NETS AND QUASI-ISOMETRIES
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1. Basic definitions and extension lemma

Let $X$ be a complete metric space with the distance function $d_X$.

**Definition 1.** A subset $\Gamma \subset X$ is called a net if:

1. $\Gamma$ is uniformly discrete, i.e. there is $r > 0$ such that for $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, $d_X(\gamma_1, \gamma_2) \geq r$

2. $\Gamma$ spans $X$, i.e. there exists $R > 0$ such that for every $x \in X$ one can find $\gamma \in \Gamma$ with $d_X(x, \gamma) < R$.

The infimum of all $R$ satisfying (1.2) will be called the spanning constant for $\Gamma$.

Now let $X, Y$ be two metric spaces.

**Definition 2.** A map $\varphi : X \to Y$ is called a quasi-isometric embedding if there exist positive numbers $A, B$ such that for any $x_1, x_2 \in X$, $x_1 \neq x_2$:

$$A < \frac{d_Y(\varphi x_1, \varphi x_2)}{d_X(x_1, x_2)} < B$$

A one-to-one quasi-isometric embedding is called a quasi-isometry. It follows from the definition that the inverse map to a quasi-isometry is also a quasi-isometry.

**Definition 3.** [4, p. 7] A continuous map $\varphi : X \to Y$ is called a pseudo-isometry if for some positive constants $A, B, C$ and for any $x_1, x_2 \in X$ one has:

$$(1.3) \quad A d_X(x_1, x_2) - C < d_Y(\varphi x_1, \varphi x_2) < B d_X(x_1, x_2) + C$$

**Lemma 1.** Let $\Gamma$ be a net in a metric space $X$, $\varphi : \Gamma \to Y$ a pseudo-isometry. Then there exists a subset $\Gamma' \subset \Gamma$ such that $\Gamma'$ is also a net in $X$ and the restriction of $\varphi$ to $\Gamma'$ is a quasi-isometric embedding.

This lemma is an almost immediate corollary of the following statement.

**Lemma 2.** Let $\Gamma$ be a net in a complete non-compact metric space $X$ and $T$ be a positive number. Then there exists a subset $\Gamma' \subset \Gamma$ which is also a net in $X$ and such that:

(A) For every $\gamma_1, \gamma_2 \in \Gamma'$, $\gamma_1 \neq \gamma_2$, $d_X(\gamma_1, \gamma_2) > T$.

**Proof.** Let $B(T)$ be the collection of all subsets of $\Gamma$ satisfying (A). It is non-empty and is partially ordered by inclusion. Any ordered subset of $B(T)$ has the maximal element (the union of all its elements); consequently by Zorn’s lemma, there is a maximal element $\Gamma' \in B(T)$. This means

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that for every point $\gamma \in \Gamma$ there is a point $\gamma' \in \Gamma'$ such that $d_X(\gamma, \gamma') < T$, because otherwise $\Gamma'$ is not maximal. But then for every $x \in X$ there is a point $\gamma' \in \Gamma$ such that $d_X(x, \gamma') \leq d_X(x, \gamma) + d_X(\gamma, \gamma') \leq R + T$, i.e. $\Gamma'$ is a net in $X$. □

Proof of Lemma 1. If $X$ is compact, let $\Gamma'$ be any one-point subset of $\Gamma$. If it is not compact, let us apply Lemma 2 with $T = \frac{dC}{2}$. Then for any $\gamma_1, \gamma_2 \in \Gamma'$ one has:

$$d_Y(\varphi \gamma_1, \varphi \gamma_2) > A d_X(\gamma_1, \gamma_2) - C \geq \frac{A}{2} d_X(\gamma_1, \gamma_2) + \left( \frac{A}{2} T - C \right) \geq \frac{A}{2} d_X(\gamma_1, \gamma_2)$$

and similarly since $B \geq A$

$$d_Y(\varphi \gamma_1, \varphi \gamma_2) < B d_X(\gamma_1, \gamma_2) + C < \frac{3}{2} B d_X(\gamma_1, \gamma_2) + \left( C - \frac{B}{2} T \right) < \frac{3}{2} B d_X(\gamma_1, \gamma_2)$$

□

Let us denote by $X^n$ the $n$th cartesian power of the space $X$ with the product topology. Furthermore, let $\Sigma_n$ be the standard $(n-1)$ simplex:

$$\Sigma_n = \left\{ (t_1, \ldots, t_n) : t_i \geq 0, i = 1, \ldots, n, \sum_{i=1}^{n} t_i = 1 \right\}.$$

Definition 4. A centroid on a metric space $X$ is a map

$$C : \bigcup_{n=1}^{\infty} X^n \times \Sigma_n \to X$$

satisfying the following properties (1.4)-(1.7):

(1.4) For $n = 1$, $C(x, 1) = x$

(1.5) $C(x_1, \ldots, x_n, t_1, \ldots, t_{k-1}, 0, t_{k+1}, \ldots, t_n) = C(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n, t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)$

(1.6) $C$ is continuous on every $X^n \times \Sigma_n$

(1.7) For every $n$ and $R > 0$ there exists $F(R, n)$ such that if $d_X(x_i, x_j) \geq R$, $i, j = 1, \ldots, n$ then

$$d_X(C(x_1, \ldots, x_n, t_1, \ldots, t_n), x_i) < F(R, n)$$

for every $i = 1, \ldots, n$ and $(t_1, \ldots, t_n) \in \Sigma_n$.

It follows from (1.4) and (1.5) that:

$$C(x_1, \ldots, x_n, 0, \ldots, 0, 1, 0, \ldots, 0) = x_i$$

(1.8)

↓

$i$th place

Definition 5. An $n$-centroid on $X$ is a map defined on $\bigcup_{k=1}^{n} X^k \times \Sigma_k$ and satisfying conditions (1.4)-(1.7).

Remark. Let us make the following identification on $\bigcup_{k=1}^{n} X^k \times \Sigma_k$: for each $k$ and $i$ identify $(x_1, \ldots, x_k, t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_k)$ with $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k, t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k)$ and call the corresponding topological space $\hat{X}^n$. Then properties (1.5) and (1.6) can be expressed by saying that there is a continuous map $\hat{C} : \hat{X}^n \to X$ such that $C = \hat{C} \circ \pi$, where $\pi : \bigcup_{k=1}^{n} X^k \times \Sigma_k \to \hat{X}^n$ is the projection provided by the identification.

A centroid $C$ is called uniform if the function $F(R, n)$ in (1.7) can be chosen independently of $n$. The following simple lemma is very useful for the construction of centroids.

Lemma 3. Let $X$ be a metric space in which balls of finite radius are compact. Then every 2-centroid on $X$ can be extended to a centroid.
Proof. We will use induction in \( n \), namely we set for \( n = 3, 4, \ldots \)

(1.9) \[
C(x_1, \ldots, x_n, t_1, \ldots, t_n) = \frac{C\left( C\left( x_1, \ldots, x_{n-1}, \frac{t_1}{t_1 + \cdots + t_{n-1}}, \ldots, \frac{t_{n-1}}{t_1 + \cdots + t_{n-1}} \right), x_n, t_1 + \cdots + t_{n-1}, t_n \right)}{t_1 + \cdots + t_{n-1}}
\]
if at least one of the numbers \( t_1, \ldots, t_{n-1} \) is positive and

(1.10) \[
C(x_1, \ldots, x_n, 0, 0, \ldots, 1) = x_n.
\]

We assume that conditions (1.5)-(1.7) hold for \( k \)-centroid \( k = 2, \ldots, n-1 \). Condition (1.5) for \( n \)-centroid follows immediately from (1.4) and (1.5) for 2-centroid. If \( k < n \) we have from (1.9) and (1.5) for \( (n-1) \)-centroid

\[
C(x_1, \ldots, x_n, t_1, \ldots, 0, \ldots, 0) = \frac{C\left( C\left( x_1, \ldots, x_{n-1}, \frac{t_1}{t_1 + \cdots + t_{n-1}}, \ldots, \frac{t_{n-1}}{t_1 + \cdots + t_{n-1}} \right), x_n, t_1 + \cdots + t_{n-1}, t_n \right)}{t_1 + \cdots + t_{n-1}}
\]

Continuity of \( n \)-centroid at every point except of \( (x_1, \ldots, x_n, 0, \ldots, 0, 1) \) follows directly from (1.9) and from the continuity of \( (n-1) \) and 2-centroids. In order to prove the continuity at \( (x_1, \ldots, x_n, 0, \ldots, 0, 1) \), let us notice that \( (n-1) \)-centroid maps a set \( U \times \sigma_{n-1} \) where \( U \) is a neighborhood of \( (x_1, \ldots, x_n) \) with compact closure into a compact closure into a compact set \( A \). Since 2-centroid is uniformly continuous on compact sets we have uniformly for \( x \in A \)

\[
\lim_{t \to 0} C(x, x', t, t') = C(x, x, 0, 1) = x_n.
\]

In order to prove (1.7) for \( n \)-centroid, let us assume that we have found a function \( F(R, n-1) \). We can also assume \( F(R, 2) \) and \( F(R, n-1) \) are non-decreasing. We then have:

(1.11) \[
d(C(x_1, \ldots, x_{n-1}, t_1, \ldots, t_{n-1}), x_n) \leq d_X(x_n, x_1) + d_X(C(x_1, \ldots, x_{n-1}, t_1, \ldots, t_{n-1}), x_1) \leq R + F(R, n-1)
\]

and from 1.9 for \( i = 1, \ldots, n-1 \)

\[
d_X(C(x_1, \ldots, x_i, t_1, \ldots, t_n), x_i) \leq \sum_{i=1}^{n-1} d_X(C(x_1, \ldots, x_i, t_1, \ldots, t_n), C(x_1, \ldots, x_{i-1}, t_1, \ldots, t_{i-1}))
\]

Using (1.7) for 2-centroid and \( (n-1) \)-centroid and 1.11 we see that the first term is estimated from above by \( F(F(R, n-1) + R, 2) \) and the second by \( F(R, n-1) \). By the same reason

\[
d_X(C(x_1, \ldots, x_n, t_1, \ldots, t_n), x_n) \leq F(F(R, n-1) + R, 2).
\]

Thus we can put

(1.12) \[
F(R, n) = F(F(R, n-1) + R, 2) + F(R, n-1)
\]

and this function is non-decreasing □
**Definition 6.** We will call a metric space $X$ uniformly locally compact if for any $r$, $R$ the maximal number of points in any $R$-ball in $X$ with pairwise distances $\geq r$ is bounded by a number depending only on $R$ and $r$.

**Lemma 4** (Extension Lemma). Let us assume that a metric space $X$ is uniformly locally compact and a space $Y$ admits a centroid. Then every pseudo-isometry $\varphi : \Gamma \to Y$ of a net $\Gamma \subset X$ into $Y$ can be extended to a pseudo-isometry $\tilde{\varphi} : X \to Y$ with the same constants $A$ and $B$ and probably different $C$.

**Proof.** Let $\alpha$ be a continuous non-negative function defined on the set of all positive numbers such that:

\begin{align}
(1.13) & \quad \alpha(t) \to \infty \quad \text{as} \quad t \to 0 \\
(1.14) & \quad \alpha(t) > 0 \quad \text{for} \quad 0 < t \leq R \quad \text{where} \quad R \quad \text{comes from} \quad (1.2) \\
(1.15) & \quad \alpha(t) = 0 \quad \text{for} \quad \text{all sufficiently large} \quad t, \quad \text{say for} \quad t \geq T.
\end{align}

Since $X$ is uniformly locally compact the net $\Gamma$ is at most countable. Actually it is infinite countable unless $X$ is compact in which case all our considerations become trivial. Let $\gamma_1, \gamma_2, \gamma_3, \ldots$ be an ordering of the elements of $\Gamma$. Let for $x \in X$

$$
\Phi_\alpha(x) = \{ \gamma \in \Gamma : \alpha(d_X(x, \gamma)) > 0 \} \cup (\Gamma \cap \{ x \}).
$$

By (1.2) and (1.14) the set $\Phi_\alpha(x)$ is non-empty for every $x$. Since $X$ is uniformly locally compact the number of elements in $\Phi_\alpha(x)$ (which we denote by $k(x)$) is bounded from above by a number $K$.

We can represent $\Phi_\alpha(x)$ in the following form $\{ \gamma_{i_1(x)}, \gamma_{i_2(x)}, \ldots, \gamma_{i_k(x)} \}$, where $i_1(x) < i_2(x) < \ldots < i_{k(x)}(x)$.

Let us denote for $j = 1, \ldots, k(x)$

\begin{align}
(1.16) & \quad \varphi(\gamma_{i_j}(x)) = \tilde{\varphi}_j(x) \quad \text{and} \\
(1.17) & \quad w_j(x) = \begin{cases} \\
\frac{\alpha(d_X(x, \gamma_{i_j}(x)))}{k(x)}, & \text{if} \quad x \notin \Gamma \\
\sum_{i=1}^{k(x)} \alpha(d_X(x, \gamma_{i}(x))), & \text{if} \quad x \in \Gamma
\end{cases}
\end{align}

It follows from (1.3) and (1.15) that all the points $\tilde{\varphi}_j(x)$ lie within at most $2(BT + C)$ from each other.

We set now

\begin{equation}
(1.18) \quad \tilde{\varphi}(x) = C(\tilde{\varphi}_1(x), \ldots, \tilde{\varphi}_{k(x)}(x), w_1(x), \ldots, w_{k(x)}(x)).
\end{equation}

The continuity of $\tilde{\varphi}$ follows directly from (1.16), (1.17) and the properties of the centroid. Note that (1.8) guarantees that $\tilde{\varphi}$ is an extension of $\varphi$, and (1.1) and (1.13) provide continuity of $\tilde{\varphi}$ at the points of $\Gamma$.

In order to check (1.3) let us take arbitrary two points $x_1, x_2 \in X$ and find points $\gamma_1, \gamma_2 \in \Gamma$ such that $d_X(x_i, \gamma_i) < R, \quad i = 1, 2$. Then all the points $\tilde{\varphi}_j(x_i), \quad j = 1, \ldots, k(x_i)$ lie within the distance $2(BT + C)$ from $\varphi(\gamma_i)$ and by (1.7) and (1.18) we have

$$
d_x(\tilde{\varphi}(x_i), \varphi(\gamma_i)) \leq F(2(BT + C), K)
$$

and furthermore
Let us assume that by uniqueness $K$:

$$d_N(\tilde{\nu}(x_1), \tilde{\nu}(x_2)) \leq d_N(\tilde{\nu}(x_1), \varphi(\gamma_1)) + d_N(\tilde{\nu}(x_2), \varphi(\gamma_2)) + d_\gamma(\varphi(\gamma_1), \varphi(\gamma_2))$$

$$< B d_X(\gamma_1, \gamma_2) + C + 2F(2(BT+C), K)$$

$$\leq B(d_X(x_1, x_2) + 2R) + C + 2F(2(BT+C), K)$$

$$= B d_N(x_1, x_2) + C'$$

where $C' = 2BR + 2F((2BT+C), K) + C$. Similarly, we obtain $d_\gamma(\tilde{\nu}(x_1), \tilde{\nu}(x_2)) \geq Ad_X(x_1, x_2) - (2AR + C + 2F(2(BT+C), K)) \geq Ad_X(x_1, x_2) - C'$.

\[\Box\]

2. Examples of Centroid

According to Lemma 3 in order to construct a centroid on a metric space $N$ it is enough to define a continuous map $C : N \times N \times [0,1] \to N$ such that

**(2.1)**

$$C(x_1, x_2, 1) = x_1, \quad C(x_1, x_2, 0) = x_2$$

and $d_N(x_1, x_2) < R$ implies

**(2.2)**

$$d_N(x_1, C(x_1, x_2, t)) < F(R)$$

We will show how to construct such a map in several important cases.

**Example 1.** Let $N$ be a complete Riemannian manifold such that every two of its points are connected by a unique geodesic.

Let us denote for $x_1, x_2 \in N$ by $G_{x_1,x_2}$ the geodesic connecting $x_1$ with $x_2$ provided with the length parameter. Since the length of this geodesic is equal to the distance $d_N(x_1, x_2)$ we can represent $G_{x_1,x_2}$ as a map

$$G_{x_1,x_2} : [0, d_N(x_1, x_2)] \to N$$

where

**(2.3)**

$$G_{x_1,x_2}(0) = x_1 \text{ and } G_{x_1,x_2}(d_N(x_1, x_2)) = x_2.$$  

Let us define a centroid $C : N \times N \times [0,1] \to N$ by

**(2.4)**

$$C(x_1, x_2, t) = G_{x_1,x_2}(td_N(x_1, x_2))$$

Property (2.1) immediately follows from (2.3); (2.2) with $F(R) = R$ folds because

$$d_N(x_1, C(x_1, x_2, t)) + d_N(x_2, G_{x_1,x_2}(t)) = d_N(x_1, x_2)$$

Thus, it is left to prove the continuity of $C$. It is obviously continuous for $x_1 = x_2$. So let us assume that $x_1 \neq x_2, x_1^{(n)} \to x_1, x_2^{(n)} \to x_2, t_n \to t$ and $G_{x_1^{(n)},x_2^{(n)}}(t_n)$ does not converge to $G_{x_1,x_2}(t)$. Since all the curves $G_{x_1^{(n)},x_2^{(n)}}$ lie in a compact part of $N$ which can be covered by a fixed number of coordinate charts one can use usual compactness argument in functional spaces to show that there is a subsequence of the sequence $G_{x_1^{(n)},x_2^{(n)}}$ which converges uniformly to a Lipschitz curve $K : [0, d_N(x_1, x_2)] \to N$ different from $G_{x_1,x_2}$. It is easy to see that $K(0) = x_1$, $K(d_N(x_1, x_2)) = x_2$. $d_N(x_1, K(t)) = t$ so that the length of $K$ is equal to $d_N(x_1, x_2)$. Thus, $K$ must be a geodesic and by uniqueness $K = G_{x_1,x_2}$.

Let us point out two particular cases to which the above construction applies.

**Example 1A.** $N$ is the universal covering of a compact Riemannian manifold of non-positive sectional curvature.

**Example 1B.** $N = G/K$ where $G$ is a connected semisimple Lie group, $K$ its maximal compact
subgroup. Any left-invariant metric on \( G \) which is also right-invariant with respect to \( K \) projects into a \( G \)-left invariant on \( N \).

**Example 2.** Let \( N \) be a connected Lie group of exponential type, i.e. the map \( \exp : \mathfrak{n} \to N \) from the Lie algebra of \( N \) into \( N \) is one-to-one, provided with a left-invariant Riemannian metric. In this case for every \( x \in N \) there is exactly one one-parameter subgroup \( \{ g_t^{(x)} \} \) of \( N \) such that \( x = g_1(x) \).

Let us define

\[
C(x_1, x_2, t) = x_1 g_t(x_1^{-1} x_2)
\]

(2.5)

Since for every \( x \in N \), \( g_0(x) = e \), \( g_1(x) = x \), condition (2.1) follows immediately from (2.5). Continuity is also obvious in this case because \( g_t(x) \) depends continuously on both \( x \) and \( t \). To verify (2.2) let us remark that since both the Riemannian metric on \( N \) and the centroid \( C \) are left-invariant one has

\[
C(x_1, x_2, t) = x_1 C(e, x_1^{-1} x_2, t)
\]

where \( d_N(e, x_1^{-1} x_2) = d_N(x_1, x_2) \) and

\[
d_N(x_1, C(x_1, x_2, t)) = d_N(e, C(e, x_1^{-1} x_2, t)),
\]

(2.6)

\[
d_N(x_2, C(x_1, x_2, t)) = d_N(x_1^{-1} x_2, C(e, x_1^{-1} x_2, t)).
\]

(2.7)

But if \( d_N(x_1, x_2) < R \), all elements present in the right hand parts of (2.6) and (2.7) lie in the image of an \( R \)-ball in \( \mathfrak{n} \) under the exponential map. This image is a compact set and consequently is contained in a \( d_N \) ball about \( e \). We can choose the radius of that ball as \( F(R) \).

**Example 3.** Examples 1B and 2 can be generalized in the following way.

Let \( N \) be a metric space which is homeomorphic to Euclidean space \( \mathbb{R}^m \) and which has a transitive group of isometries. Then \( N \) can be represented as \( I(N)/I_0 \) where \( I(N) \) is the connected component of the identity in the group of isometries of \( N \) and \( I_0 \) is the stabilizer of a point \( x_0 \) in \( I(N) \). \( I(N) \) is a Lie group and \( I_0 \) is its Lie subgroup so that \( I(N) \) is a locally trivial fibered bundle over \( N \). The fiber over a point \( x \in N \) consists of all isometries from \( I(N) \) which map \( x \) into \( x_0 \). Since \( N \) is contractible this fibered bundle is trivial, i.e. there is a continuous section \( \psi : N \to I(N) \).

Obviously, for \( x \in N \)

\[
\psi(x) \cdot x = x_0
\]

(2.8)

Let us fix a homeomorphism \( \Phi : \mathbb{R}^m \to N \) which maps the origin into \( x_0 \). Let for \( x \in N \)

\[
\Phi^{-1}(x) = (s_1(x), \ldots, s_m(x))
\]

and for \( t \in \mathbb{R} \)

\[
g_t(x) = \Phi(ts_1(x), \ldots, ts_m(x))
\]

(2.9)

Obviously

\[
g_0(x) = x_0, \quad g_1(x) = x
\]

(2.10)

We are now ready to define a centroid:

\[
C(x_1, x_2, t) = (\psi(x))^{-1}(g_{1-t}(\psi(x_1)x_2))
\]

(2.10)

Condition (2.1) follows directly from (2.10), (2.9) and (2.8). For,

\[
C(x_1, x_2, 1) = (\psi(x_1))^{-1}(g_0(\psi(x_1)x_2)) = (\psi(x_1))^{-1}x_0 = x_1
\]
and similarly
\[ C(x_1, x_2, 0) = (\psi(x_1))^{-1}(g_1(\psi(x_1)x_2)) = (\psi(x_1))^{-1}\psi(x_1)x_2 = x_2 \]
Continuity follows from the continuity of the section \( \psi \) and from (2.10). Finally, (2.2) can be proved as in Example 2. Namely,
\[ C(x_1, x_2, t) = (\psi(x_1))^{-1}C(x_0, \psi(x_1)x_2) \]
and since \( \psi(x_1) \) is an isometry
\[
\begin{align*}
  d_N(x_0, \psi(x_1)x_2) &= d_N(x_1, x_2) \\
  d_N(x_1, C(x_1, x_2, t)) &= d(x_0, C(x_0, \psi(x_1)x_2, t)) \\
  d_N(x_2, C(x_1, x_2, t)) &= d(\psi(x_1)x_2, C(x_0, \psi(x_1)x_2, t))
\end{align*}
\]
But since \( \Phi \) is a homeomorphism, the preimage of the \( R \)-ball around \( x_0 \) in \( d_N \)-metric is contained in a Euclidean ball of some radius, say \( T(R) \); the point \( C(x_0, \psi(x_1)x_2, t) \) belongs to the image of that ball which is compact a consequently is contained in some ball in \( d_N \)-metric. Let us denote the radius of this last ball by \( F(R) \).
The most important particular case of the situation described above appears in the following context:

Example 3A. Let \( G \) be a connected Lie group, \( K \subset G \) - a maximal compact subgroup \( N = G/K \) - the homogeneous space provided with a metric invariant with respect to the left action of \( G \). Then \( N \) is homeomorphic to a Euclidean space and consequently it admits a centroid.

3. HOMOTOPY ARGUMENT

In this section we find conditions which guarantee that in the situation described in Example 3, the extension described in Lemma 4 is a surjective map.

**Theorem 1.** Let \( M, N \) be two complete metric spaces homeomorphic to \( \mathbb{R}^m \) and \( \mathbb{R}^n \) correspondingly. Let us assume that \( M \) has a transitive group of isometries. Let, furthermore, \( \varphi : M \to N \) be a continuous map such that for every \( x, y \in M \)
\[ d_N(\varphi(x), \varphi(y)) \geq Ad_M(x, y) - C \]
for some constants \( A, C \). Then
\[ (3.1) \quad n \geq m \]
\[ (3.2) \quad \text{If } m = n, \text{ then } \varphi(M) = N \]

**Proof.** Let us fix a point \( x_0 \in M \), a homeomorphism \( \Phi : M \to \mathbb{R}^m \) which maps \( x_0 \) to the origin and a continuous section \( \psi : M \to I(M) \) of the fibered bundle \( I(M) \to M \).
Let us consider in the Cartesian square \( M \times M \) the “thickened diagonals” with respect to \( d_M \)
\[ \delta_R = \{(x, y) \in M \times M : d_M(x, y) \leq R\} \]
and also Euclidean “thickened diagonals”
\[ \Delta_R = \{(x, y) \in M \times M : \Phi(\psi(x)y) \in B_R\} \]
where \( B_R \) is the Euclidean \( R \)-ball in \( \mathbb{R}^m \) around the origin.
Since \( \Phi \) is a homeomorphism and \( \psi \)’s are isometries one can easily see that for every \( R > 0 \) one can find \( f(R) \) such that
\[ (3.3) \quad \Delta_R \subset \delta_{f(R)} \]
and

\[ \delta R \subset \Delta_{f(R)} \]

The set \( M_R = (M \times M) \setminus \Delta_R \) for every \( R \) is a deformation retract of \( (M \times M) \setminus \Delta = \{(x, y) : x \neq y\} \) via \( (x, y) \to x \), \( \psi(x)^{-1} \Phi^{-1}(i \Phi \psi(x)y) \) and thus \( M_R \) is homotopically equivalent to \( (M \times M) \setminus \Delta \). Furthermore, the latter set is homotopically equivalent to the sphere \( S^{m-1} = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : \sum_{i=1}^m x_i^2 = 1\} \) via the map

\[ \nu_M : (x, y) \mapsto \frac{\Phi(x) - \Phi(y)}{||\Phi(x) - \Phi(y)||}. \]

Let \( \sigma_M : M \times M \to M \times M \) be the standard involution given \( \sigma_M(x, y) = (y, x) \). Obviously, \( \nu_M(\sigma_M z) = -\nu_M(z) \).

**Lemma 5.** Given any number \( R \) there exists a continuous map \( \mu : S^{m-1} \to M_R \) such that

\[ \mu(-x) = \sigma_M(\mu x) \]

**Proof.** Since \( \sigma_M \delta R = \delta R \), it follows from (3.3) and (3.4) that one can find positive numbers \( R_0 = R < R_1 < \cdots < R_m \) such that

\[ \sigma_M \Delta_{R_i} \subset \Delta_{R_{i+1}} \quad i = 0, \ldots, m - 1. \]

Let us take a point \( z \in M_{R_m} \) so that by (3.6) \( \sigma_M z \in M_{R_{m-1}} \). Since the set \( M_{R_{m-1}} \) is homotopically equivalent to \( S^{m-1} \) it is pathwise connected so that we can connect \( z \) and \( \sigma_M z \) in \( M_{R_{m-1}} \) by a path \( \lambda_0 \). By (3.6) \( \sigma_M \lambda_0 \subset M_{m-2} \). Clearly, the union \( \lambda_0 \cup \sigma_M \lambda_0 \) defines a continuous map

\[ \mu_1 : S^1 \to M_{R_{m-2}} \]

which is skew-symmetric, i.e.

\[ \mu_1(-x) = \sigma_M \mu_1(x). \]

Continuing by induction in dimension and using (3.6) we can construct maps \( \mu_i : S^i \to M_{R_{m-i-1}} \) for \( i = 1, \ldots, m - 1 \) such that

\[ \mu_i(-x) = \sigma_M \mu_i(x) \]

For, assuming that \( \mu_{i-1} \) has been constructed and using the fact that \( \pi_{i-1}(M_{R_{m-i}}) = 0 \), we can extend \( \mu_{i-1} \) to a continuous map

\[ \lambda_{i-1} : S^i_+ \to M_{R_{m-i}} \]

where \( S^i_+ = \{(x_1, \ldots, x_{i+1}) \in S^i : x_{i+1} \geq 0\} \). Furthermore, since by (3.6) \( \sigma_M(\lambda_{i-1}(M_{R_{m-i}})) \subset M_{R_{m-i}} \), we can extend \( \lambda_{i-1} \) to a skew-symmetric continuous map \( \mu_i : S^i \to M_{R_{m-i}} \) by setting

\[ \mu_i(-x) = \begin{cases} \lambda_{i-1}(x) & \text{if } x \in S^i_+ \\ \sigma_M \lambda_{i-1}(-x) & \text{if } x \in S^i \setminus S^i_+ \end{cases} \]

Finally we can put \( \mu = \mu_{m-1} \).

If the number \( R \) in Lemma 5 is chosen sufficiently large, e.g. \( R > f \left( \frac{2C}{A} \right) \), where the function \( f \) comes from (3.3) and (3.4), then

\[ (\varphi \times \varphi)M_R \subset (N \times N) \setminus \Delta \]

Consequently we can construct the composition map:
Theorem 2. Let \( M = G/K \), \( N = H/L \) be the factors of connected Lie groups \( G \), \( H \) by their maximal compact subgroups \( K \) and \( L \). Let us fix on \( M \) and \( N \) a \( G \) and \( H \)-invariant Riemannian metrics correspondingly. Let \( \Gamma \subset M \) be a net with the spanning constant \( R \) and \( \varphi : \Gamma \to N \) be a pseudo-isometry. Then

\[
(4.1) \quad \dim N \geq \dim M
\]
\( (4.2) \) \( \varphi \) can be extended to a continuous pseudo-isometry \( \bar{\varphi} : M \to N \)

\( (4.3) \) If \( \dim N = \dim M \), then \( \bar{\varphi}(M) = N \) and the set \( \varphi(\Gamma) \) spans \( N \) with the spanning constant \( R_1 \leq BR + C \) where the constants \( B, C \) are determined by the pseudo-isometry \( \varphi \) via \((1.3)\).

**Proof.** Since the space \( M \) admits a transitive group of isometries it is uniformly locally compact (cf. Definition 6). The space \( N \) admits a centroid (cf. Example 3A). Thus, we can apply Lemma 4 and extend \( \varphi \) to a continuous pseudo-isometry \( \bar{\varphi} : M \to N \). All assumptions of Theorem 1 hold for the map \( \bar{\varphi} \). Thus \( \dim N \geq \dim M \) and if \( \dim N = \dim M \) then \( \bar{\varphi}(M) = N \) there exists \( \gamma \in \Gamma \) such that \( d_M(x, \gamma) < R \). Consequently by \((1.3)\) one has

\[
d_N(\varphi(\gamma), y) = d_N(\varphi(\gamma), \bar{\varphi}(x)) \leq Bd_M(x, \gamma) + C < BR + C
\]

\hfill \Box

**Theorem 3.** Let \( \Gamma \) be a net in a connected Lie group \( G \) provided with a left-invariant Riemannian metric

(a) If \( \varphi : \Gamma \to G \) is a pseudo-isometry, then \( \varphi(\Gamma) \) spans \( G \).

(b) If \( \varphi : \Gamma \to G \) is a quasi-isometric embedding then \( \varphi(\Gamma) \) is a net in \( G \).

In both cases \((a)\) and \((b)\) the spanning constant for \( \varphi(\Gamma) \) is determined by \( G \), the spanning constant for \( \Gamma \) and the constants \( B, C \) from \((1.3)\).

**Proof.** If \( \varphi \) is a quasi-isometric embedding, then \( \varphi(\Gamma) \) is obviously uniformly discrete. Thus, it is enough to prove \((a)\).

Let \( K \) be a maximal compact subgroup of \( G \), \( \pi : G \to N = G/K \) be the standard projection. Since all left-invariant metric on \( G \) are equivalent, we can assume that the chosen metric \( d_G \) is a two-sided \( K \)-invariant so it generates a metric \( d_N \) on \( N \) invariant with respect to the left action of \( G \). Let \( D \) be the diameter of \( K \) in \( G \). Then we have for \( g_1, g_2 \in G \)

\( (4.4) \)

\[
d_G(g_1, g_2) - D \leq d_N(\pi g_1, \pi g_2) \leq d_G(g_1, g_2)
\]

These inequalities imply that both \( \pi \) and \( \pi \circ \varphi : \Gamma \to N \) are pseudo-isometries. By Lemma 1 there exists a next \( \Gamma' \subset \Gamma \) in \( G \) such that the restriction of \( \pi \) to \( \Gamma' \) is a quasi-isometric embedding. By the same lemma there is another net \( \Gamma'' \subset \Gamma' \) such that \( \pi \circ \varphi|_{\Gamma''} : \Gamma'' \to N \) is a quasi-isometric embedding. Consequently

\[
\pi \varphi \pi^{-1} : \pi(\Gamma'') \to N
\]

is also a quasi-isometric embedding, its image being \( \pi(\varphi(\Gamma'')) \). Theorem 2 says then that \( \pi(\varphi(\Gamma'')) \) spans \( N \) and by \((4.4)\) \( \varphi(\Gamma'') \) and hence \( \varphi(\Gamma) \), spans \( G \).

\hfill \Box

5. Applications to Ergodic Theory

Let \((X, \mu)\) be a Lebesgue measure space, i.e. a separable complete non-atomic probability measure space and let \( \delta = \{ S_\gamma \}_{\gamma \in \Gamma} \) be a measurable right action of a locally compact second countable group \( \Gamma \) on \( X \) be non-singular transformations. Let \( G \) be another locally compact second countable group. A measurable function \( \alpha : X \times \Gamma \to G \) is called a \( G \)-cocycle over the action \( g \) if for a.e. \( x \in X \) and for every \( \gamma_1, \gamma_2 \in \Gamma \)

\( (5.1) \)

\[
\alpha(x, \gamma_2 \gamma_1) = \alpha(x, \gamma_1) \alpha(\delta_{\gamma_1} x, \gamma_2)
\]

The construction of Mackey range \([5], [3] \) Section 8, allows us to associate with any \( G \) cocycle \( \alpha \) over a right \( \Gamma \) action a left action of \( G \). Namely we first determine a \( G \)-extension \( g^\alpha = \{ S^\alpha_\gamma \}_{\gamma \in \Gamma} \) of \( g \) which acts on \( X \times G \) by

\( (5.2) \)

\[
S^\alpha_\gamma(x, g) = (\delta_\gamma x, g \alpha(x, \gamma))
\]
It is easy to see that the cocycle equation (5.1) is equivalent to the group property for that extension \( S^\alpha_1, S^\alpha_2 = S^\alpha_{2 \gamma_1} \).

The group \( G \) acts on \( X \times G \) by the left shifts \( L_{g_0} : L_{g_0}(x, g) = (x, g_0 g) \) and this action obviously commutes with the extension \( S^\alpha \). In particular, this action maps orbits of \( S^\alpha \) into orbits and thus we can consider the factor action of \( G \) in the space of orbits of \( S^\alpha \). This action is called Mackey range of \( \alpha \) and is denoted by \( L^\alpha \). In general the space of \( S^\alpha \) orbits may not have good measurable structure and event if it has such a structure and \( S \) is measure preserving, the natural \( L^\alpha \) invariant measure may be infinite. We will give a sufficient condition which guarantees that the factor space has a structure of Lebesgue space with a natural finite invariant measure.

Let us assume that \( \Gamma \) is finitely generated discrete group and that \( G \) is a locally compact Lie group. The word-length metrics on \( \Gamma \) determined by different systems of generators are equivalent in the sense that the identity map is a quasi-isometry. Similarly, all left-invariant Riemannian metrics on \( G \) are equivalent. Thus the notions of quasi-isometric embedding and pseudo-isometry from \( \Gamma \) to \( G \) are intrinsically defined.

Any left-invariant metric on \( \Gamma \) can be transferred to any orbit of a right \( \Gamma \) action. Thus, in our case the orbits of the action \( S \) are provided with a natural class of metrics defined up to a quasi-isometry.

**Definition 7.** A \( G \)-cocycle \( \alpha \) over a \( \Gamma \) action \( S \) is called a Lipschitz cocycle if for almost every \( x \in X \) the map \( \alpha_x : \Gamma \to G, \alpha_x(\gamma) = \alpha(x, \gamma) \) is a pseudo-isometry with constants \( A, B, C \) (cf. (1.3)) independent on \( x \).

**Theorem 4.** Let \( \Gamma \) be a uniform lattice in a connected Lie group \( G \) and let \( \alpha \) be a Lipschitz \( G \) cocycle over a right measurable non-singular action \( S \) of \( \Gamma \) on a Lebesgue space \( (X, \mu) \). Then the \( \Gamma \)-action \( S^\alpha \) has a measurable fundamental domain \( D = \bigcup_{x \in X} \{x\} \times D_x \) where all sets \( D_x \) are uniformly bounded and each \( D_x \) has a boundary of codimension one. Consequently if the action \( S \) preserves the measures \( \mu \) and if the group \( G \) is unimodular, the restriction of the measure \( \mu \times \chi_G \) (\( \chi_G \) is the Haar measure of \( G \)) to \( D \) determines a finite invariant measure for the Mackey range \( L^\alpha \).

**Proof.** It follows from the cocycle equation (5.1) that \( \alpha(x, \text{id}_\Gamma) = \text{id}_G \) and

\[
\alpha(x, \gamma^{-1}) = \alpha(S_{\gamma^{-1}}, \gamma)^{-1}
\]

By Theorem 3 the set \( \alpha_x(\Gamma) \) for almost every \( x \in X \) spans \( G \) with a uniformly bounded spanning constant with respect to any left invariant Riemannian metric on \( G \). The inversion \( g \to g^{-1} \) maps left-invariant metrics on \( G \) into right invariant ones. From now on we will work with a fixed right-invariant Riemannian metric \( d_G \) on \( G \). Thus, by the above remark for almost every \( x \in X \), the set \( (\alpha_x(\Gamma))^{-1} \) spans \( G \) with respect to \( d_G \).

Let \( D_x(\gamma) \) be the Dirichlet region of the point \( (\alpha_x(\gamma))^{-1} \), i.e.

\[
D_x(\gamma) = \{g \in G : d_G(g, \alpha(x, \gamma)^{-1}) \leq d_G(g, \alpha(x, \gamma')^{-1}) \text{ for all } \gamma' \in \Gamma\}
\]

Since \( (\alpha_x(\Gamma))^{-1} \) is a discrete set, the boundary of each set \( D_x(\gamma) \) has codimension one. Theorem 3 guarantees that all sets \( D_x(\gamma) \) are compact and have bounded diameters.

We obtain, using (5.1), (5.3) and the invariance of \( d_G \) with respect to right multiplication on \( G \), that \( D_x(\gamma) \cdot \alpha(x, \gamma) \) is exactly:

\[
\{g \in G : d_G(g(\alpha(x, \gamma)^{-1}), \alpha(x, \gamma)^{-1}) \leq d_G((g\alpha(x, \gamma)^{-1}, (\alpha(x, \gamma')^{-1}) \text{ for all } \gamma' \in \Gamma\}
= \{g \in G : d_G(g, \text{id}) \leq d_G(g, \alpha(x, \gamma')^{-1} \alpha(x, \gamma) \text{ for all } \gamma' \in \Gamma\}
= \{g \in G : d_G(g, \text{id}) \leq d_G(g, \alpha(S_x, \gamma' \gamma^{-1})^{-1} \text{ for all } \gamma' \in \Gamma\}
= D_{S^{-1}}(\text{id})
\]

and by (5.2)
Thus, every orbit of $S^\alpha$ visits the set

\begin{align}
S^\alpha_\gamma(\{x\} \times D_x(\gamma)) &= \{S_\gamma x\} \times D_{S_\gamma x}(\text{id}) \\
\text{(5.4)} \\
D &= \bigcup_{x \in X} \{x\} \times D_x(\text{id})
\end{align}

at least once.

If we assume that for every $\gamma \neq \text{id}_G$, $d(x, \gamma) \neq \text{id}_G$ almost everywhere then the sets $D_x(\gamma)$ form a partition of $X \times G$ up to a set of measure zero and the set defined by (5.4) is a fundamental domain for $S^\alpha$ satisfying all conditions of the theorem.

In a general situation the Lipschitz condition from Definition 7 guarantees that the following equivalence relation has finite equivalence classes:

$x \sim y$ if $y = S_\gamma x, \alpha(x, \gamma) = \text{id}$

Moreover, the partition of $X$ into equivalence classes is measurable. Let us choose a measurable set $A \subset X$ which intersects each equivalence class by exactly one point and define

$$D = \bigcup_{x \in A} \{x\} \times D_x(\text{id})$$

This set is obviously a fundamental domain for $S^\alpha$ and satisfies all conditions of the theorem.

Remark 1. Construction used in the proof of Theorem 4 is a straightforward modification of the construction from [2], Proposition 1, which deals with the case $\Gamma = \mathbb{Z}^n, G = \mathbb{R}^n$. Our Theorem 3 replaces the ergodic theorem and elementary index arguments in $\mathbb{R}^n$ used in [2].

Another application of our results to ergodic theory involves the extension of the notion of Kakutani equivalence of group action to various classes of non-abelian groups. The construction is described in Section 8 of [3]. Details will appear in a separate paper.

\[ \square \]

References