OPEN PROBLEMS IN ELLIPTIC DYNAMICS

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Elliptic dynamics: difficult to define rigorously. Key phenomenon is slow orbit growth. See [10, Section 7] for a detailed discussion.

A not unrelated area is *parabolic dynamics*: intermediate (polynomial) orbit growth. For example, the growth of vectors under action of a unipotent matrix. In the nonlinear case the matrix Df still looks like a Jordan block. See [10, Section 8]

The problem for elliptic dynamics is that the linear model is rotation, with no growth at all, which is too restrictive. We will come to that later.

Characterize slow orbit growth of a smooth system by $h_{\text{top}} f = 0$. This of course includes both elliptic and parabolic situations. How can this happen?

Example 1. Translation on a torus; the ultimate example of an elliptic system. A continuous-time version of this is linear flow; any time change also works, since a time change preserves zero entropy. That is, $\mathbf{v}$ a vector field, $g\colon \mathbb{T}^n \to \mathbb{R}$ a function, then $g\mathbf{v}$ is a vector field defining a system which still has zero entropy.

Example 2. If $f\colon M \to M$ has zero entropy, let B be a Riemannian bundle over M with fiber N with projection map π, and let $F\colon B \to B$ be such that $f\pi = \pi F$ and the action on N is by isometries. Then the entropy of this *isometric extension* is zero.

Example 3. In *low dimension* (2 for maps, 3 for flows), carry out a limit process. (Invertibility plus *very low dimension*, i.e. 1 for maps, 2 for flows, implies zero entropy). h_{top} is lower semicontinuous on $C^{1+\varepsilon}$, so

$$h_{\text{top}}(\lim f_n) \leq \liminf h_{\text{top}} f_n.$$

Proof of this: $h_{\text{top}} f = \sup\{h_{\text{top}}(f|\Lambda) \mid \Lambda \text{ an invariant hyperbolic set}\}$ [12], and such sets persist under small perturbations. (In C^∞, h_{top} is upper semi-comtinuous and hence continuous in low dimension).

Example 4. This gives non-trivial examples constructed by approximation by conjugation method [1, 3]. Let S_t an S^1 action on M, $\mathcal{A}$ the closure of

$$\{\varphi S_t \varphi^{-1} \mid \varphi \in \text{Diff}(M),\ t \in S^1\}$$

in $C^{1+\varepsilon}$. Then every $f \in \mathcal{A}$ has zero entropy. Can construct topologically transitive, ergodic examples, etc. Basic low-dimensional cases are
(i) rotations on a disc, annulus, sphere, projective plane, and torus;

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(ii) S^1 fiber actions on S^1 principal bundles over surfaces, such as the unit tangent bundle.

$\mathcal{A} \supset \bigcup_{\alpha\in S^1} \mathcal{A}_\alpha$, where $\mathcal{A}_\alpha$ is the closure of $\{\varphi S_\alpha \varphi^{-1} \mid \varphi \in \mathrm{Diff}(M)\}$. It is not known but very likely that inclusion is proper, We will return to that.

Example 5. More general than last example. Integrable systems; i.e systems that have a first integral, that is not constant on any open set; singular levels can create entropy only in higher dimensions.

Problem 1. *Is it true that in low dimensions, every conservative (e.g. volume-preserving) map/flow with zero entropy is a limit of integrable systems?*

I hesitate to call this "conjecture" since at present I have no bias towards one answer over the other.

Example 6. Horocycle flow (no counterpart in two-dimensional discrete-time case). $M = SL(2,\mathbb{R})/\Gamma$, where Γ is cocompact with no elliptic elements. Action on left by $\left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)$. This is a parabolic system. The vector field is generated by $h = \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$. A parabolic matrix is a limit of elliptic matrices, so this system is a limit of integrable systems: take

$$\begin{pmatrix} 1-\varepsilon & t \\ (-2\varepsilon+\varepsilon^2)/t & 1-\varepsilon \end{pmatrix}$$

Integrable system – periods of all orbits are the same – *periodic system*. Time-t map H_t of the horocycle flow is a limit of periodic maps. In fact, this fits into the scheme of Example 4 with the action of S^1 of type (ii). Notice that $H_t = \lim \varphi_n S_{\alpha_n} \varphi_n^{-1}$ where $\alpha_n \to$).

Conjecture 1. $H_t \notin \bigcup_{\alpha\in S^1} \mathcal{A}_\alpha$

Proof of this in [3] contains an error, see
http://www.math.psu.edu/katok_a/pub/Herman-survey-horo-corrected.pdf

Example 7. Flows on surfaces of genus $g \geq 2$. Return map to a section has discontinuities, preserves a smooth parameter: *interval exchange transformation*. The original flow is a suspension flow under a function over an IET. IETs are approximable by integrable systems; approximate the points of discontinuity by rational numbers.

All this fails in higher dimensions, as semicontinuity of entropy no longer holds.

Example 8. A family of four-dim maps $f_t \colon M \to M$ such that $h_{\mathrm{top}} f_t = 0$ for irrational t, and positive entropy for rational t. Continuous at irrationals. For Diophantine t, f_t is integrable, i.e. splits into invariant two-dimensional tori with translations. Perhaps, this is true even for *all* irrationals.

$M = S^1 \times (SL(2,\mathbb{R})/\Gamma)$, and $\varphi \colon S^1 \to S^1$ a smooth function with zero average. $g_t = \left(\begin{smallmatrix}e^{t/2} & 0\\ 0 & e^{-t/2}\end{smallmatrix}\right)$ is geodesic flow. Then

$$f_t(\theta, x) = (\theta + t, g_{\varphi(\theta)} x).$$

t irrational implies no exponential growth (Birkhoff, Ruelle). If $\varphi(\theta) = H(\theta + t) - H(\theta)$, we can straighten out the orbits and get an integrable system. φ a trigonometric polynomial implies this is possible for all t. If t is rational the entropy is the average of φ^+.

Related to this are examples of integrable systems (in fact geodesic flows with positive topological entropy, [2]. Interestingly, the first integrals are C^∞, not C^ω.)

Elliptic dynamics splits into two paradigms: Diophantine ("regular" or "stable") and Liouvillean ("exotic" or "pathological") [10, Section 7]. Dichotomy for irrational numbers:

Diophantine: $\alpha \in \mathcal{D}$ if $|\alpha - p/q| \geq Cq^{-r}$.

Liouvillean: $\alpha \in \mathcal{L}$ if $<$ for all r.

The space $\mathcal{A}$ and its subspaces $\mathcal{A}_\alpha$, $\alpha \in \mathcal{L}$ from before is a Liouvillean playground.

$\mathcal{A}_\alpha = \overline{\{\varphi S_\alpha \varphi^{-1}\}}$; $\alpha \in \mathcal{L}$ implies topological transitivity, mixing, etc. are generic in $\mathcal{A}_\alpha$.

Question: *IS THERE NON-TRIVIAL DIOPHANTINE BEHAVIOR?*

"Non-trivial" is probably better rendered as "non-standard". The power of Diophantine paradigm is that it asserts remarkable stability of the the single simple model: translation or a vector field with a *Diophantine vector*: $\alpha = (\alpha_\mathbf{1}, \dots, \alpha_\mathbf{n})$, $\sum_{i=1}^n k_i\alpha_i - k > C|k|^{-r}$ for some positive C and r and any integers $k_1, \dots, k_n, k$ for maps, and $\sum_{i=1}^n k_i\alpha_i > C|k|^{-r}$ for flows.

Famous illustrations of that are KAM, Hermann's last geometric theorem, etc.)

Example 9. S_α is rotation of the disc. Nothing in $\mathcal{A}_\alpha$ is topologically transitive if $\alpha \in \mathcal{D}$, while for $\mathcal{L}$, topologically transitive are dense G_δ. Smooth map on the boundary with Diophantine rotation number, by Hermann–Yoccoz, is conjugate to rotation. By "Herman last geometric theorem" [5] invariant curves accumulate on the boundary.

Now consider $M = \mathbb{T}^2$, $S_t(x, y) = (x + t, y)$. For $\alpha \in \mathcal{D}$ what can we say about $\mathcal{A}_\alpha$? Does it have "interesting behavior"? Specifically,

Problem 2. *Are there topologically transitive (or ergodic area-preserving) transformations in $\mathcal{A}_\alpha, \alpha \in \mathcal{D}$?*

This is somewhat exotic. In more natural terms (Franks, etc.): If the set of rotation vectors has an interior, then the map has positive entropy (the spread of directions produces entropy). If $h_{\text{top}} = 0$, all rotation vectors are on a single interval—is the slope of this interval in $\mathcal{D}$ or $\mathcal{L}$? Non-trivial things happen when slope is in $\mathcal{L}$ or $\mathbb{Q}$. What if $\mathcal{D}$? Translations work. But is there anything else? Do there exist non-trivial models for Diophantine dynamics?

Problem 3. *Does there exist a volume-preserving C^∞ map f on M, dimension of M greater than one, such that f is measurably conjugate to a Diophantine rotation R_α?*

For $\alpha \in \mathcal{L}$, Yes, On every manifold admitting an S^1 action [4].

Rather than the linearized system, one can look at typical properties for elliptic/parabolic behavior. In both cases, irreducible (of some form) should imply strictly ergodic (or at most finitely many ergodic measures). For homogeneous parabolic, this is Ratner theory (up to measures on an algebraic submanifold). Not proven in the general parabolic setting. I plan to discuss this and related issue in a separate note.

Better statement: finitely many ergodic invariant measures with support of positive volume. For an IET can bound the number of such measures by a constant. For homogeneous parabolic (Ratner): G a Lie group, G/Γ a factor, g_t a one-parameter action from left, unipotent adjoint representation. Consider invariant distributions. Measures are $C(X)^*$, distributions are $C^\infty(X)^*$.

Elliptic: few invariant distributions. Parabolic: infinitely many invariant distributions *even if uniquely ergodic.* (Paradigmatic statement, not theorem, supported by all known examples, see [11, 6, 7]).

Don't have geometric methods (Livsic, etc.) from hyperbolicity. Harmonic analysis; need algebraic structure. Have a dual object—large symmetry group. Baby example (from book on combinatorial constructions); $f(x, y) = (x + \alpha, y + x)$ on torus. Dual space is $\mathbb{Z}^2$. f acts on characters

$$\chi_{m,n}(x, y) = \exp(2\pi i(mx + ny)).$$

(Fourier decomposition). Construct invariant distributions by seeing how f acts on $\mathbb{Z}^2$. To get *measures*, need a rate of decay of coefficients. Only one measure here, but every dual orbit carries a distribution. Forni–Flaminio worked out prediction from Katok's book regarding horocycle flow; found infinitely many independent invariant distributions (obstructions to cohomological equation). Decompose dual space into dual orbits, look at irreducible representations of G.

Group representations: a powerful tool, but capricious (like early cars). Technical obstacles to generalizations.

What systems have very few invariant distributions?

There is always a measure. If nothing else, say *strongly* uniquely ergodic. $\mathcal{D}$ vs. $\mathcal{L}$ dichotomy returns. For example, linear flow on torus with a time change (preserves distribution). $\mathcal{D}$: straighten time change (Kolmogorov rigidity) by solving cohomological equation to straighten the roof function. Not true for $\mathcal{L}$.

Can play the same game for group extensions.

Theorem 1 (Avila, Forni, Kocsard, not yet written)**.** *Any C^∞ diffeomorphism of S^1 with irrational rotation number is strongly uniquely ergodic.*

(Topologically conjugate to rotation implies uniquely ergodic, measure on S^1 may be smooth or singular).

Geometrically appearing distributions.

Example 10. Fixed point with an eigenvalue $= 1$, eigenvector $\mathbf{v}$. $\varphi \mapsto D_{\mathbf{v}}\varphi$ is an invariant distribution.

Conjecture 2. *If S^1 acts freely on M (ie. M is an S^1 bundle) and $\mathcal{A} = \overline{\{\varphi S_\alpha \varphi^{-1}\}}$ then a dense G_δ subset of $\mathcal{A}$ comprises strongly uniquely ergodic actions.*

Suppose φ preserves area. Uniquely ergodic a dense G_δ: Fathi-Herman (see [3]).

Can prove: there exists $f \in \mathcal{A}_\alpha$ for $\alpha \in \mathcal{L}$ which is uniquely ergodic and has extra linearly independent invariant distributions (k of them, where $k = \dim M$). Based on Katok–Gunesch [9]: show there exists meas. inv. Riemannian metric, get invariant vector fields, not globally integrable—differentiate along these, integrate with respect to invariant measure.

Need stronger notion than strong unique ergodicity to deal with Liouvillean pathology.

Suppose f is SUE and invariant measure is only obstruction to solving cohomological equation. That is, $\int \varphi \, d\mu = 0$ implies there exists H such that $\varphi = H \circ f - H$. For torus and linear flow, this happens exactly in $\mathcal{D}$.

Conjecture 3 (Katok conjecture)**.** *In this case $M = \mathbb{T}$ and f is C^∞ conjugate to a Diophantine translation.*

In dimension 2, easy exercise. Uniquely ergodic implies torus. (Lefschetz) homotopy type, homotopic to identity, lift to cover, get displacement φ. F is lift, $F(x) = x + \varphi(x)$, φ periodic, $\varphi = \varphi_0 + H \circ f - H$, take coordinate change $y = x + H(x)$.

Dimension 3, big achievement. Solved, but uses Weinstein conjecture from symplectic geometry, [8, 13]

Jacobian $J(x) = H(f(x))/H(x)$ important for invariant densities. Can exclude many homotopy types, but unipotent maps dog you. (Need symplectic techniques for dimension 3, not developed enough for higher dimensions).

All known constructions are algebraic, or more precisely, built from algebraic blocks, or perturbative.

Question: *WHAT LIES BEYOND PERTURBATIVE CONSTRUCTIONS?*

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