Brin Prize
On the work of Artur Avila on Quasi-periodic Schrödinger operators

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Plan

Quasi-periodic 1D Schrödinger operator

We shall describe the **impressive** collection of **deep** results obtained by **Artur Avila** on the spectral properties of 1D quasi-periodic Schrödinger operators.

- 1D Schrödinger operator

$$H : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$$
$$(u_n)_{n\in\mathbb{Z}} \mapsto (Hu)_{n\in\mathbb{Z}} := (u_{n+1} + u_{n-1} + V_n u_n)_{n\in\mathbb{Z}}$$

  The sequence $(V_n)_{n\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ is called the *potential*.

- Quasi-periodic : means

$$V_n = V(\theta + n\alpha)$$

  where $V : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ is, say, a continuous map, $\theta \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is *the phase* and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{T}^d$ is the *frequency vector*.

Quasi-periodic 1D Schrödinger operator

Remarks :

- We shall mainly concentrate on the case $d = 1$ (one frequency) and V real analytic.
- The potential is defined by a dynamics : we say that it is dynamically defined.
- Observe that the potential H_θ depends on the phase θ ; we shall denote it by H_θ.

Spectrum

- In that framework : H_θ is a symmetric bounded operator.
- **Spectrum** of H_θ,
  $\Sigma(H_\theta) = Spec(H_\theta) := \mathbb{C} \setminus \{E \in \mathbb{C} : (H_\theta - E)^{-1} \text{exists and is continuous}\}$
- It is a compact subset of $\mathbb{R}$.

Spectral measures

Spectral measures

- $\forall u \in l^2(\mathbb{Z})$, $\exists \mu_{u,\theta}$ probability measure s.t. $\forall z \in \mathbb{C}$, $\Im z > 0$

$$\langle (H_\theta - z)^{-1} u, u \rangle = \int_{\mathbb{R}} \frac{d\mu_{u,\theta}(t)}{t - z}.$$

  μ_u is called the spectral measure associated to u.
- This measures allow Functional Calculus (Spectral Theorem).
- Since H is of Schrödinger type, to understand H_θ it is enough to know

$$\mu_\theta = (1/2)(\mu_{\delta_0,\theta} + \mu_{\delta_1,\theta})$$

  which is called the spectral measure of H_θ.
- One has $supp(\mu_{H_\theta}) = Spec(H_\theta)$.
- One defines $Spec_{ac}(H_\theta)$, $Spec_{sc}(H_\theta)$, $Spec_{pp}(H_\theta)$ as the support of the ac, sc, pp part of μ_θ.

The model

Aim : Understand

- the Spectrum of H_θ
- The nature of the spectrum which means whether H_θ has some ac, sc, pp component
- The spectral measures μ_θ for most/all phases.

The model

Important facts and concepts

- Since the potential is dynamically defined the spectrum $Spec(H_\theta)$ does not depend on θ : denote it by Σ.
- The Integrated Density of States (IDS) :
  - This is the θ-independent probability measure

$$N = \int_{\mathbb{T}^d} \mu_\theta d\theta$$

  - If we denote by $(E_{n,j}(\theta))_{-n \leqslant j \leqslant n}$ the eigenvalues (which are all real) of the finite range operator H_θ^n, then the sequence of measures

$$N_n^\theta = \frac{1}{2n+1} \sum_{j=-n}^{n} \delta_{E_{n,j(\theta)}}$$

    converges weakly to N.
- On the other hand, the spectral measures μ_θ strongly depend on θ.

Schrödinger operators
Berezansky's theorem

Though the eigenvalue equation does not always have a solution in $l^2(\mathbb{Z})$ there exist a lot of solutions to this equation with moderate growth.

This is the content of Berezansky's theorem : For μ_θ-a.e. $E \in \Sigma(H_\theta) = \Sigma$ there exists a solution $(u_k)_{k\in\mathbb{Z}}$ of $H_\theta u = Eu$

$$\forall n \in \mathbb{Z},\ u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n = Eu_n$$

such that $|u_n| = O((1 + |n|)^{1/2+\epsilon})$.

Almost Mathieu operator

- A very studied case is when $V(\theta) = 2\lambda\cos(2\pi\theta)$. One then speaks of the Almost-Mathieu Operator.
- λ is called the coupling constant : it determines different regimes.
- In that case we denote the operator by $H_{\lambda,\alpha,\theta}$.
- Aubry-André duality : $(u_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$, $f(\varphi) = \sum_{k\in\mathbb{Z}} u_k e^{2\pi i k\varphi}$

$$H_{\lambda,\alpha,\theta} u = Eu \iff$$
$$e^{2\pi i\theta} f(\varphi + \alpha) + e^{-2\pi i\theta} f(\varphi - \alpha) + \frac{1}{\lambda}\cos(2\pi\varphi) f(\varphi) = \frac{E}{2\lambda} f(\varphi).$$

Ex. of consequence :

$$\Sigma(H_\lambda) = \frac{1}{2\lambda}\Sigma(H_{1/\lambda}).$$

Plan

Results by Artur Avila

Artur Avila has obtained outstanding results

- on the Almost Mathieu Operator for the
  - Spectrum : measure and Cantor nature ;
  - IDS : absolute continuity ;
  - Spectral measures : Hölder continuity.
- on the Dynamics of the related Schrödinger cocycle (see below).
- He has developed a theory that shows that the intuition given by the study of the Almost Mathieu Operator is a good guide to study general potentials and at the same time it shows that the Almost-Mathieu potential is very particular.

Artur and the Almost Mathieu Operator : The spectrum

Results on **the spectrum** of the Almost-Mathieu Operator

Theorem (Aubry-André conjecture, Avila-K)

The spectrum of $H_{\lambda,\alpha,\theta}$ has Lebesgue measure $4|1-\lambda|$ for any λ, θ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Ann. of Maths, 2006
Previous results by Jitomirskaya-Krasovskii ($\lambda \neq 2$) and Last ($\lambda = 1$,α not CT).
Pb 5 of Simon's list
Techniques : Kotani theory+Renormalization+KAM

Artur and the Almost Mathieu Operator : the spectrum

Results on **the spectrum** of the Almost-Mathieu Operator

Theorem (Ten Martini problem, Avila-Jitomirskaya)

For any α irrational, and $\lambda \neq 0$ the spectrum of the Almost Mathieu operator is a Cantor set.

Ann. of Mahs, 2009
Pb 4 of Simon's list.
Previous results by Bellissard-Simon, Sinai, Hellfer-Sjöstrand, Choi-Eliott-Yui, Avron-van Moche-Simon, Last, Puig...

Theorem (Dry Ten Martini problem, Avila-Jitomirskaya)

For any α in DC, and $\lambda \neq -1, 0, 1$ all the gaps of the spectrum of the Almost Mathieu operator predicted by the gap labeling theorem are open.

JEMS, 2010

Artur and the Almost Mathieu Operator : the IDS

Results on **the IDS** of the Almost-Mathieu Operator

Theorem (Avila-Damanik)

For any α, and $|\lambda| \neq 1$ the Integrated Density of States of $H_{\lambda,\alpha,\theta}$ is absolutely continuous.

Invent. Math., 2008.
Previous and related results by Jitomirskaya, Goldstein-Schlag.

Theorem (Avila-Jitomirskaya)

For any α in DC, and $\lambda \neq -1, 0, 1$ the Integrated Density of States of the Almost Mathieu operator $H_{\lambda,\alpha,\theta}$ is $1/2$-Hölder.

JEMS, 2010

Artur and the Almost Mathieu Operator : regularity of spectral measures

Results on **the Spectral measures** of the Almost-Mathieu Operator

Theorem (Problem 6 of Simon's list, Avila-Damanik)

For any α, $|\lambda| < 1$ and almost every θ, the spectral measures of $H_{\lambda,\alpha,\theta}$ are absolutely continuous.

Theorem (Problem 6 of Simon's list, Avila-Jitomirskaya)

For any α in DC, $0 < |\lambda| < 1$ and all θ the spectral measures of the Almost Mathieu operator $H_{\lambda,\alpha,\theta}$ are absolutely continuous.

Artur and the Almost Mathieu Operator

How to attack these problems ?
Techniques

(a) As usual in spectral theory : Harmonic and complex analysis (complexify the energy E).

(a') Kotani's theory (projective Schrödinger cocycle for complex energies).

(b) Dynamical techniques : study of Schrödinger cocycles : KAM Theory

(c) How generalized eigenfunctions given by Berezansky's theorem decay : (Anderson) localization.

(d) In the case of the Almost Mathieu Operator : Aubry-André duality.

Generally parts (b) and (c) are sensitive to arithmetic conditions : diophantine conditions (in particular on α).
In the case of Almost Mathieu, Point (d) establishes a bridge between (b) and (c).

Plan

Dynamics of the Schrödinger cocycle

- Schrödinger cocycle : the map $(\alpha, S_E(\cdot))$:

$$\mathbb{T}^d \times \mathbb{R}^2 \to \mathbb{T}^d \times \mathbb{R}^2 \tag{1}$$

$$(\theta, v) \mapsto \left(\theta + \alpha, \underbrace{\begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}}_{S_E(\theta)} v\right) \tag{2}$$

- Iterates : $(\alpha, S_E(\cdot))^n = (n\alpha, S_E^n(\cdot))$ where $(n \geqslant 1)$

$$S_E^{(n)}(\theta) = S_E(\theta + n\alpha) \cdots S_E(\theta)$$

- Berezansky theorem : θ-fixed, μ-ae E there exist generalized eigenfunctions $H_\theta u = Eu$; equivalent to

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = S_E(\theta) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = S_E^{(n)}(\theta) \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$$

Dynamics of the Schrödinger cocycle

- Projective cocycle : the map $(\alpha, \hat{S}_E(\cdot))$ $(\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\})$:

$$\mathbb{T}^d \times \mathbb{H} \to \mathbb{T}^d \times \mathbb{H} \tag{3}$$

$$(\theta, m) \mapsto \left(\theta + \alpha, \underbrace{E - V(\theta) - \frac{1}{m}}_{\hat{S}_E(\theta)\cdot m}\right) \tag{4}$$

- Complexification : $(\alpha, \hat{S}_{E+i\epsilon}(\cdot))$ $(\epsilon > 0)$ contracts the Poincaré metric on $\mathbb{H}$ ($(\alpha, S_{E+i\epsilon})$ is uniformly hyperbolic).
- Gives existence of invariant sections $m^{\pm}_{E+i\epsilon}(\cdot) : \mathbb{T}^d \to \mathbb{H}$:

$$\hat{S}_{E+i\epsilon}(\theta) \cdot m^{\pm}_{E+i\epsilon}(\theta) = m^{\pm}_{E+i\epsilon}(\theta + \alpha)$$

such that $\begin{pmatrix} \mp m^{\pm}_{E+i\epsilon}(\cdot) \\ 1 \end{pmatrix} \mathbb{C}$ is the stable/unstable bundle of the uniformly hyperbolic cocycle $(\alpha, S_{E+i\epsilon})$.

Dynamical invariants :Rotation number and Lyapunov exponent

- The fibered rotation number :

$$\rho(\alpha, S_E) = \lim_{n\to\infty} \frac{arg(S_n(x,v),v)}{n}$$

- Lyapunov exponent :

$$LE(\alpha, S_E) = ae - \lim_{n\to\infty} \frac{1}{n} \|S_E^{(n)}(x)\|$$

(if α minimal, otherwise integrate).

Links between the spectral and dynamical objects

SPECTRAL OBJECTS		DYNAMICAL OBJECTS
$Hu = Eu$	$\longleftrightarrow$	$S_E \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}$
RESOLVENT SET $\mathbb{C} \setminus \Sigma(H)$ $\mathbb{C} \setminus \Sigma(H)$	$\longleftrightarrow$ $=$	UNIFORM HYPERBOLICITY $\{E : (\alpha, S_E)$Uniformly hyperbolic$\}$
IDS $N(-\infty, E])$	$\longleftrightarrow$ $=$	ROTATION NUMBER $1 - 2\rho(\alpha, S_E)$
DECAY OF GREEN FUNCTION	$\longleftrightarrow$	LYAPUNOV EXPONENT
SPECTRAL MEASURES $\int_{\mathbb{R}} \frac{d\mu_{\theta,0}(t)}{t-(E+i\epsilon)}$	$\longleftrightarrow$ $=$	m-FUNCTIONS $\frac{m^+_{E+i\epsilon}(\theta)m^-_{E+i\epsilon}(\theta) - 1}{m^+_{E+i\epsilon}(\theta) + m^-_{E+i\epsilon}(\theta)}$
THOULESS FORMULA $LE(\alpha, S_E) = \int_{\mathbb{R}} \log\|t-E\| dN(E)$	$\longleftrightarrow$	COMPLEX ROT. NB. IS HOLOM. $z \mapsto LE(z) + 2\pi i \rho(z)$, $\Im z > 0$

Links between the spectral and dynamical aspects

- Cocycle (α, S_E) is reducible if conjugated to constant : $\exists B : \mathbb{R}^d/2\mathbb{Z}^d \to SL(2,\mathbb{R})$, $\exists A_0 \in SL(2,\mathbb{R})$

$$S_E(\theta) = B(\theta+\alpha)A_0 B(\theta)^{-1}.$$

- Almost reducible if $\exists B_n : \mathbb{R}^d/2\mathbb{Z}^d \to SL(2,\mathbb{R})$, $\exists A_n \in SL(2,\mathbb{R})$ s.t. in the appropriate topology

$$B_n(\theta+\alpha)^{-1} S_E(\theta) B_n(\theta) - A_n \to 0$$

Does not always hold due to possible Non-Uniform Hyperbolicity but when true very useful.

Typical results on reducibility

"Reducibility Theorem" : α diophantine. There exists $\epsilon_0(\alpha)$ such that if $\rho(\alpha, S_E(\cdot))$ is diophantine w.r.t. α and if $\|V\| < \epsilon_0$ then $(\alpha, S_E(\cdot))$ is reducible.
"Almost Reducibility Theorem" : There exists ϵ_0 such that if $\|V\| < \epsilon_0$ then $(\alpha, S_E(\cdot))$ is almost-reducible.

- Reducibility Theorem : is a theorem due to Eliasson in the analytic category (also true C^k). It is a extension of Dinaburg-Sinai (ϵ_0 depended on α and ρ).
- The almost reducibility theorem is a theorem due to Eliasson when α is diophantine and in the analytic or smooth category : $\epsilon_0 = \epsilon_0(\alpha)$.
- There are extensions of both theorems when α is any irrational (but then the notion of reducibility has to be changed).
- Non-Uniform Hyperbolicity is a clear obstruction for these theorems to be true.

Links between the spectral and dynamical aspects

SPECTRAL ASPECTS $\Sigma_\theta(\alpha, V)$		DYNAMICAL ASPECTS $(\alpha, S_E(\cdot))$
AC PART OF Σ	$\longleftrightarrow$	REDUCIBILITY
PP PART OF Σ	$\longleftrightarrow$	NON-UNIFORM HYPERBOLICITY
SC PART OF Σ	$\longleftrightarrow$	?

A peculiar but rich model : Almost Mathieu

$$(Hu)_n = u_{n+1} + u_{n-1} + 2\lambda \cos(2\pi n\alpha + \theta)u_n.$$

Three regimes

- **Subcritical** $|\lambda| < 1$: $LE(\alpha, S_E) = 0$, ac-spectrum, reducibility (most of the time).
- **Supercritical** $|\lambda| > 1$: $LE(\alpha, S_E) > 0$, pp-spectrum, Non-Uniform Hyperbolicity and Anderson Localization (when dioph. condition).
- **Critical** $|\lambda| = 1$: $LE(\alpha, S_E) = 0$, sc-spectrum.

Plan

Sketch of the proof of the Aubry-André conjecture

Theorem (Aubry-André conjecture, Avila-K)

The spectrum of $H_{\lambda,\alpha,\theta}$ has Lebesgue measure $4|1-\lambda|$ for any λ, θ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Case that remained α CT and $\lambda = 1$: prove $Leb(\Sigma(H_{\lambda=1,\alpha})) = 0$.

Sketch of the proof of the Aubry-André conjecture

- Otherwise, Σ has positive Lebesgue measure.
- By Bourgain-Jitomirskaya : $LE(\alpha, S_{E,\lambda=1}) = 0$ for E in a positive Leb. meas. set.
- By a result of Kotani : for a.e E in that set the cocycle is L^2-conjugated to an $SO(2,\mathbb{R})$-valued cocycle.
- This is enough (*via a priori* estimates) to implement a renormalization scheme ;
- Allows a global$\longrightarrow$ local reduction.
- Then use Eliasson's (KAM) reducibility theorem to reduce to constant case.
- Aubry-André duality implies existence of exponentially decaying eigenfunctions.
- Contradicts $LE(\alpha, S_E) = 0$. □

Sketch of the proof of the Ten Martini Conjecture

Theorem (Ten Martini problem, Avila-Jitomirskaya)

For any α irrational, and $\lambda \neq 0$ the spectrum of the Almost Mathieu operator is a Cantor set.

Important quantity $\beta = \limsup_{n\to\infty} \frac{\log q_{n+1}}{q_n}$.
Important Theorem : Bourgain-Jitomirskaya theorem : $(\alpha, E) \mapsto LE(\alpha, E)$ is continuous on $(\mathbb{R}\setminus\mathbb{Q}) \times \mathbb{R}$.

Sketch of the proof of the Ten Martini Conjecture

(1) Have to prove that Σ cannot contain an interval. Otherwise :
(2) By an improved version of a theorem of Kotani : (α, S_E) is analytically $SO(2,\mathbb{R})$-reducible on an interval I.
(3) This and AA-duality already implies the result when $\beta = 0$.
(4) This and AA-duality also implies that Anderson Localization$\Longrightarrow$ Cantor spectrum.
(5) In any case this implies that the IDS is lipschitz w.r.t. α.

Now : two different regime

(a) Diophantine side : by Jitomirskaya's technique localization holds for $\lambda \geqslant e^{(16/9)\beta}$.
(b) Liouvillian side : If $e^{-2\beta} < \lambda \leqslant 1$: point (5) allows to use periodic approximations which are known to give good understanding of the spectrum.

In any case Cantor spectrum. □

Plan

The global theory of Artur Avila for one-frequency Schrödinger operators

Different regimes (analytic case)

- **Subcritical** : Uniform subexponential bound on the growth of $(\alpha, S_E(z))$ for $|\Im z| < \epsilon$ (implies $LE(\alpha, S_E(\cdot + is)) = 0$ for all $|s| < \epsilon$).
- **Supercritical** : $LE(\alpha, S_E(\cdot) > 0$.
- **Critical** : otherwise.

The global theory of Artur Avila for one-frequency q.p. Schrödinger operators

Theorem

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\delta > 0$. The set of (V, E) s.t. E is critical for $H_{\alpha,V}$, is contained in a contable union of codimension one analytic submanifolds of $C^\omega_\delta(\mathbb{R}/\mathbb{Z}, R) \times \mathbb{R}$.

Conjecture

For a typical operator H, the spectral measures have no singular continuous component.

Stratified analyticity of the Lyapunov exponent

Definition : If X is a topological space, a stratification of X is a finite or countable strictly decreasing sequence of closed sets $X = X_0 \supset X_1 \supset \cdot$ s.t $\cap_i X_i = \emptyset$. $X_i \setminus X_{i+1}$ is the i-th stratum of the stratification. If X is a subset of a real analytic manifold and $f : X \to \mathbb{R}$ a continuous function, we say that f is C^r-stratified if there exists a stratification s.t the restriction of f to each stratum is C^r.

Theorem

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $V : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ real analytic. Then the Lyapunov exponent is a C^ω-stratified analytic function of the energy ! ! ! ! !

Quantization of the acceleration and regularity

- **Quantification of the acceleration** : the map $\epsilon \mapsto LE(\alpha, A(\cdot + i\epsilon)$ is piecewise linear with integer slopes.
- **Regularity** : if $\epsilon \mapsto LE(\alpha, A(\cdot + i\epsilon)$ is affine for ϵ in a neighborhood of 0.
- **Regularity theorem** : if (α, A) is regular and $LE(\alpha, A) > 0$ then (α, A) is uniformly hyperbolic.

The Almost reducibility conjecture

Conjecture

$$LE(\alpha, A) = 0 + \textit{regularity} \Longrightarrow \textit{almost reducibility}$$

Conjecture proved for

- Almost Mathieu
- Cocycles close to constants
- α exponentially well approximated by rationals ($\limsup \frac{\log q_{n+1}}{q_n} > 0$)