On the work of Omri Sarig on Markov partitions for surface diffeomorphisms and applications

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M is a closed surface,

$f : M \to M$ a $C^{1+\alpha}$ diffeomorphism of M

with *positive topological entropy* h_{top}.

Theorem 1 [S. 2013] *Assume there exists a measure with maximal entropy. Then, there exists an integer p such that*

$$\liminf_{k\to\infty} e^{-kph_{top}} P_{kp}(f) > 0,$$

where $P_n(f) := \#\{$ hyperbolic periodic points with period $n\}$.

Remarks

1. For a general $C^{1+\alpha} f$, Katok [80] proved:

$$\limsup_n \frac{1}{n} \ln P_n(f) \geq h_{top}.$$

2. If f is C^∞, then there exists a measure of maximal entropy (Newhouse/ Yomdin).

3. Not true if $h_{top} = 0$ (rotations, but also Feigenbaum-like (Franks-Young 81) and other Franks phenomena)

4. For a generic f in $C^r, r \geq 2$ with a Newhouse domain, $P_n(f)$ can grow arbitrarily fast (Kaloshin 00),

5. but for all $\delta > 0$, $\ln P_n(f) \leq Cn^{1+\delta}$ is prevalent (Hunt-Kaloshin 07).

Theorem 2[S.] *There is at most a countable number of measures of maximal entropy. For each one of them, the measured dynamical system* (M, m, f) *is isomorphic to a Bernoulli times a finite rotation.*

Theorem 2 answers a question from Jérôme Buzzi.

Theorem 2 also holds for equilibrium states of Hölder continuous functions, as soon as they have positive entropy.

Theorem 2 was known for hyperbolic systems in all dimensions (Anosov, Sinai, Bowen, Ruelle).

There are 1-dimensional analogs (Takahashi, Hofbauer, Keller)

Theorem 2 was known for special systems (Hénon like) and special values of the parameters: for the BRS measure (Benedicks-Carleson, Mora-Viana, Benedicks-Young, Wang-Young), for the measure of maximal entropy (Pierre Berger shows that there is only one measure of maximal entropy).

Partial analogs of Theorem 2 can be proven in the presence of Young towers,

or for systems with singularities (Chernov).

A a countable alphabet. $A^{\mathbb{Z}}$ the shift space.

A *subshift of finite type* $\Sigma \subset A^{\mathbb{Z}}$ is a closed shift invariant set defined by neighboring conditions. Σ is locally compact if each state has only a finite number of neighbors. Set

$\Sigma^{\#}$ for the set of sequences $\underline{x} = \{x_i\}_{i \in \mathbb{Z}}$ such that

$\exists u, v \in A, x_i = u$ i.o. for $i > 0, x_i = v$ i.o. for $i < 0$.

By Poincaré recurrence, $\Sigma^{\#}$ supports all shift invariant probability measures.

Theorem 3[S.] *Let* $0 < \chi < h_{top}(f)$. *There exist a locally compact Markov shift* Σ *and a map* $\varphi : \Sigma^{\#} \to M$ *such that*

- $\varphi \circ \sigma = f \circ \varphi$,

- φ *is finite-to-one, Hölder continuous and*

- $\varphi(\Sigma^{\#})$ *has full measure for all invariant probability measures of entropy* $> \chi$.

Finite-to-one: there is a function $\rho : A \times A \to \mathbb{N}$ such that if u, v are such that $x_i = u$ i.o. for $i > 0, x_i = v$ i.o. for $i < 0$, then

$$\text{Card}(\varphi^{-1}(\varphi(\underline{x})) \leq \rho(u, v).$$

Theorem 3 $\Longrightarrow$ Theorem 1,2 by properties of locally compact subshifts of finite type (Gurevich, Sarig).

The proof of Theorem 3 rests on the construction of a countable Markov partition with good properties.

Theorem 4[Main result, S.2013] *M, f as above, $0 < \chi < h_{top}(f)$. Then, there exists a pairwise collection $\mathcal{R} = \{R_i, i \in \mathbb{N}\}$ of subsets of M such that:*

- *for any invariant probability measure m of entropy $> \chi$, $m(\cup_i R_i) = 1$,*

- *the sets R_i are rectangles: they have a local product structure, and*

- *the sets R_i have the Markov property.*

Namely, each R is partitioned into pieces of stable manifolds $W^s(x,R)$ and into pieces of unstable manifolds $W^u(x,R)$ with:

- $W^u(x,R) \cap W^s(x,R) = \{x\}$,

- $\forall x, y \in R, \exists z \in R, W^u(x,R) \cap W^s(y,R) = \{z\}$, and

- if $x \in R_1, f(x) \in R_2$, then

  $f(W^s(x)) \subset W^s(f(x)),\ f^{-1}(W^u(f(x))) \subset W^u(x).$

Classical Bowen-Sinai construction in the uniformly hyperbolic case:

An ε-pseudo orbit is a sequence $\{x_n\}_{n\in\mathbb{Z}}$ such that

$$d(f(x_i), x_{i+1}) < \varepsilon, \text{ for all } \; i \in \mathbb{Z}.$$

For all $\delta > 0$ sufficiently small, there is $\varepsilon > 0$ such that for any ε-pseudo orbit, there is a unique y such that $d(f^i y, x_i) < \delta$ for all $i \in \mathbb{Z}$.

Choose as alphabet A a covering of M by ε-balls. Define the subshift of finite type Σ_0 by $A_i \sim A_j$ if $f(A_i) \cap A_j \neq \emptyset$. Define $\pi : \Sigma_0 \to M$ by the ε-pseudo orbit property.

The sets $\pi([x_0 = a]) = \pi([x_+]) \cap \pi([x_-])$ have the product structure and the Markov property, but

they do not form a partition and $\mathrm{Card}(\pi^{-1}(\underline{x})) = \infty$.

There is a clever way of subdividing the $\pi([x_0 = a])$ that overcomes these two problems (Bowen, Sinai)

Want to do the same for a general $C^{1+\alpha}$ diffeo of a surface, at least on a set of full measure for any measure with entropy $> \chi$.

Dim 2 + entropy $> \chi \implies$ exponents $> \chi$ and $< -\chi$.

Pesin theory applies. There is a set of full measure for all measures with entropy $> \chi$, which is "measurably" hyperbolic.

Two nice features of Pesin theory:

All sizes are controlled by a slowly varying measurable function ℓ:

$$\ell(f^{\pm}x) \leq e^{\varepsilon}\ell(x)$$

and (Brin) all object are Hölder continuous on sets of arbitrarily large measure.

Omri has a non-uniform shadowing property which will do by putting all the necessary features in the definition of ε-pseudo orbit:

- $d(fx_i, x_{i+1}) \leq \varepsilon \ell(x_{i+1})^{-big\ power}$
  $d(x_i, f^{-1}x_{i+1}) \leq \varepsilon \ell(x_i)^{-big\ power}$

- unstable curves at x_{i+1} are $\ell(x_{i+1})^{-some\ power}$ comparable to images of unstable curves at x_i

- stable curves at x_i are $\ell(x_i)^{-some\ power}$ comparable to inverse images of stable curves at x_{i+1}

- $e^{-\varepsilon}\ell(x_i) \leq \ell(x_{i+1}) \leq e^{\varepsilon}\ell(x_i)$.

Then for $\delta > 0$ small enough, there is $\varepsilon > 0$ so that any ε-pseudo orbit has a Oseledets regular point y with

$$d(x_i, f^i y) \leq \delta \ell(f^i y)^{-small\ power} \quad \text{for all } i \in \mathbb{Z}.$$

Such point y is unique satisfying

$$\liminf_{n \to +\infty} \ell(f^n y) < +\infty, \quad \liminf_{n \to -\infty} \ell(f^n y) < +\infty.$$

The set $M_\chi^\#$ of such points has full measure for all probability measures with entropy $> \chi$.

Define as before a mapping from a subshift of finite type Σ_0 (in fact from the recurrent part $\Sigma_0^{\#}$) by associating to a sequence of symbols satisfying the ε-pseudo orbit condition the unique point in $M^{\#}$ that δ-shadows the pseudo orbit.

Almost done:

- Σ_0 is locally compact,
- semi-conjugacy from Σ_0 to $M^{\#}$,
- Hölder coefficient controlled all over Σ_0,
- for points in $\Sigma_0^{\#}$, local product structure.

Remains to do: combinatorial construction of Bowen-Sinai. It can be done.

On the work of Omri Sarig on invariant measures for horocycle flow on hyperbolic surfaces

(M, g) a surface of constant curvature -1

SM the unit tangent bundle, $SM \sim \Gamma \backslash PSL(2, \mathbb{R})$, where Γ is a discrete subgroup of $PSL(2, \mathbb{R})$.

$\{h_t\}_{t \in \mathbb{R}}$ the stable horocycle flow, the right action of $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$.

Question: describe locally finite invariant ergodic measures under the horocyclic flow.

Furstenberg; M compact, only Liouville measure

Dani-Smillie: M finite volume, Liouville measure and closed horocycles.

Babillot-L.: M abelian infinite cover of a compact manifold. There are new locally finite invariant ergodic measures.

Sarig: In that case, there are no other ones

Babillot: question about a general M: is any locally finite horocyle invariant ergodic measure which is not supported by a closed horocycle quasi-invariant under the geodesic flow?

Nice: if m is such a measure, let $\widetilde{m}$ the Γ-invariant measure associated on $PSL(2, \mathbb{R})$. Write $\widetilde{m}$ in KAN coordinates

$$\widetilde{m} = d\nu e^{\alpha s} ds dt.$$

Then $F(z) = \int (P(\xi, z))^{\alpha} d\nu(\xi)$ is a Γ-invariant positive eigenfunction of the Laplacian, minimal with those properties.

Omri: true for some M, not true for others. Let M be an infinite surface, it admits pants decompositions. Once chosen the decomposition, the metric yields the lengths of the cuffs.

M is called *tame* if it admits a pants decomposition with bounded lengths of cuffs.
M is called *weakly tame* if it admits a pants decomposition such that, for any geodesic ray which crosses an infinite number of cuffs, $\{c_n\}_{n\in\mathbb{N}}$,

$$\liminf_n \text{ length } (c_n) < +\infty$$

Theorem [Sarig] *A weakly tame surface has Babillot property.*

Corollary (from the proof) If Γ has Poincaré exponent $< 1/2$ and limit set the whole circle, then $M = \Gamma \setminus PSL(2, \mathbb{R})$ does not have Babillot property.