Chapter 11.- Mixed-Strategy Nash Equilibrium

• As we have seen, some games do not have a Nash equilibrium in pure strategies.

• However, existence of Nash equilibrium would follow if we extend this notion to mixed strategies.

• All we need is for each player’s mixed strategy to be a best response to the mixed strategies of all other players.
• **Example: Matching pennies game.**- We saw before that this game does not have a Nash equilibrium in pure strategies.

- Intuitively: Given the “pure conflict” nature of the matching pennies game, letting my opponent know for sure which strategy I will choose is never optimal, since this will give my opponent the ability to hurt me for sure.

- This is why **randomizing is optimal.**
• Consider the following profile of mixed strategies:

\[ \sigma_1 = \left( \frac{1}{2}, \frac{1}{2} \right) \text{ and } \sigma_2 = \left( \frac{1}{2}, \frac{1}{2} \right) \]

• Note that

\[ u_1(H, \sigma_2) = 1 \cdot \frac{1}{2} - 1 \cdot \frac{1}{2} = 0 \]
\[ u_1(T, \sigma_2) = -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0 \]

• And therefore,

\[ u_1(\sigma_1, \sigma_2) = \frac{1}{2} \cdot u_1(H, \sigma_2) + \frac{1}{2} \cdot u_1(T, \sigma_2) = 0 \]
Since payoffs are symmetrical, we also have
\[ u_2(\sigma_1, \sigma_2) = 0 \]

Note that:

Each player is \textit{indifferent} between his two strategies (H or T) if the other player randomizes according to \( \sigma_j = \left( \frac{1}{2}, \frac{1}{2} \right) \) (both H and T yield a payoff of zero). \textit{Both strategies are best responses to} \( \sigma_j = \left( \frac{1}{2}, \frac{1}{2} \right) \).

Playing the mixed strategy \( \sigma_i = \left( \frac{1}{2}, \frac{1}{2} \right) \) also yields a payoff of zero and therefore is also a best response to \( \sigma_j = \left( \frac{1}{2}, \frac{1}{2} \right) \).
• Therefore, if the other player chooses H or T with probability \( \frac{1}{2} \) each, then each player is perfectly content with also randomizing between H and T with probability \( \frac{1}{2} \).

• This constitutes a Nash equilibrium in mixed strategies.
• **Definition:** Consider a (mixed) strategy profile
\[
\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)
\]
where \(\sigma_i\) is a mixed strategy for player \(i\). The profile \(\sigma\) is a **mixed-strategy Nash equilibrium** if and only if playing \(\sigma_i\) is a best response to \(\sigma_{-i}\). That is:
\[
u_i(\sigma_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \quad \text{for each } s'_i \in S_i
\]

• **Fact #1 about mixed-strategy Nash Equilibrium:** A mixed strategy is \(\sigma_i\) is a best response to \(\sigma_{-i}\) only if \(\sigma_i\) assigns positive probability exclusively to strategies \(s_i \in S_i\) that are best-responses to \(\sigma_{-i}\).
• Facts about mixed-strategy Nash equilibria:

1. In any mixed-strategy Nash equilibrium \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \), players assign positive probability only to rationalizable strategies. That is, \( \sigma_i(s_i) > 0 \) only if \( s_i \) is rationalizable.

2. In any mixed-strategy Nash equilibrium \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \), the mixed strategy \( \sigma_i \) assigns positive probability exclusively to strategies \( s_i \in S_i \) that are best-responses to \( \sigma_{-i} \). That is:

   \[
   \text{If } \sigma_i(s_i) > 0, \text{ then it must be that: } \quad u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \text{ for every } s'_i \in S_i.
   \]
3. In any mixed-strategy Nash equilibrium $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, each player $i$ is indiffereent between all the strategies $s_i$ that he can play with positive probability according to $\sigma_i$. That is, for each $i = 1, \ldots, n$:

$$u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i})$$

for all $s_i, s'_i$ such that $\sigma_i(s_i) > 0$ and $\sigma_i(s'_i) > 0$.

- Using these facts, we can characterize a step-by-step procedure to find mixed-strategy Nash equilibria in two player games (things get a bit more complicated in games with three or more players).
• Procedure for finding mixed-strategy equilibria in discrete, two-player games:

1. **Step 1:** Find the set of rationalizable strategies in the game using iterated dominance.

2. **Step 2:** Restricting attention to rationalizable strategies, write equations for each player to characterize mixing distributions that make each player indifferent between the relevant pure strategies.

3. **Step 3:** Solve these equations to determine equilibrium mixing distributions.
• **Example: A lobbying game.** Suppose two firms simultaneously and independently decide whether to lobby (L) or not lobby (N) the government in hopes of trying to generate favorable legislation. Suppose payoffs are:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>X L</td>
<td>−5, −5</td>
<td>25, 0</td>
</tr>
<tr>
<td>X N</td>
<td>0, 15</td>
<td>10, 10</td>
</tr>
</tbody>
</table>
• This game has two pure-strategy Nash equilibria:

<table>
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<tr>
<th></th>
<th>L</th>
<th>N</th>
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</thead>
<tbody>
<tr>
<td>X</td>
<td>-5, -5</td>
<td>0, 15</td>
</tr>
<tr>
<td>Y</td>
<td>25, 0</td>
<td>10, 10</td>
</tr>
</tbody>
</table>

• **Question:** Does it also have a mixed-strategy Nash equilibrium?

• Since this game has only two players and two strategies, this question is easy to answer.

• **Step 1:** Note that both strategies are rationalizable for each player.
• **Step 2:** With only two players and two strategies, a profile of mixed strategies $\sigma_1, \sigma_2$ is a Nash equilibrium if and only if:

I. Player 1 is indifferent between L and N when player 2 uses $\sigma_2$.

II. Player 2 is indifferent between L and N when player 1 uses $\sigma_1$.

• That is, if and only if $\sigma_1, \sigma_2$ are such that:

$$u_1(L, \sigma_2) = u_1(N, \sigma_2)$$

and

$$u_2(\sigma_1, L) = u_2(\sigma_1, N)$$
• Since each player has only two strategies (L and N), any mixed strategy is fully described by
  \[ \sigma_i = (\sigma_i(L), 1 - \sigma_i(L)) \]
  
• Where:
  \[ \sigma_i(L) = \Pr(\text{Player } i \text{ chooses } L) \]
  \[ 1 - \sigma_i(L) = \Pr(\text{Player } i \text{ chooses } N) \]

• Therefore,
  \[ u_1(L, \sigma_2) = -5 \cdot \sigma_2(L) + 25 \cdot (1 - \sigma_2(L)) = 25 - 30 \cdot \sigma_2(L) \]
  \[ u_1(N, \sigma_2) = 0 \cdot \sigma_2(L) + 10 \cdot (1 - \sigma_2(L)) = 10 - 10 \cdot \sigma_2(L) \]
  \[ u_2(\sigma_1, L) = -5 \cdot \sigma_1(L) + 15 \cdot (1 - \sigma_1(L)) = 15 - 20 \cdot \sigma_1(L) \]
  \[ u_2(\sigma_1, N) = 0 \cdot \sigma_1(L) + 10 \cdot (1 - \sigma_1(L)) = 10 - 10 \cdot \sigma_1(L) \]
• In any mixed-strategy Nash equilibrium, we must have $u_1(L, \sigma_2) = u_1(N, \sigma_2)$. That is:
  $$25 - 30 \cdot \sigma_2(L) = 10 - 10 \cdot \sigma_2(L)$$
• This will be satisfied if:
  $$\sigma_2(L) = \frac{3}{4}$$
• And we also must have $u_2(\sigma_1, L) = u_2(\sigma_1, N)$. That is:
  $$15 - 20 \cdot \sigma_1(L) = 10 - 10 \cdot \sigma_1(L)$$
• This will be satisfied if:
  $$\sigma_1(L) = \frac{1}{2}$$
• Therefore, this game has a mixed-strategy equilibrium \((\sigma_1, \sigma_2)\), where:

\[
\sigma_1 = \left( \frac{1}{2}, \frac{1}{2} \right)
\]

and

\[
\sigma_2 = \left( \frac{3}{4}, \frac{1}{4} \right)
\]

• This example also illustrates that some games may have Nash equilibria in pure strategies AND also in mixed strategies.
• **Example: A tennis-service game.** Consider two tennis players.

• Player 1 (the server) must decide whether to serve to the opponent’s forehand (F), center (C) or backhand (B).

• Simultaneously, Player 2 (the receiver) must decide whether to favor the forehand, center of backhand side.
• Suppose payoffs are given by:

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<thead>
<tr>
<th></th>
<th>F</th>
<th>C</th>
<th>B</th>
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<tbody>
<tr>
<td>F</td>
<td>0, 5</td>
<td>2, 3</td>
<td>2, 3</td>
</tr>
<tr>
<td>C</td>
<td>2, 3</td>
<td>0, 5</td>
<td>3, 2</td>
</tr>
<tr>
<td>B</td>
<td>5, 0</td>
<td>3, 2</td>
<td>2, 3</td>
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• We begin by noting that this game does not have any pure-strategy Nash equilibrium.
• To see why, note that best-responses are given by:

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<th>F</th>
<th>C</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>0, 5</td>
<td>2, 3</td>
<td>2, 3</td>
</tr>
<tr>
<td>C</td>
<td>2, 3</td>
<td>0, 5</td>
<td>3, 2</td>
</tr>
<tr>
<td>B</td>
<td>5, 0</td>
<td>3, 2</td>
<td>2, 3</td>
</tr>
</tbody>
</table>

• So there is no pair of mutual best-responses in pure strategies.
• **Question:** Find the mixed-strategy Nash equilibria in this game.

• **Step 1:** Using iterated dominance, find the set of rationalizable strategies $R$.
  
  – To find the reduced game $R^1$:
  
  – Note first that all three strategies $\{F, C, B\}$ are best-responses for player 2, so they will all survive.

  – For player 1, $\{C, B\}$ are best-responses. And we can show easily that $F$ is dominated by a mixed strategy between $\{C, B\}$. From here, we have:

$$R^1 = \{C, B\} \times \{F, C, B\}$$
(cont...)  

– To find $R^2$, we note that in the reduced game $R^1$, the only dominated strategy is $F$, for player 2. Player 1 does not have any dominated strategy in $R^1$. Therefore,  

\[ R^2 = \{C, B\} \times \{C, B\} \]

– It is easy to verify that there are no dominated strategies in $R^2$. Therefore the game cannot be reduced any further and we have  

\[ R = \{C, B\} \times \{C, B\} \]
• The set of rationalizable strategies is:

![Game Matrix]

• To find mixed-strategy Nash equilibria, we need to look for mixing distributions:

\[ \sigma_1 = (0, \sigma_1(C), 1 - \sigma_1(C)) \]

\[ \sigma_2 = (0, \sigma_2(C), 1 - \sigma_2(C)) \]

(where each player randomizes only between “C” and “B” and play “F” with zero probability) such that both players are indifferent between C and B.
• That is, we must have:

\[ u_1(C, \sigma_2) = u_1(B, \sigma_2) \]

and

\[ u_2(\sigma_1, C) = u_2(\sigma_1, B) \]

• Expected payoffs are given by:

\[ u_1(C, \sigma_2) = 0 \cdot \sigma_2(C) + 3 \cdot (1 - \sigma_2(C)) = 3 - 3 \cdot \sigma_2(C) \]
\[ u_1(B, \sigma_2) = 3 \cdot \sigma_2(C) + 2 \cdot (1 - \sigma_2(C)) = 2 + 1 \cdot \sigma_2(C) \]
\[ u_2(\sigma_1, C) = 5 \cdot \sigma_1(C) + 2 \cdot (1 - \sigma_1(C)) = 2 + 3 \cdot \sigma_1(C) \]
\[ u_2(\sigma_1, B) = 2 \cdot \sigma_1(C) + 3 \cdot (1 - \sigma_1(C)) = 3 - 1 \cdot \sigma_1(C) \]
• Therefore, \( \sigma_1(C) \) and \( \sigma_2(C) \) need to satisfy:

\[
3 - 3 \cdot \sigma_2(C) = 2 + 1 \cdot \sigma_2(C)
\]

and

\[
2 + 3 \cdot \sigma_1(C) = 3 - 1 \cdot \sigma_1(C)
\]

• This yields:

\[
\sigma_2(C) = \frac{1}{4} \quad \text{and} \quad \sigma_1(C) = \frac{1}{4}
\]
Therefore, the mixed-strategy Nash equilibrium in this game is given by the mixing distributions:

\[
\begin{align*}
\sigma_1 &= \left(0, \frac{1}{4}, \frac{3}{4}\right) \\
\sigma_2 &= \left(0, \frac{1}{4}, \frac{3}{4}\right)
\end{align*}
\]
• **Example:** Find the set of Nash equilibria (pure and mixed) in this game:

```
  2
  |
  |
|
 U | X | Y | Z |
|
 U | 2,0| 1,1| 4,2|
|
 M | 3,4| 1,2| 2,3|
|
 D | 1,3| 0,2| 3,0|
```
• We begin with the pure-strategy equilibria:

```
   1  2
 U  2,0  1,1  4,2
 M  3,4  1,2  2,3
 D  1,3  0,2  3,0
```

(b)
• **Mixed-strategy equilibria**: first, using iterated dominance we look for the set of rationalizable strategies $R$

  – Player 1: “M” is a best response to “X” and “Y”, while “U” is a best response to “Z”. The strategy “D” is dominated by “U”.
  – Player 2: “Z” is a best response to “U”, “X” is a best response to “M” and “D”.
  – We need to check if “Y” is a dominated strategy. Same procedure we followed in Chapter 6 shows that it is NOT a dominated strategy.
  – Therefore:

$$R^1 = \{U, M\} \times \{X, Y, Z\}$$
• Matrix form of the reduced game $R^1$ is:

\[
\begin{array}{ccc}
1 & 2 & \\
\hline
U & 2,0 & 1,1 & 4,2 \\
M & 3,4 & 1,2 & 2,3 \\
D & 1,5 & 6,2 & 5,6 \\
\end{array}
\]

(b)

– Player 1 has no dominated strategies in the reduced game given by $R^1$.
– For Player 2, “Y” is dominated by “Z” in the reduced game given by $R^1$.
– Therefore, $R^2 = \{U, M\} \times \{X, Z\}$. 
• $R^2 = \{U, M\} \times \{X, Z\}$. Reduced game:

- Player 1 has no dominated strategies in the reduced game given by $R^2$.
- Player 2 has no dominated strategies in the reduced game given by $R^2$.
- Therefore, no further reduction can be done and we have $R^2 = R$. Therefore,

$$R = \{U, M\} \times \{X, Z\} = \{(U, X), (U, Z), (M, X), (M, Z)\}$$
• Focusing on the rationalizable strategies $R$, we now need to find well-defined mixing probabilities

\[
\sigma_1 = (\sigma_1(U), 1 - \sigma_1(U), 0)
\]

\[
\sigma_2 = (\sigma_2(X), 0, 1 - \sigma_2(X))
\]

such that both players are indifferent between their actions (X and Z for player 2, and U and M for player 1). That is:

\[
u_1(U, \sigma_2) = u_1(M, \sigma_2)
\]

and

\[
u_2(\sigma_1, X) = u_2(\sigma_1, Z)
\]
• The reduced game \( R \) looks like this:

\[
\begin{array}{ll}
  & 1 & 2 \\
U & 2, 0 & 4, 2 \\
M & 3, 4 & 2, 3 \\
\end{array}
\]

• From here we have:

\[
u_1(U, \sigma_2) = 2 \cdot \sigma_2(X) + 4 \cdot (1 - \sigma_2(X)) = 4 - 2 \cdot \sigma_2(X)
\]

\[
u_1(M, \sigma_2) = 3 \cdot \sigma_2(X) + 2 \cdot (1 - \sigma_2(X)) = 2 + 1 \cdot \sigma_2(X)
\]

• And:

\[
u_2(\sigma_1, X) = 0 \cdot \sigma_1(U) + 4 \cdot (1 - \sigma_1(U)) = 4 - 4 \cdot \sigma_1(U)
\]

\[
u_2(\sigma_1, Z) = 2 \cdot \sigma_1(U) + 3 \cdot (1 - \sigma_1(U)) = 3 - 1 \cdot \sigma_1(U)
\]
• Both players will be indifferent between their relevant strategies if and only if:

\[ 4 - 2 \cdot \sigma_2(X) = 2 + 1 \cdot \sigma_2(X) \] (for player 1)

\[ 4 - 4 \cdot \sigma_1(U) = 3 - 1 \cdot \sigma_1(U) \] (for player 2)

• The first condition will hold if and only if

\[ \sigma_2(X) = \frac{2}{3} \]

• And the second condition will hold if and only if

\[ \sigma_1(U) = \frac{1}{3} \]
Therefore, this game has one mixed-strategy Nash equilibrium where players randomize according to the distributions:

\[
\sigma_1 = \left( \frac{1}{3}, \frac{2}{3}, 0 \right)
\]

\[
\sigma_2 = \left( \frac{2}{3}, 0, \frac{1}{3} \right)
\]
• **Mixed-strategy Nash Equilibrium in Continuous Games:** As in discrete games, the key feature is that players must randomize in a way that makes other players indifferent between their relevant strategies.

• **Example: Bertrand competition with capacity constraints.**

• Consider a duopoly industry of a homogenous good with two firms who compete in prices.

• Suppose the market consists of **10 consumers**, each of which will purchase **one unit of the good**. Suppose that **each consumer is willing to pay at most $1 for the good.**
• For simplicity, suppose the production cost is zero for both firms.
• If this setup fully describes the model, then it is a very simple case of Bertrand competition. As we learned previously, the equilibrium prices would be those that yield a profit of zero.
• Since production cost is zero, this mean that the Nash equilibrium prices would be:

\[ p_1 = 0 \quad \text{and} \quad p_2 = 0 \]

as we learned previously, this would be the UNIQUE Nash equilibrium in the game.
• Suppose now that both firms have a **capacity constraint**. Specifically, suppose each firm can produce **at most eight units of the good**.

• This will change the features of the model drastically: Now **the firm with the cheapest price cannot capture the entire market because of the capacity constraint**.

• Conversely, **the firm with the highest price can still capture two consumers**.

• As a result, the Nash equilibrium properties of this model will change. As we will see, it will no longer have an equilibrium in pure strategies. Instead, it will have a unique equilibrium in mixed strategies.
• With capacity constraints, the game no longer has an equilibrium in pure strategies: We begin by noting that by setting the highest possible price ($p_i = 1$), firm $i$ ensures itself a profit of at least $2$ (since at the very least it will sell two units due to the capacity constraint of the opponent).

• Suppose $p_1 = p_2 > 0$. Can this be a Nash equilibrium? No, because it would be better for either firm to undercut the other firm’s price by an infinitesimal amount. This will always yield a higher payoff than choosing the same price as the opponent.
• Suppose $p_1 = p_2 = 0$. Can this be a Nash equilibrium? **It used to be the Nash equilibrium without capacity constraints, but not any more.** Why? Because if my opponent sets a price of zero, my best response now is to set a price of $1$. This will ensure me a profit of $2$ instead of $0$, which is what I would obtain if I set my price to zero.

• Therefore, combining the two cases above, there cannot be a Nash equilibrium in pure strategies where $p_1 = p_2$.
• Can there be a pure-strategy Nash equilibrium in which \( p_i < p_j \leq 1 \)? First note that if one firm chooses a price higher than the other firm, then the only rational price to choose is the highest possible price (since you would have two captive costumers).

• That is, if \( p_i < p_j \) in equilibrium, then it must be the case that \( p_j = 1 \). But if \( p_j = 1 \), it is not optimal for firm \( i \) to charge strictly less than 1. Firm \( i \) would like to keep raising \( p_i \) by infinitesimal amounts to become closer and closer to $1. So the best response by \( i \) would not be well-defined.
• Therefore since there is no pure-strategy Nash equilibrium where \( p_1 = p_2 \) and there is no pure-strategy Nash equilibrium where \( p_i < p_j \), we conclude that this game does not possess a pure-strategy Nash equilibrium.

• How about a mixed-strategy Nash equilibrium?

• Notice that the strategy space is continuous, which makes the problem a bit “trickier”. Still, we can describe the mixed-strategy Nash equilibrium using the same principle as in discrete games: In equilibrium, both players must be indifferent between all their relevant strategies.