Robust and Chance-Constrained Optimization under Polynomial Uncertainty

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Joint work with
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Robust Optimization

Find solution $x_{\text{RO}}$ of

$$(\text{RO}): \min_{x \in X} c^\top x$$

subject to $f(x, q) \leq 0$ for all $q \in Q$

- In this talk
  
  $f(x, q)$ convex in $x$ for fixed $q$
  polynomial in $q$ for fixed $x$

- This problem is in general very difficult – NP hard
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Solution Approaches

- Polytopic and Multilinear Dependence on Uncertainty $q$
  - Enough to Use Extreme Points of the Support Set of $q$

- LFT Dependence on Uncertainty $q$
  - Conditions for Norm Bounded $q$

- Polynomial Dependence on Uncertainty $q$
  - Sum of Squares (SOS) or moment-based approaches
    - idea: ‘lift’ the problem adding variables (monomials) and look for an SOS representation
    - yields to a hierarchy of convex LMI relaxations
    - guaranteed asymptotic convergence, but the dimension grows large
A Probabilistic Viewpoint
Probabilistic Robust Optimization

Assume that \( q \) is random with known pdf \( \mu(\cdot) \)

Probabilistic Optimization

Find solution \( x_{PO} \) of

\[
(PO): \min_{x \in X} c^T x \\
\text{subject to } \text{Prob}\{q \in Q : f(x, q) > 0\} \leq \epsilon
\]

- We require that the set of \( q \)'s that violate the constraints is small
- The above problem is a \textit{chance constrained problem} – even harder!
Based on random sampling of $q$

They lead to *polynomial-time* algorithms

They provide soft solutions:

$\Rightarrow$ The solution is guaranteed only in probability

Goal of this talk is to derive computable hard bounds
Randomized Algorithms
A Probabilistic Viewpoint

- Based on random sampling of $q$
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Randomized Algorithms
A Probabilistic Viewpoint

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  \( \Rightarrow \) The solution is guaranteed only in probability

Goal of this talk is to derive computable hard bounds
Preliminaries

Assumptions

Function \( f(\cdot) \)

i) For fixed \( q \in Q \subset \mathbb{R}^d \), the function \( f(\cdot, q) \) is **convex** in \( x \)

ii) For fixed \( x \in \mathcal{X} \), the function \( f(x, \cdot) \) is a **polynomial** of total degree \( \sigma \) in \( x \)

iii) The set \( \mathcal{X} \subset \mathbb{R}^{nx} \) is a bounded convex set

iv) The function \( f(\cdot, \cdot) \) is a bounded; i.e. \( f : \mathcal{X} \times Q \to [-1, \infty) \)

Random uncertainty \( q \)

i) \( q_i \)'s are independent
   - \( \mu(q) = \mu_1(q_1)\mu_2(q_2)\cdots\mu_d(q_d) \)
   - \( Q = Q_1 \times Q_2 \times \cdots \times Q_d \)

ii) The moments of \( \mu \) are finite
Preliminaries

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Function $f()$

i) For fixed $q \in Q \subset \mathbb{R}^d$, the function $f(\cdot, q)$ is convex in $x$

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ii) The moments of $\mu$ are finite
A novel concept: Kinship Functions
Towards Convex Relaxation of PO

Definition (Kinship function)

A function $\kappa : \mathbb{R} \rightarrow \mathbb{R}$ is said to be a kinship function (KF) if

(a) $\kappa(0) = 1$

(b) $\kappa(y)$ is a convex, nonnegative and nondecreasing function for $y \in [-1, \infty)$

Formulation motivated by the dilation integral approach
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Formulation motivated by the dilation integral approach
A Convex Relaxation

Theorem

Let $\kappa(\cdot)$ be a kinship function, define

$$V_\kappa(x) \doteq \int_Q \kappa[f(x, q)] \mu(q) dq$$

then

$$\text{Prob}\{q \in Q : f(x, q) > 0\} \leq V_\kappa(x)$$

Relaxation

$$\min_{x \in X} c^T x$$

subject to

$$V(x) \leq \epsilon$$

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The relaxation above is convex
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**Relaxation**

$$\min_{x \in \mathcal{X}} c^\top x$$

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**Theorem**

*The relaxation above is convex*
Optimal Choice of Kinship Function

Definition (Optimal Polynomial Kinship Function)

An optimal polynomial kinship function or order $\varrho$ is defined as a solution of the following optimization problem

$$\min_{\rho} \int_{-1}^{0} \rho(y) \, dy$$

subject to $\rho(y)$ is a polynomial of order $\varrho$

$\rho(y)$ is a kinship function

Optimal polynomial KFs can be used for any problem

$\Rightarrow$ Need to solve the problem above only once
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Optimal polynomial KFs can be used for any problem

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Designing optimal polynomial KFs is a finite dimensional convex problem.

Lemma (Design of $k_\varrho$)

The optimal $k_\varrho(y) = a_0 + a_1 y + \cdots + a_\varrho y^\varrho$ can be obtained as the solution of the following LMI optimization problem:

$$\min_{(a_0, \ldots, a_\varrho), (Y_1, Y_2) \succeq 0} \sum_{i=0}^{\varrho} \frac{(-1)^i}{i+1} a_i$$

subject to

$$a_0 = 1, \quad \sum_{i=0}^{\varrho} (-1)^i a_i = 0, \quad \sum_{i=1}^{\varrho} i(-1)^{i-1} a_i = 0$$

$$Y_1 H_{1,k} + Y_2 H_{2,k} - \sum_{i=k+2}^{\varrho} \frac{i!(-1)^{i-k-2} a_i}{k!(i-k-2)!} = 0,$$

$$k = 0, \ldots, \varrho - 2$$

where $H_{1,k} \in \mathcal{R}^{(n_1+1) \times (n_1+1)}$ and $H_{2,k} \in \mathcal{R}^{(n_2+1) \times (n_2+1)}$, with $n_1 = \left\lfloor \frac{\varrho - 2}{2} \right\rfloor$ and $n_2 = \left\lfloor \frac{\varrho - 3}{2} \right\rfloor$, are given series of Hankel matrices.
IDEA: seeking a polynomial that minimizes its $\mathcal{L}_1$ norm on $[-1, 0]$

In this way the subset $\{q \in Q : f(x, q) < 0\}$ contributes less possible to the integral $V_{\kappa}(x)$
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Tightness of the relaxation
Optimal Polynomial Kinship Functions (cont.)

Let

\[ V_\kappa(x, \varrho) = \int_Q \kappa_\varrho[f(x, q)]\mu(q)\,dq \]

where \( \kappa_\varrho(\cdot) \) is an optimal polynomial kinship function of order \( \varrho \)

Theorem (Convergence)

Assume that the original robust problem (RO) admits a (unique) solution \( x_{RO} \).
Let \( x_{\varrho}^* \) be the solution of the optimization problem

\[
\min_{x \in \mathcal{X}} c^T x \\
\text{s.t.} \quad V_\kappa(x, \varrho) \leq \epsilon
\]

Then

\[
\lim_{\varrho \to \infty} x_{\varrho}^* = x_{RO}
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Convex Relaxation – Recap

- We have “relaxed” the original problem and obtained the following convex optimization problem:

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad V_\kappa(x, \rho) \leq \epsilon
\end{align*}
\]

where \( \kappa_\rho(\cdot) \) is an optimal polynomial KF of order \( \rho \)

- The relaxation is tight w.r.t. RO for \( \rho \to \infty \)

- If you stop at fixed \( \rho \), you have a guaranteed solution to PO

Can one find an efficient way of computing/solving this?
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\min_{x \in \mathcal{X}} & \quad c^T x \\
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Numerical Integration – Quadrature Formulae

One Dimensional Integration

Given $g : \mathbb{R} \rightarrow \mathbb{R}$, consider first the one dimensional integration ($d = 1$) problem:

$$I[g] = \int_{-1}^{1} g(q) \mu(q) dq$$

**Quadrature formula**

An $N$-point QF with nodes $\theta_k$ and weights $w_k$ is defined as

$$Q_N[g] = \sum_{k=1}^{N} w_k g(\theta_k)$$

- $R_N[g] = I[g] - Q_N[g]$ is the error, or residual, of the QF
- The QF consists in evaluating $g(q)$ on the $N$-point grid

$$\Theta_N = \{\theta_1, \ldots, \theta_N\}$$
Degree of Exactness (DoE)

The DoE, \(\text{deg}(Q_N)\), of a quadrature formula is the maximum integer \(s\) such that the QF is exact for all polynomials of degree less than or equal to \(s\), and there exists a polynomial \(p\) of degree \(s + 1\) such that \(R_N[p] \neq 0\)

Gauss Formulae

The maximum DoE is achievable by the Gauss Formulae:

\[
\text{deg}(Q_N) = 2N - 1
\]

In our case:

- The nodes are chosen as the zeros of the \(N\)-th order Legendre orthogonal polynomial \(P_N(x)\)
- The weights are computed by integrating the associated Lagrange polynomials
Cubature Rule: Multidimensional QF

The QF approach can be extended in a straightforward way to multidimensional integration using the so-called Cubature Rules.

- Assume $g(q)$ has degree $\nu_i$ in the variable $q_i$, $i = 1, \ldots, d$.
- Choose $N_i : \deg(Q_{N_i}) \geq \nu_i$. Then, the tensor product
  
  $$ (Q_{N_1} \otimes \cdots \otimes Q_{N_d})[g] = \sum_{k_1=1}^{N_1} \cdots \sum_{k_d=1}^{N_d} (w_{k_1} \cdots w_{k_d})g(\theta_{k_1} \cdots \theta_{k_d}) $$

  evaluates exactly the integral $\int_{\Omega_d} g(q) dq$
- This corresponds to evaluating $g(q)$ on the multi-grid
  
  $$ H_d \doteq (\Theta_{N_1} \times \cdots \times \Theta_{N_d}) \subset \Omega_d $$

---

**However:** The number of nodes in this formula is given by

$$ |H_d| = \prod_{i=1}^{d} N_i $$

It depends exponentially on the dimension $d$. 
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It depends exponentially on the dimension \( d \).
Sequences of (low-dim) QF

Consider particular sequences $Q^{(1)}, Q^{(2)}, \ldots$ of QF with increasing index of precision $i$:

$$Q^{(i)}[g] = Q_{N_i}[g] = \sum_{k=1}^{N_d} w_k^{(i)} g(\theta_k^{(i)}), \quad i = 1, 2, \ldots$$

with nodes $\Theta^{(i)} \doteq \{\theta_1^{(i)}, \ldots, \theta_{N}^{(i)}\}$

These QF should satisfy the following properties:

(i) **Nested nodes**: $\Theta^{(i-1)} \subset \Theta^{(i)} \quad i = 1, 2, \ldots$

(ii) **Precision**: $\deg(Q^{(i)}) \geq 2i - 1 \quad i = 1, 2, \ldots$

(iii) **Initial condition**: $Q^{(1)}[f] = 2f(0)$ (one node formula)
Smolyak Cubature Rules

Smolyak cubature rule is a linear combination of product formulae involving only relatively low-precision QF:

**Smolyak formula**

For a given *precision level* $\ell > 0$, define

$$S_{\ell,d}[g] = \sum_{\ell+1 \leq \|i\|_1 \leq \ell+d} (-1)^{\ell+d-\|i\|_1} \left( \left\| i \right\|_1 - \ell - 1 \right)^{d - 1} (Q^{(i_1)} \otimes \cdots \otimes Q^{(i_d)})[g]$$

where $i \doteq [i_1 \cdots i_d]^\top$, $i_k > 0$, is the vector of precision indices for each dimension.
Smolyak Cubature Rules: Key Properties

Number of nodes
The number of nodes is polynomial in $d$: $N_{\ell,d} \approx \frac{2^\ell}{\ell!} d^\ell$

Degree of exactness
The formula has high DoE: $\deg(S_{\ell,d}) \geq 2\ell + 1$

- $S_{\ell,d}$ does not depend on the integrand, but only on $\ell, d$
- For given $\ell$ and $d$, the $N_{\ell,d}$ nodes and weights of $S_{\ell,d}$ can be computed once for all and stored for successive computations
- Once these nodes $\theta_j$ and weights $w_j, j = 1, \ldots, N_{\ell,d}$, are obtained, we end up with a standard $N_{\ell,d}$-point cubature formula:

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Sparse grids

The nodes $\theta_j$ of the Smolyak formula form a so-called sparse grid

$$H_{\ell,d} \doteq \bigcup_{\|i\|_1 = \ell + d} (\Theta^{(i_1)} \times \cdots \times \Theta^{(i_d)}) \subset \Omega_d$$

Two and three dimensional sparse grids for $\ell = 7$
Numerical Integration - Recap

- We have universal QF: works for any polynomial with total degree less than or equal to DoE

- Can compute the integral

\[ V_{\kappa}(x, \varrho) = \int_{Q} \kappa_{\varrho}[f(x, q)]\mu(q)\,dq \]

in polynomial time, setting \( \ell = \varrho \sigma \) and letting

\[ V_{\kappa}(x, \varrho) = \sum_{i=1}^{N_{\ell,d}} w_i \kappa_{\varrho}[f(x, \theta_i)] \]

- Can now compute

\[ \min_{x \in X} c^T x \]

s.t. \[ \sum_{i=1}^{N_{\ell,d}} w_i \kappa_{\varrho}[f(x, \theta_i)] \leq \epsilon \]

hence, solving the convex relaxation of the original problem
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hence, solving the convex relaxation of the original problem.
Solution of the relaxed problem

Subgradient computation

Proposition (Subgradient of $V_\kappa$)

For given $q \in Q$, let $\partial f(x, q)$ be a (sub)gradient of the function $f$ with respect to $x$. Consider a polynomial $KF k_\varrho(y) = a_0 + a_1 y + \cdots + a_\varrho y^\varrho$ of degree $\varrho$. Then

$$
\partial V_\kappa(x) = \sum_{k=1}^{N_{\ell,d}} w_k \partial f(x, \theta_k) \left( \sum_{j=0}^{\varrho} j a_j [f(x, \theta_k)]^{j-1} \right)
$$

is a (sub)gradient of $V_\kappa(x, \varrho)$

At this point, standard tools as (sub)gradient descent or ellipsoidal/cutting plane localization methods can be applied for his solution.
In BSRD09\(^1\) a novel approach is developed for checking positivity of a polynomial

\[ p(x) > 0 \]

via the following sequence of dilation integrals, for \( \rho = 2, 4, \ldots \)

\[ \int_x (1 - \alpha p(x))^\rho \]

It can be shown that, for fixed \( \alpha \), the function \( (1 - \alpha y)^\rho \) is a particular kinship function (not optimal!)

This is an analysis problem: we extend to design

\(^1\)B. Barmish, P.S. Shcherbakov, S. Ross and F. Dabene, On Positivity of Polynomials Via Dilation Integrals, IEEE Transactions on Automatic Control, 2009
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SOS, similarly to our method, constructs a sequence of solutions that converges to $x_{RO}$

SOS suffers from an exponential explosion (in the number of monomials) as dimension grows.

In terms of cost function $c^T x$, the convergence is from above

Our approach is polynomial in the dimension

But it is exponential in $\varrho$ – this is unavoidable (remind, it’s NP hard)

If we choose uniform grids, it’s the other way around

Since we adopt a probabilistic relaxation, we converge from below

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Application Example: A Portfolio Selection Problem

Consider portfolio selection problem using factor models via the Value at Risk (VaR) approach

\[
\min_{\gamma, x \in \mathcal{X}} \gamma
\]

subject to

\[
k(\eta) \sqrt{x^T (A(q) \Sigma A(q)^T + D) x - x^T A(q) \hat{\phi}} \leq \gamma \quad \text{for all } q \in Q
\]

- \( \eta \): the risk level
- \( \gamma \): the loss level, upper bound of the VaR
- \( x \): the investment policy, constraint in some convex set \( \mathcal{X} \)
- \( \phi \): the vector of factors, with mean \( \hat{\phi} \) and covariance matrix \( \Sigma \)
- \( D \): diagonal covariance matrix of the residuals \( u \), assumed to be zero mean
- \( A(q) \): the sensitivity matrix, polynomial in the uncertain parameters \( q \in Q \)

Remark

If one can solve this, one can guarantee that, with a risk level less than \( \eta \), one has \( \text{VaR} \leq \gamma \) for all instances consistent with the uncertainty model above.
Consider portfolio selection problem using factor models via the Value at Risk (VaR) approach

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- \( \phi \): the vector of factors, with mean \( \hat{\phi} \) and covariance matrix \( \Sigma \)
- \( D \): diagonal covariance matrix of the residuals \( u \), assumed to be zero mean
- \( A(q) \): the sensitivity matrix, polynomial in the uncertain parameters \( q \in Q \)

**Remark**

If one can solve this, one can guarantee that, with a risk level less than \( \eta \), one has \( \text{VaR} \leq \gamma \) for all instances consistent with the uncertainty model above.
Application Example: A Portfolio Selection Problem

The feasibility problem for a given $\gamma$:

Find a policy $x \in X$, such that $f(x, q) \leq 0$ for all $q \in Q$ where

$$f(x, q) = x^T \left( A(q) \left( k^2(\eta)\Sigma - \hat{\phi}\hat{\phi}^T \right) A(q)^T + D \right) x - 2\gamma x^T A(q)\hat{\phi} - \gamma^2$$

The function $f(x, q)$ is polynomial in $q$, and convex in $x$ if

$$k^2(\eta)\Sigma - \hat{\phi}\hat{\phi}^T \succ 0$$

Remark: This condition is satisfied in real market data for low risk lever $\eta$; e.g., see Table 2 in Fama and Frence’s paper (1993) for $\eta = 5\%$. 
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The corresponding relaxed convex optimization problem

\[ V^*_{\kappa_\varrho}(\gamma) = \min_{x \in X} \int_{Q} \kappa_\varrho(\alpha f(x, q)) \, dq \]

- \( \alpha \): a positive number chosen such that \( \alpha f(x, q) \geq -1 \)
- \( \kappa_\varrho(\cdot) \): the optimal polynomial kinship function of order \( \varrho \)
- \( V^*_{\kappa_\varrho}(\gamma) \): the upper bound of the volume percentage of the ‘bad’ set in \( Q \)

Remark

This problem can be easily recasted into our original problem.
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Remark

This problem can be easily recasted into our original problem.
Numerical Results

The sensitivity matrix \( A(q) \) is, in our example, assumed to be an interval matrix of the form

\[
A(q) = A_0 + \sum_{i,j} q_{ij} A_{ij}
\]

where \( \|q\|_\infty \leq q_{\text{max}} \) and \( A_{ij} \) has all elements equal to zero except the \( i,j \)-th one where

\[
[A_{ij}]_{i,j} = [A_0]_{i,j}
\]

\[
A_0 = \begin{pmatrix}
0.4666 & -0.6952 \\
0.2447 & -0.5934 \\
0.9796 & 0.6386
\end{pmatrix}
\]

\[
\Sigma = \begin{pmatrix}
0.2009 & 0.1791 \\
0.1791 & 0.4489
\end{pmatrix}
\]

\[
D = \begin{pmatrix}
0.1902 & 0 & 0 \\
0 & 0.5995 & 0 \\
0 & 0 & 0.2923
\end{pmatrix}
\]

\[
\hat{\phi} = \begin{pmatrix}
0.0584 \\
0.5385
\end{pmatrix}
\]
Numerical Results

- Risk level $\eta = 5\%$
- Uncertainty magnitude $q_{\text{max}} = 0.05$
- Degree $\varrho = 3$ leading to a total degree of 6
- Implemented cubature rules with $\ell = 3$ and $d = 6$

$$V^{*}_{k_3}(\gamma) \text{ v.s. } \gamma \text{ for } q_{\text{max}} = 0.05$$

1,000,000 randomly generated samples (for $\gamma = 2.6$) $\implies$ VaR $\leq \gamma$ satisfied for all
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![Graph](image)

$V_{\kappa_{3}}(\gamma)$ v.s. $\gamma$ for $q_{max} = 0.05$

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Numerical Results

- Risk level $\eta = 5\%$
- Degree $\varrho = 3$ leading to a total degree of 6
- Implemented cubature rules with $\ell = 3$ and $d = 6$
- Set threshold of the volume percentage of the "bad" set in $Q$ to be 0.1%.

The loss level $\gamma$ v.s. the uncertainty magnitude $q_{\text{max}}$

Dabbene, Feng, Lagoa (Torino, Penn State)  Robust and Chance-Constrained Optimization  Barmish's Workshop, 2009
Conclusions

In this talk:

- Polynomial-time algorithm for (approximate) solution convex optimization problems with polynomial dependence on uncertainty

- We build a succession of solutions that eventually converges to the solution of RO

- If we stop at some point, we have a guaranteed solution to PO

- Numerical examples show the efficacy of the procedure were presented

Further research directions:

- Specialize to Robust Linear Matrix Inequalities

- Kinship functions are a general concept: we may improve computational efficiency by looking at other types of KFs
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