Problem 3: Paying off Bonds
The state has $5,000,000 worth of bonds that are due in 20 years. A sinking fund is established to pay off the debt. If the state can earn 10% annual interest compounded annually, what is the annual sinking fund deposit needed?

Problem 4: Managing a Condo
The Crown Colony Association is required by law to set aside funds to replace its roof. It is estimated that the roof will need to be replaced in 20 years at a cost of $180,000. The condo can invest in treasuries yielding 6% paid semiannually. If the condo invests in the treasuries, what semiannual payment is required to have the funds to replace the roof in 20 years?

***Problem 5: Time Needed for a Million Dollars
If Josh deposits $10,000 every year in an account paying 8% compounded annually, how long will it take him to accumulate $1,000,000?

Homework: 5.3: #1-25 every other odd, 35-36
V. Present Value of an Annuity

**Definition:** The present value of an annuity is the amount of money needed now so that if it is invested at $i$ percent, $n$ equal dollar amounts can be withdrawn without any money left over.

**Problem: College Tuition**
Jessica's parents will be paying her college tuition of $20,000 per year for 4 years. If they currently have the money invested at 6% compounded annually, how much money do they need to have in the account to pay the tuition? (Assume that "now" is the beginning of the year and the payment will be made at the end of the year.)

a. Using the compound interest formula, $A_n = P(1+i)^n$, find the amount needed now to make the first payment.

b. Using the compound interest formula, $A_n = P(1+i)^n$, find the amount needed now to make the second payment.

c. Using the compound interest formula, $A_n = P(1+i)^n$, find the amount needed now to make the third payment.

d. Using the compound interest formula, $A_n = P(1+i)^n$, find the amount needed now to make the fourth payment.

e. Sum the results of (a)-(d) to find the amount needed now to pay the college tuition.

The results can be generalized with the following formula:

<table>
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<tr>
<td>If $P$ represents the withdrawal per payment period, the present value $V$ of an annuity at a rate of interest $i$ per payment period for $n$ payment periods is given by</td>
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<tr>
<td>$V = P \frac{1 - (1+i)^{-n}}{i}$</td>
</tr>
</tbody>
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**Example**
Now use the formula for the present value of an annuity to find the amount needed now to pay the college tuition in the previous problem.

VI. Amortization

We say a loan with a fixed rate of interested is amortized if both principal and interest are paid by a sequence of equal payments made over equal periods of time. (The word comes from the Greek root "mort" meaning death; we are, in a sense, "killing" the loan by paying it off.)
When a loan of \( V \) dollars is amortized at a rate of interest \( i \) per payment period over \( n \) payment periods, the customary question is "What is the payment \( P \)?" In other words, in amortization problems, we want to find the amount of payment \( P \) that, after \( n \) payment periods at the rate of interest \( i \) per payment period, given us a present value equal to the amount of the loan.

Thus, to solve amortization problems, find \( P \) in the present value formula:

\[
V = P \frac{1 - (1 + i)^{-n}}{i}
\]

Here, the payment \( P \) represents the amount required to pay off a loan of \( V \) borrowed at a rate of interest \( i \) per payment period for \( n \) payment periods.

**Example: College Continues**
Jessica needs to attend college for a fifth year to finish the requirements for a double major. Since her parents did not anticipate this expense, they take out a loan for the $20,000 at 9\% compounded monthly.

a. What monthly payment is necessary for them to pay off the loan in 3 years?

b. What monthly payment is necessary for them to pay off the loan in 5 years?

c. What total amount is paid out for the 3-year loan?

d. What total amount is paid out for the 5-year loan?

**VII. Application Problems**

**Problem 1: College expenses**
Allison works during the summer to help with expenses at school the following year. She is able to save $800 per month for 3 months, and she invests at 4\% compounded monthly.

a. How much has Allison saved after 3 months?

b. At the end of each month, Allison will begin to withdraw equal amounts each month for the next 8 months. What is the most Allison can withdraw each month?
Problem 2: Car payments
James obtains a loan for his brand new car. His car costs $18,000 and he puts $1000 down and amortizes the rest with equal monthly payments over a 5-year period at 6% to be compounded monthly.
   a. What will the monthly payment be?
   
   b. How much interest will be paid?

Problem 3: Mortgage Payments
Suppose your parents are interested in buying a house that will cost $180,000. They will use the equity in their present house to make a 20% down payment on the new house and finance the rest with a 30-year mortgage at an interest rate of 6.5% compounded monthly.
   a. What will the monthly payment be?
   
   b. How much interest will be paid?

Homework: 5.4: #1-5 odd, 9, 15