Robust Control of Linear Quantum Systems

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Introduction

- Recent developments in quantum and nano technology have provided a great impetus for research in quantum feedback control systems.

- In particular, it is being realized that robustness is critical in quantum feedback control systems, as it is in non-quantum feedback control systems.

- We present an overview of some recent results in the area of robust feedback control of linear quantum systems.

- We consider a class of quantum systems described by linear Heisenberg dynamics driven by quantum Gaussian noise processes, and controlled by a linear feedback controller which is also a quantum system.
The most common area in which such systems arise is in the area of quantum optics; e.g.,

Photo courtesy of Elanor Huntington (UNSW@ADFA)
Analysis of Linear Quantum Systems

In the analysis of linear quantum systems, we consider noncommutative stochastic models of the form

\[
\begin{align*}
    dx(t) &= Ax(t)dt + [ B \ G \ ] [ dw(t)^T \ dv(t)^T ]^T; \\
    dz(t) &= Cx(t)dt + [ D \ H \ ] [ dw(t)^T \ dv(t)^T ]^T.
\end{align*}
\]

(1)

where \( x(t) = [ x_1(t) \ldots x_n(t) ]^T \) is a vector of self-adjoint possibly non-commutative system variables. Also, \( A, B, G, C, D \) are real matrices.

The vector quantity \( w \) describes the input signals and is assumed to admit the decomposition

\[
    dw(t) = \beta_w(t)dt + d\tilde{w}(t)
\]

(2)

where \( \tilde{w}(t) \) is the noise part of \( w(t) \) and \( \beta_w(t) \) is a self-adjoint process.
In this quantum system, the input channel has two components, $dw = \beta_w dt + d\tilde{w}$ which represents disturbance signals, and $dv$, which represents additional noise sources.

The process $\beta_w(t)$ serves to represent variables of other systems which may be passed to the system.

This system may represent the closed loop quantum classical system in a quantum $H^\infty$ control system.
Definition. The above quantum stochastic system is said to be Strictly Bounded Real with disturbance attenuation $g$ if there exists a positive operator valued quadratic form $V(x) = x^T X x$ (where $X$ is a real positive definite symmetric matrix) and constants $\lambda > 0$, $\epsilon > 0$ such that

\[
\langle V(x(t)) \rangle + \int_0^t \langle \beta_z^T \beta_z - (g^2 - \epsilon)\beta_w^T \beta_w + \epsilon x^T x \rangle ds \\
\leq \langle V(x(0)) \rangle + \lambda t \quad \forall t > 0,
\]

for all Gaussian states $\rho$ for the initial variables $x(0)$.

Here we use the shorthand notation $\langle \cdot \rangle$ for quantum expectation over all initial variables and noises.
Theorem. [Quantum Strict Bounded Real Lemma]

The following statements are equivalent

(i) The above quantum stochastic system is strictly bounded real with disturbance attenuation \( g \).

(ii) \( A \) is a stable matrix and \( \| C(sI - A)^{-1}B + D \|_\infty < g \).

(iii) \( g^2 I - D^T D > 0 \) and there exists a positive definite matrix \( \tilde{X} > 0 \):

\[
A^T \tilde{X} + \tilde{X} A + C^T C + (\tilde{X} B + C^T D) \times (g^2 I - D^T D)^{-1}(B^T \tilde{X} + D^T C) < 0.
\]

(iv) \( g^2 I - D^T D > 0 \) and the algebraic Riccati equation

\[
A^T X + X A + C^T C + (X B + C^T D) \times (g^2 I - D^T D)^{-1}(B^T X + D^T C) = 0
\]

has a stabilizing solution \( X \geq 0 \).

Furthermore, if these statements hold then \( X < \tilde{X} \).
$H^\infty$ of linear quantum systems

- In the quantum $H^\infty$ control problem we consider a \textit{plant} which is described by noncommutative stochastic models of the form

$$
\begin{align*}
    dx(t) &= Ax(t)dt + B_0 dv(t) + B_1 dw(t) + B_2 du(t); \quad x(0) = x; \\
    dz(t) &= C_1 x(t)dt + D_{12} du(t); \\
    dy(t) &= C_2 x(t)dt + D_{20} dv(t) + D_{21} dw(t)
\end{align*}
$$

- We assume that the “control” input is of the form

$$
    du(t) = \beta_u(t) dt + d\tilde{u}(t)
$$

where $\tilde{u}(t)$ is the noise part of $u(t)$. 

Assumptions

1. $D_{12}^T D_{12} = E_1 > 0$.

2. $D_{21} D_{21}^T = E_2 > 0$.

3. The matrix

$$
\begin{bmatrix}
A - j\omega I & B_2 \\
C_1 & D_{12}
\end{bmatrix}
$$

is full rank for all $\omega \geq 0$.

4. The matrix

$$
\begin{bmatrix}
A - j\omega I & B_1 \\
C_2 & D_{21}
\end{bmatrix}
$$

is full rank for all $\omega \geq 0$. 
Riccati Equations Our quantum $H^\infty$ result is stated in terms of the following pair of algebraic Riccati equations:

\[
(A - B_2 E_1^{-1} D_{12}^T C_1)^T X + X (A - B_2 E_1^{-1} D_{12}^T C_1) \\
+ X (B_1 B_1^T - g^2 B_2 E_1^{-1} B_2^T) X \\
+ g^{-2} C_1^T (I - D_{12} E_1^{-1} D_{12}^T) C_1 = 0;
\]

\[
(A - B_1 D_{21}^T E_2^{-1} C_2) Y + Y (A - B_1 D_{21}^T E_2^{-1} C_2) \\
+ Y (g^{-2} C_1^T C_1 - C_2^T E_2^{-1} C_2) Y \\
+ B_1 (I - D_{21}^T E_2^{-1} D_{21}) B_1^T = 0.
\]
**Controller** Our quantum $H^\infty$ control is defined by state space matrices constructed from the Riccati equations as follows:

\[
A_K = A + B_2 C_K - B_K C_2 + (B_1 - B_K D_{21}) B_1^T X; \\
B_K = (I - Y X)^{-1} (Y C_2^T + B_1 D_{21}^T) E_2^{-1}; \\
C_K = -E_1^{-1} (g^2 B_2^T X + D_{12}^T C_1).
\]
Theorem.

Necessity.
If there exists a controller such that the resulting closed loop system is strictly bounded real with disturbance attenuation $g$, then the Riccati equations will have stabilizing solutions $X \geq 0$ and $Y \geq 0$ such that the matrix $XY$ has a spectral radius strictly less than one.

Sufficiency.
Suppose the Riccati equations have stabilizing solutions $X \geq 0$ and $Y \geq 0$ such that the matrix $XY$ has a spectral radius strictly less than one. If the controller is such that the matrices $A_K, B_K, C_K$ are as defined in terms of the Riccati solutions as above, then the resulting closed loop system will be strictly bounded real with disturbance attenuation $g$. 
Physical Realizability of Linear Quantum Systems

- In the quantum $H^\infty$ control problem, the controller can be a classical system constructed from digital or analog electronics, a quantum system constructed from cavities, beamsplitters and phase shifters, or a combination of both.

- If the controller is to be a purely quantum system (coherent quantum feedback control), the question arises as to whether a given controller transfer function can be physically realized as a quantum system.
Example Quantum $H^\infty$ control system consisting of an optical cavity plant and an optical cavity controller:

Photo courtesy of Hideo Mabuchi, Stanford.
Physical Realizability of Quantum System Models

To consider the issue of whether a state space quantum system model (controller) is physically realizable, it is most convenient to consider complex quantum systems of the form

\[
da = F_0 \begin{bmatrix} a \\ a^* \end{bmatrix} \, dt + G_0 \begin{bmatrix} dv \\ dv^* \end{bmatrix},
\]

\[
\db = H_0 \begin{bmatrix} a \\ a^* \end{bmatrix} \, dt + K_0 \begin{bmatrix} dv \\ dv^* \end{bmatrix}.
\]

We assume without loss of generality that the number of inputs equals the number of outputs.
Also, we write
\[ F_0 = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \in \mathbb{C}^{n_a \times 2n_a}, \]
\[ G_0 = \begin{bmatrix} G_1 & G_2 \end{bmatrix} \in \mathbb{C}^{n_a \times 2n_v}, \]
\[ H_0 = \begin{bmatrix} H_1 & H_2 \end{bmatrix} \in \mathbb{C}^{n_v \times 2n_a}, \]
\[ K_0 = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \in \mathbb{C}^{n_v \times 2n_v}, \]

\[ F = \begin{bmatrix} F_1 & F_2 \\ F_2^* & F_1^* \end{bmatrix}, \quad G = \begin{bmatrix} G_1 & G_2 \\ G_2^* & G_1^* \end{bmatrix}, \]
\[ H = \begin{bmatrix} H_1 & H_2 \\ H_2^* & H_1^* \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & K_2 \\ K_2^* & K_1^* \end{bmatrix}. \]
This complex linear quantum system is equivalent to a real linear quantum system of the form considered previously via the substitutions:

\[ x_k = a_k + a_k^*, \quad x_{n\alpha + k} = -i(a_k - a_k^*). \]

\[ y_\ell = b_\ell + b_\ell^*, \quad y_{n_v + \ell} = -i(b_\ell - b_\ell^*), \quad 1 \leq \ell \leq n_v, \]

\[ w_j = v_j + v_j^*, \quad w_{n_v + j} = -i(v_j - v_j^*), \quad 1 \leq j \leq n_v. \]
This leads to the following system with self-adjoint variables and real coefficient matrices:

\[
\begin{align*}
\frac{dx}{dt} &= Ax + B dw, \\
\frac{dy}{dt} &= Cx + D dw,
\end{align*}
\]

where \(x = [x_1 \ x_2 \ \cdots \ x_{2n_a}]^\top\), \(y = [y_1 \ y_2 \ \cdots \ y_{2n_v}]^\top\),

\[
A = \frac{1}{2} \Phi F \Phi^\dagger, \quad B = \frac{1}{2} \Phi G \Phi^\dagger,
\]

\[
C = \frac{1}{2} \Phi H \Phi^\dagger, \quad D = \frac{1}{2} \Phi K \Phi^\dagger,
\]

\[
\Phi = \begin{bmatrix} I & I \\ -iI & iI \end{bmatrix}, \quad J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.
\]
Our notion of physical realizability is defined in terms of the preservation of certain quantum commutation relations and the equivalence of a quantum linear system to a quantum harmonic oscillator.

An alternative algebraic condition is given in the following theorem.

**Theorem.** The above complex linear quantum system is physically realizable if and only if there exists a hermitian matrix $\Omega = \Omega^\dagger$ such that

$$F\Omega + \Omega F^\dagger - GJG^\dagger = 0,$$

$$G = \Omega H^\dagger J,$$

$$K = I.$$
Our main result on physical realizability relates this property to the system theory notion of a \((J, J^\dagger)\)-unitary transfer function.

Consider the complex transfer function matrix corresponding to the above complex linear quantum system

\[
\Psi(s) = H(sI - F)^{-1}G + K.
\]

**Definition.** A system with transfer function matrix \(\Psi(s)\) is said to be is said to be dual \((J, J^\dagger)\)-unitary if

\[
\Psi(s)J\Psi^\dagger(-s^*) = J,
\]

for \(s \in \mathbb{C} : \text{Re}(s) \geq 0\).
Theorem. The above complex linear quantum system is physically realizable if and only if its transfer function matrix is dual $(J, J)$-unitary and $K = I$.

This theorem shows the equivalence between the quantum mechanical notion of physical realizability and the frequency domain systems theory notion of dual $(J, J)$-unitary.