CHAPTER 10

SMOOTH MANIFOLDS REVISITED

We return to the general discussion of differentiable manifolds started in Chapter 4.

10.1. Basics, notation

Here, for the reader’s convenience, we recall the main notions related to smooth manifolds that we studied in Chapter 4, but only to refresh the terminology and fix the notation.

Let $M$ be a smooth manifold of dimension $n$. Then by

\[ A_M = \left\{ (U_i; h_i = (x^1, \ldots, x^n) : U_i \to \mathbb{R}^n) \right\} \]

we denote some atlas of $M$; here the $h_i$ are the charts (local coordinates) of the atlas (we assume that each $h_i$ is a homeomorphism of $U_i$ onto $\mathbb{R}^n$); by $\varphi_{i,j}$ we denote the transition functions (coordinate transformations) given by $\varphi_{i,j} = h_j \circ h_i^{-1}$.

Further we denote the tangent bundle of $M$ by $\tau : TM \to M$; if $p \in M$, then $T_pM = \tau^{-1}(p)$ is the tangent space at $p$. The space $T_pM$ is an $n$-dimensional vector space with basis

\[ \partial_{p,1} = \frac{\partial}{\partial x^1}\bigg|_p, \ldots, \partial_{p,n} = \frac{\partial}{\partial x^n}\bigg|_p; \]

we will use the shorter notation $\partial_{p,i}$ rather than the cumbersome (but more often used) notation $(\partial / \partial x^i)|_p$. Recall that any element (vector) $v_p$ of $T_pM$ is a derivation, i.e., an $\mathbb{R}$-valued linear functional defined on functions (given near $p$) and satisfying the local Leibnitz rule:

\[ v_p(f \cdot g) = f(p) \cdot v_p(g) + v_p(f) \cdot g(p). \]

The value of a given vector on a given function can be calculated as the linear combination of the partial derivatives of the function $f \circ h^{-1} : \mathbb{R}^n \to \mathbb{R}$ at the point $h^{-1}(p)$.

The set of smooth sections of the tangent bundle $\tau : TM \to M$, i.e., smooth maps $V : M \to TM$ such that $\tau \circ X = \Id_M$, is denoted by $\Gamma(M)$; its elements are called (smooth) vector fields. The value of the vector field $V$ at the point $p \in M$ is a tangent vector at $p$ that we will denote by $V_p$ (rather than $V(p)$). The set $\Gamma(M)$ of all smooth vector fields has the natural structure of a module over $C^\infty(M)$. Locally, in a fixed coordinate system $(U, (x^1, \ldots, x^n))$, the module
\( \Gamma(U) \) is finitely generated with basis
\[
\partial_1 = \frac{\partial}{\partial x^1}, \ldots, \partial_n = \frac{\partial}{\partial x^n}.
\]

Of course, as a topological space, \( \Gamma(M) \) (and even the space \( \Gamma(\mathbb{R}^n) \)) is infinite-dimensional.

The **trajectory** of a vector field \( V \) is a smooth map \( \alpha : \mathbb{R} \to M \) whose tangent vector at each point \( p \in M \) coincides with \( V_p \).

We denote the algebra of smooth functions on a manifold \( M \) by \( F_M \). Recall that in the local coordinates \( (U, h = x^1, \ldots, x^n) \) any function \( f \in F \) can be expressed locally (in the neighborhood \( U \)) in coordinate form; we write \( f(q) = f(x^1(q), \ldots, x^n(q)) \) for any \( q \in U \). For any \( f \in F \), the function \( f \circ h^{-1} : \mathbb{R}^n \to \mathbb{R} \) is infinitely differentiable.

The algebra \( F_M \) is a commutative associative infinite-dimensional (provided \( n = \dim M \geq 1 \)) algebra over \( \mathbb{R} \) with unit. It has numerous zero divisors: the product of two nonzero functions \( f \) and \( g \) can be zero (this occurs if their supports do not intersect: \( \text{supp } f \cap \text{supp } g = \emptyset \)).

### 10.2. Vector fields, flows and differential operators

By the theorems of existence, uniqueness, and smooth dependence for solutions of ordinary differential equations a \( C^1 \) vector field on \( M \) induces a local flow, that is, for every \( p \in M \) there is a curve \( c_{v,p} : (-\varepsilon, \varepsilon) \to M \) such that \( c_{v,p}(0) = p \) and \( \dot{c}_{v,p}(t) := \frac{d}{dt}c_{v,p}(t) = v(c_{v,p}(t)) \). Here \( \varepsilon \) can be chosen to depend continuously on \( p \). Where defined the map \( \varphi_v : (p, t) \to \varphi^t(p) := c_{v,p}(t) \) is smooth as \( v \). By continuity of \( \varphi_v \) it is bounded on any compact manifold and hence by the group property \( c_{v,p}(t+s) = c_{v,c_{v,p}(t)}c_{v,p}(t)(s) \) (which follows from uniqueness) every vector field on a compact manifold induces a complete flow, that is, \( \varphi^t_v \) is defined for all times. If \( \varphi^t_v \) and \( \varphi^t_w \) are the flows for vector fields \( v \) and \( w \), respectively, then usually the diffeomorphisms \( \varphi^t_v \) and \( \varphi^t_w \) do not commute, that is, \( \varphi^t_v \circ \varphi^t_w \neq \varphi^t_w \circ \varphi^t_v \). If they do, the vector fields \( v \) and \( w \) are said to commute. The extent to which two vector fields \( v, w \) fail to commute is measured by their Lie bracket \( [v, w] \) which can be computed as \( [v, w](p) = \lim_{t \to 0} \frac{(w - d\varphi^t_w)(\varphi^t_v(p))}{t} \).

Let us now show briefly how these invariant notions appear in local coordinates. If \( (U, h) \) is a chart then we say that we have coordinates \( (x^1, \ldots, x^n) \) on \( U \). For \( p \in U \) the canonical basis of \( T_pM \) is the set of derivations \( \partial/\partial x^i \) induced by the curves \( \gamma_i(t) := h^{-1}(h(p) + te_i) \), where \( e_i \) is the \( i \)-th standard basis vector in \( \mathbb{R}^n \). A tangent vector \( v \in T_pM \) can then be written as \( v = \sum_{i=1}^n v^i \partial/\partial x^i \) and if \( f : M \to \mathbb{R} \) is smooth then \( v f = \sum_{i=1}^n v^i \partial(f \circ h^{-1})/\partial x^i \). Thus the induced coordinates of \( TM \) are \( (x^1, \ldots, x^n, v^1, \ldots, v^n) \), where the \( v^i \) are the components we just defined. Likewise a vector field is locally given by a representation \( v(p) = \sum_{i=1}^n v^i(p) \partial/\partial x^i \) and it is smooth if and only if the \( v^i \) are. To see that the Lie bracket of two vector fields \( v, w \) defines a derivation, that is, a vector field, we calculate in local coordinates. Namely, write \( v = \sum_{i=1}^n v^i \partial/\partial x^i \),
\[ w = \sum_{i=1}^{n} w^i \partial / \partial x^i \] and for convenience write \( f \) for \( f \circ h \). Then using the theorem of H. A. Schwarz that second partial derivatives commute we obtain

\[
(vw - wv)f = v \sum_{i=1}^{n} w^i \frac{\partial f}{\partial x^i} - w \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}
\]

\[
= \sum_{i,j=1}^{n} v^j \frac{\partial w^i}{\partial x^j} \frac{\partial f}{\partial x^i} + \sum_{i,j=1}^{n} v^j w^i \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{i,j=1}^{n} w^i \frac{\partial v^i}{\partial x^j} \frac{\partial f}{\partial x^i} + \sum_{i,j=1}^{n} v^j w^i \frac{\partial^2 f}{\partial x^i \partial x^j}
\]

\[
= \sum_{i,j=1}^{n} \left( v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \right) \frac{\partial f}{\partial x^i},
\]

that is, \([v, w] \) is indeed a vector field given locally by \( v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \). In particular \([\partial / \partial x^i, \partial / \partial x^j] = 0\). There are several important properties of Lie brackets that are not hard to check in local coordinates. By definition we obviously have \([v, w] = -[w, v] \) and \([\cdot, \cdot] \) is \( \mathbb{R} \)-bilinear, that is, \([\alpha v + \beta w, z] = \alpha [v, z] + \beta [w, z] \) for \( \alpha, \beta \in \mathbb{R} \). Next observe that for functions as coefficients we get \([fv, gw] = fg[v, w] + f(vg)w - g(wf)v \) by a coordinate calculation similar to the preceding one. This means in particular (for \( f \equiv 1 \)) that the Lie derivative is a derivation, that is, satisfies the product rule \( \mathcal{L}_v(gw) = g \mathcal{L}_v w + \mathcal{L}_v g w \). Furthermore there is the fundamental Jacobi identity

\[
(10.2.1) \quad [v, [w, z]] + [w, [z, v]] + [z, [v, w]] = 0.
\]

This is straightforward in coordinates. Namely, we know that only first-order derivatives occur, so we may simplify the calculation by discarding all higher-order derivatives. The symmetry then makes the remaining terms cancel. Alternatively write \([v, w] = vw - wv \) and expand (10.2.1) accordingly to see that all terms cancel.

Differentiating differentiable maps between manifolds is also straightforward calculus on local coordinates: If \( f: M \to N \) and \((U, h), (V, k)\) are local charts around \( p \in M \) and \( f(p) \in N \), respectively, then the differential of \( f \) at \( p \) is represented by the matrix of partial derivatives of the map \( k \circ f \circ h^{-1} \) in Euclidean space with respect to the standard bases.

### 10.3. Tensor bundles

The tangent bundle is an example of the following:

**Definition 10.3.1.** A differentiable vector bundle with structure group \( G \), a subgroup of \( GL(m, \mathbb{R}) \), over \( M \) (the base space) is a manifold \( P \), called the total space or bundle space, such that the projection \( \pi: P \to M \) is differentiable and furthermore locally \( P = M \times \mathbb{R}^m \), that is, every \( x \in M \) has a neighborhood \( U \) such that there is a diffeomorphism \( h: \pi^{-1}(U) \to U \times \mathbb{R}^m, u \mapsto (\pi(u), \varphi(u)) \) and such that for any point \( x \) in the intersection \( U_1 \cap U_2 \) of two such neighborhoods the trivialization differs by an element of \( G \). A subbundle or distribution is a bundle
whose fibers are contained in those of $P$. For two distributions $E$, $F$ we define the Whitney sum $E + F$ to be the distribution with $(E + F)_p = E_p + F_p$. We use “⊕” if the sum is (pointwise) direct, that is, $E_p \cap F_p = \{0\}$ for all $p \in M$. A section of $P$ is a map $v: M \to P$ such that $\pi \circ v = \text{Id}_M$.

**Example 10.3.2.** The tangent bundle $TM$ of $M$ is of this form: Here $m$ is the dimension of $M$ and $G = GL(m, \mathbb{R})$ acts by the linear coordinate changes in the tangent fibers induced by coordinate change in the base. The sections are the vector fields. If there is a nonvanishing vector field on $M$ then the one-dimensional subspace it spans at every point define a one-dimensional distribution.

Note that the differentiable manifold $TM$ has in turn a tangent bundle $TTM$. This is an important object. On one hand it allows us to differentiate vector fields. On the other hand classical mechanics involves second-order differential equations and the natural setting for second derivatives is the second (or double) tangent bundle $TTM$.

The second tangent bundle $TTM$ is obviously a vector bundle over $TM$, but it is, in fact, a vector bundle over $M$ as well. To that end notice that coordinate changes in $M$ change coordinates in $TTM$ by a coordinate change determined again by the linear part of the coordinate change in $M$. We will return to this in the setting of Riemannian manifolds.

From the linear structure in the tangent spaces arise linear objects other than vectors and linear maps (for example, differentials). Namely, it is often important to consider multilinear maps. The easiest examples, and a building block, are 1-forms.

**Definition 10.3.3.** We denote by $T^*M$ the cotangent bundle consisting of the spaces $T^*_pM = (T_pM)^*$ of linear maps (covectors) $T_pM \to \mathbb{R}$. A section of $T^*_pM$ is called a 1-form. A multilinear map $\underbrace{T^*_pM \oplus \cdots \oplus T^*_pM}_{k \text{ times}} \oplus \underbrace{T_pM \oplus \cdots \oplus T_pM}_{l \text{ times}} \to \mathbb{R}$ (that is, linear in each entry independently) is called a $(k, l)$-tensor. A section of the bundle $TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M = (TM)^{\otimes k} \otimes (T^*M)^{\otimes l}$ is a $(k, l)$-tensor field (or tensor). A tensor is called smooth if its values on smooth vector and covector fields define a smooth function. (Alternatively, if its coefficients in local coordinates are smooth.)

Thus a vector is a $(1, 0)$-tensor, a 1-form is a $(0, 1)$-tensor, and the Riemannian metrics defined in Definition 13.2.1 are $(0, 2)$-tensors. A basis for the space of 1-forms on $T_pM$ is given by the forms $dx^i$ which are given by the derivatives of the coordinate functions $x^i$, that is, $dx^i(\partial/\partial x^j) = \delta^i_j := \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$ The derivative of a function $f$ is a 1-form $Df(v) := vf = \sum_{i=1}^n \partial f/\partial x_i \, dx^i$. If $T$ is a $(k, l)$-tensor then $T = T^{i_1, \ldots, i_k}_{j_1, \ldots, j_l} \partial/\partial x^{i_1} \otimes \cdots \otimes \partial/\partial x^{i_k} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l}$ with $T^{i_1, \ldots, i_k}_{j_1, \ldots, j_l} = T(dx^{i_1}, \ldots, dx^{i_k}, \partial/\partial x^{j_1}, \ldots, \partial/\partial x^{j_l})$. There is a natural way to extend the Lie derivative to tensors. Namely, note first that for $(1, 0)$-tensors (vector
fields) it is already defined and that for \((0,0)\)-tensors (functions) we can define 
\[ \mathcal{L}_v f := v f. \]
Now extend to \((0,1)\)-tensors \(\xi\) by setting 
\[ \mathcal{L}_v(\xi(w)) = \mathcal{L}_v(\xi)(w) + \xi(\mathcal{L}_v w). \]
Likewise one can extend \(\mathcal{L}_v\) to any tensor field by postulating the product rule 
\[ \mathcal{L}_v(\xi \otimes \eta) = \mathcal{L}_v \xi \otimes \eta + \mathcal{L}_v \eta \otimes \xi. \]
If \(\omega\) is a \((0,1)\)-tensor on \(N\) and \(f: M \to N\) differentiable then we can define the pullback \(f^* \omega\) of \(\omega\) on \(M\) by 
\[ f^* \omega(v) := \omega(Df v). \]
This, of course, works for \((0,k)\)-tensors just as well. Likewise one can send vectors from \(M\) to \(N\) via \(Df\), but this can be expected to send vector fields to vector fields only if \(f\) is injective (if \(f(p) = f(q)\) and \(v\) is a vector field such that \(Df v(p) \neq Df v(q)\) then there is no well-defined vector field “\(f_* v\)” on \(f(M)\)).

If \(f\) is a diffeomorphism then this is no problem, however. Using pullbacks the Lie derivative of a \((0,k)\)-tensor can be computed by using the flow \(\varphi^t\) defined by the vector field \(v\) to write
\[ \mathcal{L}_v \omega = \lim_{t \to 0} (1/t)((\varphi^t)^* \omega - \omega). \]
The Lie derivative of any \((k,l)\)-tensor can be computed similarly.

An important special class of tensors is that of alternating ones:

** Definition 10.3.4. ** A \((0,k)\)-tensor \(\omega\) on a linear space is said to be an alternating tensor or an (exterior) form if \(\omega(v_1, \ldots, v_k) = 0\) whenever \(v_i = v_j\) for some \(i \neq j\). A \((0,k)\)-tensor field is said to be alternating if it is alternating at every point. Alternating \((0,k)\)-tensor fields are called \(k\)-forms, and the space of \(k\)-forms is denoted by \(\Gamma(\wedge^k T^* M)\). In analogy to the asymmetric part of a matrix the alternating part \(A\eta\) of a \((0,k)\)-tensor \(\eta\) is defined by 
\[ A\eta = 1/k! \sum_{\pi \in S_k} \text{sgn } \pi \eta \circ \pi, \]
where \(\pi\) permutes the entries and \(\text{sgn } \pi\) is its sign, that is, \(-1\) if \(\pi\) is odd, 1 otherwise. Thus \(A\) is a projection of \((T^* M)^{\otimes k}\) to \(\wedge^k T^* M\). We define the wedge product or exterior product of \(\omega \in \wedge^k T^* M\) and \(\eta \in \wedge^l T^* M\) by 
\[ \omega \wedge \eta := \frac{(k + l)!}{k! l!} A(\omega \otimes \eta) \in \wedge^{k+l} T^* M. \]
Nonzero elements of \(\Gamma(\wedge^n T^* M)\) are called volume elements and two volume elements \(\Omega, \Omega'\) are said to be equivalent if \(\Omega' = f \Omega\) for some \(f \in C^\infty(M), f > 0\). An equivalence class of volume forms is called an orientation of \(M\) and \(M\) is called orientable if there exists an orientation on \(M\).

With these definitions one gets the following standard facts: \(\omega \wedge \eta\) is \(\mathbb{R}\)-bilinear in \(\omega\) and \(\eta\), \(\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta\) (hence \(\omega \wedge \omega = 0\) for odd \(k\)), \(f^*(\omega \wedge \eta) = (f^* \omega) \wedge (f^* \eta)\), and \(\omega \wedge (\eta \wedge \lambda) = (\omega \wedge \eta) \wedge \lambda =: \omega \wedge \eta \wedge \lambda\). A basis for \(\wedge^k T^*_p M\) is given by \(\{dx^{i_1} \wedge \cdots \wedge dx^{i_k} | 1 \leq i_j \leq n\}\), where \(\{dx^i | 1 \leq i \leq n\}\) is the dual basis for \(\{\partial/\partial x^i | 1 \leq i \leq n\}\). Thus dim \(\wedge^k T^*_p M = \binom{n}{k}\). In fact, \(\beta^1 \wedge \cdots \wedge \beta^k \neq 0\) if and only if \(\{\beta^1, \ldots, \beta^k\} \subset T^*_p M\) is linearly independent.

A manifold is orientable if and only if \(\Gamma(\wedge^n T^* M)\) is one-dimensional over \(C^\infty(M)\). (Namely, there exists a volume, hence the dimension is at least one, and for two volumes \(\Omega\) and \(\Omega'\) the function \(\varphi := \Omega'/\Omega\) is well defined, since \(\Gamma(\wedge^n T^*_p M)\) is one-dimensional, and smooth as well.) One can also check that orientability is equivalent to the existence of an oriented atlas, that is, an atlas
where $h \circ h'$ preserves the orientation of $\mathbb{R}^n$ for any two charts $h, h'$. On a compact manifold a volume form can be integrated to give the total volume. This is done via charts as follows. In $\mathbb{R}^n$ we define $\int \Omega := \int \Omega_1 \cdots \Omega_n dx^1 \cdots dx^n$ for any volume $\Omega = \Omega_1 \cdots \Omega_n dx^1 \cdots dx^n$. For orientation-preserving diffeomorphisms $f$ we get $\int f^* \Omega = \int \Omega$. Thus we can define $\int \Omega$ for a manifold $M$ by taking a partition of unity $\{U_i, \psi_i\}$ subordinate to a covering by charts $(V_i, h_i)$ and define $\int \Omega := \sum_i \int (h_i)_*(\psi_i \Omega)$, and this definition via charts is coordinate independent.

### 10.4. Exterior calculus and de Rham differential

Next we want to study the calculus of exterior forms, also called exterior calculus.

**DEFINITION 10.4.1.** The exterior derivative $d$: $\Gamma(\bigwedge^k T^*M) \to \Gamma(\bigwedge^{k+1} T^*M)$ (for any $k$) is defined by the following axioms (which uniquely determine $d$): $df = Df$ for functions, $d$ is $\mathbb{R}$-linear and $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$, $d \circ d = 0$, and $d$ is locally defined, that is, if two forms coincide on an open set $O$ then their derivatives coincide on $O$ as well.

By induction on dimension one sees that this is well defined. Namely, if $\omega = \varphi dx^1 \wedge \cdots \wedge dx^k$ then necessarily $d\omega = d\varphi \wedge dx^1 \wedge \cdots \wedge dx^k$. The last property is also satisfied inductively since it holds for functions: $dd\varphi = \sum_{ij} (DD\varphi)_{ij} dx^i \wedge dx^j \wedge dx^j = 0$. Furthermore $d$ commutes with pullback and the Lie derivative: $f^*d\omega = d(f^*\omega)$ (and $f_*d\omega = d(f_*\omega)$ if $f$ is a diffeomorphism) and $L_v(\omega^1 \wedge \cdots \wedge \omega^k) = L_v\omega^1 \wedge \cdots \wedge L_v\omega^k + \cdots + \omega^1 \wedge \cdots \wedge L_v\omega^k$, whence $dL_v = L_v d$.

We occasionally use the convenient notation of the contraction of $\omega$ with a vector $v$ defined by $v \cdot \omega := \omega(v, \cdot, \ldots , \cdot)$. This is $\mathbb{R}$-linear and $C^\infty(M)$-linear in $v$. Furthermore $v \cdot (\omega \wedge \eta) = (v \cdot \omega) \wedge \eta + (-1)^k \omega \wedge (v \cdot \eta)$ and $v \cdot df = L_v f$ and

(10.4.1) $L_v \omega = v \cdot d\omega + d(v \cdot \omega)$.

Finally $f^*v \cdot f^*\omega = f^*(v \cdot \omega)$ and $f_*v \cdot f_*\omega = f_*(v \cdot \omega)$ for any diffeomorphism $f$.

### 10.5. De Rham cohomology

Associated with forms is a cohomology theory which is based on the following notion and theorem:

**DEFINITION 10.5.1.** $\omega \in \Gamma(\bigwedge^k T^*M)$ is said to be closed if $d\omega = 0$ and exact if $\omega = d\eta$ for some $\eta \in \Gamma(\bigwedge^{k-1} T^*M)$.

Since $d^2 = 0$ every exact form is closed. Locally the converse holds:

**THEOREM 10.5.2 (Poincaré Lemma).** If $\omega$ is closed then for all $p \in M$ there is a neighborhood $U$ of $p$ on which $\omega$ is exact.

**PROOF.** We use the homotopy trick (see ??, ??): Assume $p = 0 \in \mathbb{R}^n$ and let $v_t(x) = x/t$. $v_t$ generates the flow $\varphi^t(x) = tx$ for $t > 0$, so $d/dt(\varphi^t)^*\omega = (\varphi^t)^*L_{v_t} \omega = (\varphi^t)^*(d(v_t \cdot \omega)) = d((\varphi^t)^*(v_t \cdot \omega))$ since $d\omega = 0$, and $\omega - (\varphi^0)^*\omega =$