FIVE MOST RESISTANT PROBLEMS
IN DYNAMICS

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1. Coexistence of KAM circles and positive entropy in area preserving twist maps

The standard area preserving map $f_\lambda$ of the cylinder $C = S^1 \times \mathbb{R}$:

$$f_\lambda(x, y) = (x + y, y + \lambda \sin 2\pi(x + y)).$$

**Problem 1** Is metric entropy $h_{\text{area}}(f_\lambda)$ positive

(i) for small $\lambda$, or (ii) for any $\lambda$ (assuming $y$ is periodic too)?

"Yes" implies existence of ergodic components of positive measure. (Pesin, 1977.)

1954–1962  **Kolmogorov, Moser.** Existence of invariant curves for small $\lambda$ and around elliptic points.

1960s  **Sinai** attempts to solve (i) and formulates first important ideas of smooth ergodic theory.
EXPECTED ANSWER: Coexistence possible ("yes" for (i)).

HEART OF THE DIFFICULTY: if true, estimates unimaginable.

RELATED EVIDENCE: Positive topological entropy, homoclinic points, Melnikov method

HOPE FOR PROBLEM (ii): Parameter exclusion techniques,

ATTEMPTS: Lazutkin, Noble, Kosygin–Sinai.

POSSIBLE SHORTCUT: random perturbations.
2. Smooth realization of measure preserving transformations

**Problem 2** Given an ergodic measure preserving transformation $T$ of a Lebesgue measure space $X$ with measure $\mu$, under what conditions does there exist a diffeomorphism $f$ of a compact manifold $M$ preserving a smooth volume $v$ such that $(f, v)$ is measurably isomorphic to $(T, \mu)$? In particular, is there any $T$ with finite entropy, $h_\mu(T) < \infty$, for which such an $f$ does not exist?

1965 Kushnirenko. Finiteness of entropy: $h_v(f) < \infty$.
True for any Borel $f$-invariant measure $\nu$.

1970 Anosov–A.K. Non-standard smooth realizations, e.g. translations on $\infty$-dimensional torus on the disc.

1977 Pesin. $\dim M = 2$, $f$ weakly mixing implies $f$ is Bernoulli.
EXPECTED ANSWER: there are universal obstructions.

HEART OF THE DIFFICULTY: no good candidates for invariants exist or even imagined.

RESTRICTION IN LOW DIMENSION: circle maps, flows on surfaces. Most interesting: (Almost) No smooth realization for a Diophantine rotation on the disc (Herman’s last geometric theorem).

HOPE IN DIMENSION THREE: $K$ implies Bernoullii (stable ergodicity theory).
POSSIBLE REALIZATION RESULTS.

Several characteristic problems which in our view can be approached by a version of the approximation by conjugation method or its modification with decreasing chances of success.

Find a smooth realization of:

A Gaussian dynamical system with simple (Kronecker) spectrum (just solved: A.K.-A.Windsor, 2007);

A dense $G_\delta$ set of minimal interval exchange transformations;

An adding machine;

The time-one map of the horocycle flow on the modular surface $SO(2)\backslash SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ (which is not compact, so the standard realization cannot be used).
3. Orbit growth, existence of periodic orbits and ergodicity of billiards in polygons

Consider the billiard motion inside a polygon $P \subset \mathbb{R}^2$. Let $S(T)$ be the number of orbits of length $\leq T$ which begin and end in vertices.

**Problem 3** (i) Find above and below estimates for $S(T)$.

(ii) Is there a periodic billiard orbit for any $P$?

(iii) Find conditions for ergodicity of the billiard flow with respect to Liouville measure. In particular is the billiard ergodic for almost every $P$?


Dense $G_\delta$ set of ergodic polygons.

1987 A.K. Subexponential estimate: $T^{-1} \log S(T) = 0$.

1990 Masur. For rational polygons $C_1T^2 \leq S(T) \leq C_2T^2$. 
EXPECTED ANSWERS: (i) Orbit growth slower than $T^{2+\epsilon}$ (although not quadratic)

(ii) More likely than not periodic orbit exist; proved for triangles with angels $< 100^\circ$ (R. Schwartz, 2005)

(iii) Typical billiards are ergodic.

HEART OF THE DIFFICULTY: Lack of structure (unlike the rational case)

For the periodic orbit problem the behavior is parabolic; no periodic orbits in other (more characteristic) parabolic systems.
4. Invariant measures for hyperbolic actions of higher rank abelian groups

Key examples:

(i) $\times 2, \times 3$ is the action of $\mathbb{Z}^2_+$ on $S^1$ generated by $E_2 : x \mapsto 2x \pmod{1}$ and $E_3 : x \mapsto 3x, \pmod{1}$.

(ii) Let $M = SL(n, \mathbb{R})/\Gamma$, $n \geq 3$, $\Gamma$ a lattice in $SL(n, \mathbb{R})$, $D \subset SL(n, \mathbb{R})$ positive diagonals isomorphic to $\mathbb{R}^{n-1}$. The Weyl chamber flow (WCF), is the action of $D$ on $M$ by left translations.

**Problem 4** Find all ergodic invariant measures for the $\times 2, \times 3$ and the Weyl chamber flow.

1967 Furstenberg asks the question for $\times 2, \times 3$.

1990 Rudolph. The only positive entropy (P. E.) ergodic measure for $\times 2, \times 3$ is Lebesgue.

POSSIBLE APPROACHES: harmonic analysis, geometric.

HEART OF THE DIFFICULTY: For harmonic analysis approach: poor description of measures among distributions; there are many of those.

For geometric approach: lack of structure to use hyperbolicity in the zero entropy case.

HOPE FOR INTERMEDIATE RESULTS: Larger semi-groups, restriction on slow entropy of invariant measures.
5. Topological classification of Anosov diffeomorphisms and differentiable classification of Anosov actions of $\mathbb{Z}^k k \geq 2$

Let $N$ be a simply connected nilpotent group, $\Gamma \subset N$ a lattice in $N$.

**Problem 5** Is every Anosov diffeomorphism of a compact manifold $M$ topologically conjugate to a finite factor of an automorphism of a nil-manifold $N/\Gamma$?

1967 **Smale** (acknowledging A. Borel’s contribution) constructs examples on nilmanifolds which are not tori and asks the question.

1970 **Franks** develops crucial machinery and solves special cases.

1974 **Manning** solves the problem if $M$ is a finite factor of $N/\Gamma$. 
HEART OF THE DIFFICULTY: Lack of understanding of topology for possible counterexamples.

RELATED PROBLEMS (Possibly more accessible):

(i) Is every Anosov action of $\mathbb{Z}^k$, $k \geq 2$ a compact manifold $M$ without rank one factors differentiably conjugate to a finite factor of an action by affine maps of a nil-manifold $N/\Gamma$?

(ii) Is every Anosov diffeomorphism of a compact manifold $M$ with smooth stable and unstable foliations differentiably conjugate to a finite factor of an automorphism of a nil-manifold $N/\Gamma$?

HOPE for (i) and (ii): find invariant geometric structures.