Plan

1. The model
2. Results by Artur Avila
3. Dynamics of the Schrödinger cocycle
4. Sketches of some proofs
5. The global theory of Artur Avila for one-frequency Schrödinger operators

Quasi-periodic 1D Schrödinger operator

We shall describe the impressive collection of deep results obtained by Artur Avila on the spectral properties of 1D quasi-periodic Schrödinger operators.

- 1D Schrödinger operator
  \[ H : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \]
  \[ (u_n)_{n \in \mathbb{Z}} \mapsto (Hu)_n := (u_{n+1} + u_{n-1} + V_n u_n)_{n \in \mathbb{Z}} \]

  The sequence \((V_n)_{n \in \mathbb{Z}} \in \mathbb{R}^\mathbb{Z}\) is called the potential.

- Quasi-periodic: means
  \[ V_n = V(\theta + n\alpha) \]

  where \(V : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}\) is, say, a continuous map, \(\theta \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d\) is the phase and \(\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{T}^d\) is the frequency vector.

Remarks:
- We shall mainly concentrate on the case \(d = 1\) (one frequency) and \(V\) real analytic.
- The potential is defined by a dynamics: we say that it is dynamically defined.
- Observe that the potential \(H_\theta\) depends on the phase \(\theta\); we shall denote it by \(H_\theta\).
Spectrum

In that framework: $H_\theta$ is a symmetric bounded operator.

Spectrum of $H_\theta$,

$$\Sigma(H_\theta) = \text{Spec}(H_\theta) := \{ E \in \mathbb{C} : (H_\theta - E)^{-1} \text{ exists and is continuous} \}$$

It is a compact subset of $\mathbb{R}$.

The model

Aim: Understand

- the Spectrum of $H_\theta$
- The nature of the spectrum which means whether $H_\theta$ has some ac, sc, pp component
- The spectral measures $\mu_\theta$ for most/all phases.

Spectral measures

- $\forall u \in l^2(\mathbb{Z}), \exists \mu_{u,\theta}$ probability measure s.t. $\forall z \in \mathbb{C}, \exists z > 0$

$$\langle (H_\theta - z)^{-1}u, u \rangle = \int_{\mathbb{R}} \frac{d\mu_{u,\theta}(t)}{t - z}.$$

$\mu_u$ is called the spectral measure associated to $u$.

- This measures allow Functional Calculus (Spectral Theorem).
- Since $H$ is of Schrödinger type, to understand $H_\theta$ it is enough to know

$$\mu_\theta = \frac{1}{2}(\mu_{\delta_0,\theta} + \mu_{\delta_1,\theta})$$

which is called the spectral measure of $H_\theta$.

- One has $\text{supp}(\mu_{H_\theta}) = \text{Spec}(H_\theta)$.
- One defines $\text{Spec}_{ac}(H_\theta), \text{Spec}_{sc}(H_\theta), \text{Spec}_{pp}(H_\theta)$ as the support of the ac, sc, pp part of $\mu_\theta$.

Important facts and concepts

- Since the potential is dynamically defined the spectrum $\text{Spec}(H_\theta)$ does not depend on $\theta$: denote it by $\Sigma$.
- The Integrated Density of States (IDS):

$$N = \int_{\Sigma} \mu_\theta d\theta$$

If we denote by $(E_n(\theta))_{-n \leq n \leq n}$ the eigenvalues (which are all real) of the finite range operator $H_\theta^n$, then the sequence of measures

$$N_\theta^n = \frac{1}{2n+1} \sum_{j=-n}^{n} \delta_{E_{n,j}(\theta)}$$

converges weakly to $N$.

- On the other hand, the spectral measures $\mu_\theta$ strongly depend on $\theta$. 
Schrödinger operators

Berezansky’s theorem

Though the eigenvalue equation does not always have a solution in \( l^2(\mathbb{Z}) \) there exist a lot of solutions to this equation with moderate growth.

This is the content of Berezansky’s theorem: For \( \mu_{\theta}-\text{a.e.} \) \( E \in \Sigma(H_\theta) = \Sigma \) there exists a solution \((u_k)_{k \in \mathbb{Z}} \) of \( H_\theta u = Eu \)

\[ \forall n \in \mathbb{Z}, \ u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n = Eu_n \]

such that \( |u_n| = O((1 + |n|)^{1/2+\epsilon}) \).

Almost Mathieu operator

- A very studied case is when \( V(\theta) = 2\lambda \cos(2\pi \theta) \). One then speaks of the Almost-Mathieu Operator.
- \( \lambda \) is called the coupling constant: it determines different regimes.
- In that case we denote the operator by \( H_{\lambda,\alpha,\theta} \).
- Aubry-André duality: \((u_k)_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}), f(\varphi) = \sum_{k \in \mathbb{Z}} u_k e^{2\pi i k \varphi} \)

\[ H_{\lambda,\alpha,\theta} u = Eu \iff e^{2\pi i \theta} f(\varphi + \alpha) + e^{-2\pi i \theta} f(\varphi - \alpha) + \frac{1}{\lambda} \cos(2\pi \varphi) f(\varphi) = E \frac{2}{\lambda} f(\varphi). \]

Ex. of consequence:

\[ \Sigma(H_\lambda) = \frac{1}{2\lambda} \Sigma(H_{1/\lambda}). \]

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Results by Artur Avila

Artur Avila has obtained outstanding results
- on the Almost Mathieu Operator for the
  - Spectrum: measure and Cantor nature;
  - IDS: absolute continuity;
  - Spectral measures: Hölder continuity.
- on the Dynamics of the related Schrödinger cocycle (see below).
- He has developed a theory that shows that the intuition given by the study of the Almost Mathieu Operator is a good guide to study general potentials and at the same time it shows that the Almost-Mathieu potential is very particular.
Results on **the spectrum** of the Almost-Mathieu Operator

**Theorem (Aubry-André conjecture, Avila-K)**

The spectrum of $H_{\lambda,\alpha,\theta}$ has Lebesgue measure $4|1 - \lambda|$ for any $\lambda$, $\theta$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Ann. of Maths, 2006

Previous results by Jitomirskaya-Krasovskii ($\lambda \neq 2$) and Last ($\lambda = 1$, $\alpha$ not CT).

Pb 5 of Simon’s list

Techniques: Kotani theory+Renormalization+KAM

**Theorem (Ten Martini problem, Avila-Jitomirskaya)**

For any $\alpha$ irrational, and $\lambda \neq 0$ the spectrum of the Almost Mathieu operator is a Cantor set.

Ann. of Math, 2009

Pb 4 of Simon’s list.

Previous results by Bellissard-Simon, Sinai, Hellfer-Sjöstrand, Choi-Elliott-Yui, Avron-van Moche-Simon, Last, Puig...

**Theorem (Dry Ten Martini problem, Avila-Jitomirskaya)**

For any $\alpha$ in DC, and $\lambda \neq -1, 0, 1$ all the gaps of the spectrum of the Almost Mathieu operator predicted by the gap labeling theorem are open.

JEMS, 2010

Results on **the IDS** of the Almost-Mathieu Operator

**Theorem (Avila-Damanik)**

For any $\alpha$, and $|\lambda| \neq 1$ the Integrated Density of States of $H_{\lambda,\alpha,\theta}$ is absolutely continuous.


Previous and related results by Jitomirskaya, Goldstein-Schlag.

**Theorem (Avila-Jitomirskaya)**

For any $\alpha$ in DC, and $\lambda \neq -1, 0, 1$ the Integrated Density of States of the Almost Mathieu operator $H_{\lambda,\alpha,\theta}$ is $1/2$-Hölder.

JEMS, 2010

Results on **the Spectral measures** of the Almost-Mathieu Operator

**Theorem (Problem 6 of Simon’s list, Avila-Damanik)**

For any $\alpha$, $|\lambda| < 1$ and almost every $\theta$, the spectral measures of $H_{\lambda,\alpha,\theta}$ are absolutely continuous.

**Theorem (Problem 6 of Simon’s list, Avila-Jitomirskaya)**

For any $\alpha$ in DC, $0 < |\lambda| < 1$ and all $\theta$ the spectral measures of the Almost Mathieu operator $H_{\lambda,\alpha,\theta}$ are absolutely continuous.
Artur and the Almost Mathieu Operator

How to attack these problems?

Techniques

(a) As usual in spectral theory: Harmonic and complex analysis (complexify the energy $E$).

(a') Kotani's theory (projective Schrödinger cocycle for complex energies).

(b) Dynamical techniques: study of Schrödinger cocycles: KAM Theory

(c) How generalized eigenfunctions given by Berezansky’s theorem decay: (Anderson) localization.

(d) In the case of the Almost Mathieu Operator: Aubry-André duality.

Generally parts (b) and (c) are sensitive to arithmetic conditions: diophantine conditions (in particular on $\alpha$).
In the case of Almost Mathieu, Point (d) establishes a bridge between (b) and (c).

Plan

1. The model
2. Results by Artur Avila
3. Dynamics of the Schrödinger cocycle
   - Links between spectral and dynamical objects
   - Reducibility
   - Different regimes of the Almost Mathieu
4. Sketches of some proofs
5. The global theory of Artur Avila for one-frequency Schrödinger operators

Dynamics of the Schrödinger cocycle

- Schrödinger cocycle: the map $(\alpha, S_E(\cdot))$:
  \[ T^d \times \mathbb{R}^2 \rightarrow T^d \times \mathbb{R}^2 \]
  \[ (\theta, \nu) \mapsto \left( \theta + \alpha, \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix} \nu \right) \]

- Iterates: $(\alpha, S_E(\cdot))^n = (n\alpha, S_E^n(\cdot))$ where $n \geq 1$

- Berezansky theorem: $\theta$-fixed, $\mu$-ae $E$ there exist generalized eigenfunctions $H_{\mu}u = Eu$; equivalent to
  \[ \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = S_E(\theta) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = S_E^{(n)}(\theta) \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} \]

- Projective cocycle: the map $(\alpha, \hat{S}_E(\cdot)) (\mathbb{H} = \{ z \in \mathbb{C} : \Im z > 0 \})$:
  \[ T^d \times \mathbb{H} \rightarrow T^d \times \mathbb{H} \]
  \[ (\theta, m) \mapsto \left( \theta + \alpha, E - V(\theta) - \frac{1}{m} \right) \]

- Complexification: $(\alpha, \hat{S}_{E+i\epsilon}(\cdot)) (\epsilon > 0)$ contracts the Poincaré metric on $\mathbb{H}$ ($(\alpha, S_{E+i\epsilon})$ is uniformly hyperbolic).
- Gives existence of invariant sections $m_{E+i\epsilon}^{\pm}(\cdot): T^d \rightarrow \mathbb{H}$:
  \[ \hat{S}_{E+i\epsilon}(\theta) \cdot m_{E+i\epsilon}^{\pm}(\theta) = m_{E+i\epsilon}^{\pm}(\theta + \alpha) \]
  such that $\begin{pmatrix} \mp m_{E+i\epsilon}^{\pm}(\cdot) \end{pmatrix}$ is the stable/unstable bundle of the uniformly hyperbolic cocycle $(\alpha, S_{E+i\epsilon})$. 
Dynamical invariants: Rotation number and Lyapunov exponent

- The fibered rotation number:
  \[ \rho(\alpha, S_E) = \lim_{n \to \infty} \frac{\arg(S_n(x, \nu), \nu)}{n} \]

- Lyapunov exponent:
  \[ LE(\alpha, S_E) = ae - \lim_{n \to \infty} \frac{1}{n}||S_E^n(x)|| \]
  (if \( \alpha \) minimal, otherwise integrate).

Links between the spectral and dynamical objects

<table>
<thead>
<tr>
<th>SPECTRAL OBJECTS</th>
<th>DYNAMICAL OBJECTS</th>
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<tbody>
<tr>
<td>( Hu = Eu )</td>
<td>( S_E \begin{pmatrix} u_n \ u_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1} \ u_n \end{pmatrix} )</td>
</tr>
<tr>
<td>( \text{RESOLVENT SET} \ C \setminus \Sigma(H) ) ( \subset \Sigma(H) )</td>
<td>( \text{UNIFORM HYPERBOLICITY} ) ( { E : (\alpha, S_E) \text{Uniformly hyperbolic} } )</td>
</tr>
<tr>
<td>( \text{IDS} ) ( N(-\infty, E) )</td>
<td>( \text{ROTATION NUMBER} ) ( 1 - 2\rho(\alpha, S_E) )</td>
</tr>
<tr>
<td>( \text{DECAY OF GREEN FUNCTION} )</td>
<td>( \text{LYAPUNOV EXponent} )</td>
</tr>
<tr>
<td>( \int_{\mathbb{R}} \frac{d\mu_{\theta,0}(t)}{t - (E + i\epsilon)} )</td>
<td>( m\text{-FUNCTIONS} )</td>
</tr>
<tr>
<td>( \text{THOULESS FORMULA} )</td>
<td>( \text{COMPLEX ROT. NB. IS HOLOM.} )</td>
</tr>
<tr>
<td>( LE(\alpha, S_E) = \int_{\mathbb{R}} \log</td>
<td>t - E</td>
</tr>
</tbody>
</table>

Links between the spectral and dynamical aspects

- Cocycle \( (\alpha, S_E) \) is reducible if conjugated to constant:
  \( \exists B : \mathbb{R}^d / 2\mathbb{Z}^d \to SL(2, \mathbb{R}), \exists A_0 \in SL(2, \mathbb{R}) \)
  \[ S_E(\theta) = B(\theta + \alpha)A_0B(\theta)^{-1}. \]

- Almost reducible if \( \exists B_n : \mathbb{R}^d / 2\mathbb{Z}^d \to SL(2, \mathbb{R}), \exists A_n \in SL(2, \mathbb{R}) \) s.t. in the appropriate topology
  \[ B_n(\theta + \alpha)^{-1}S_E(\theta)B_n(\theta) - A_n \to 0 \]

Does not always hold due to possible Non-Uniform Hyperbolicity but when true very useful.

Typical results on reducibility

"Reducibility Theorem": \( \alpha \) diophantine. There exists \( \epsilon_0(\alpha) \) such that if \( \rho(\alpha, S_E(\cdot)) \) is diophantine w.r.t. \( \alpha \) and if \( \| V \| < \epsilon_0 \) then \( (\alpha, S_E(\cdot)) \) is reducible.

"Almost Reducibility Theorem": There exists \( \epsilon_0 \) such that if \( \| V \| < \epsilon_0 \) then \( (\alpha, S_E(\cdot)) \) is almost-reducible.

- Reducibility Theorem: is a theorem due to Eliasson in the analytic category (also true \( C^k \)). It is a extension of Dinaburg-Sinai (\( \epsilon_0 \) depended on \( \alpha \) and \( \rho \)).
- The almost reducibility theorem is a theorem due to Eliasson when \( \alpha \) is diophantine and in the analytic or smooth category: \( \epsilon_0 = \epsilon_0(\alpha) \).
- There are extensions of both theorems when \( \alpha \) is any irrational (but then the notion of reducibility has to be changed).
- Non-Uniform Hyperbolicity is a clear obstruction for these theorems to be true.
Links between the spectral and dynamical aspects

<table>
<thead>
<tr>
<th>SPECTRAL ASPECTS $\Sigma_\theta(\alpha, V)$</th>
<th>DYNAMICAL ASPECTS $(\alpha, S_E(\cdot))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC PART OF $\Sigma$</td>
<td>REDUCIBILITY</td>
</tr>
<tr>
<td>PP PART OF $\Sigma$</td>
<td>NON-UNIFORM HYPERBOLICITY</td>
</tr>
<tr>
<td>SC PART OF $\Sigma$</td>
<td>?</td>
</tr>
</tbody>
</table>

A peculiar but rich model: Almost Mathieu

$$(Hu)_n = u_{n+1} + u_{n-1} + 2\lambda \cos(2\pi n \alpha + \theta) u_n.$$ 

Three regimes

- **Subcritical** $|\lambda| < 1$ : $LE(\alpha, S_E) = 0$, ac-spectrum, reducibility (most of the time).
- **Supercritical** $|\lambda| > 1$ : $LE(\alpha, S_E) > 0$, pp-spectrum, Non-Uniform Hyperbolicity and Anderson Localization (when dioph. condition).
- **Critical** $|\lambda| = 1$ : $LE(\alpha, S_E) = 0$, sc-spectrum.

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Sketch of the proof of the Aubry-André conjecture

Theorem (Aubry-André conjecture, Avila-K)

The spectrum of $H_{\lambda, \alpha, \theta}$ has Lebesgue measure $4|1 - \lambda|$ for any $\lambda, \theta$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Case that remained $\alpha$ CT and $\lambda = 1$ : prove $\text{Leb}(\Sigma(H_{1, \alpha, \cdot})) = 0.$
Sketch of the proof of the Aubry-André conjecture

- Otherwise, $\Sigma$ has positive Lebesgue measure.
- By Bourgain-Jitomirskaya: $LE(\alpha, S_E, \lambda=1) = 0$ for $E$ in a positive Leb. meas. set.
- By a result of Kotani: for a.e $E$ in that set the cocycle is $L^2$-conjugated to an $SO(2, \mathbb{R})$-valued cocycle.
- This is enough (via a priori estimates) to implement a renormalization scheme.
- Allows a global $\to$ local reduction.
- Then use Eliasson’s (KAM) reducibility theorem to reduce to constant case.
- Aubry-André duality implies existence of exponentially decaying eigenfunctions.
- Contradicts $LE(\alpha, S_E) = 0$. □

Sketch of the proof of the Ten Martini Conjecture

(1) Have to prove that $\Sigma$ cannot contain an interval. Otherwise:
(2) By an improved version of a theorem of Kotani: $(\alpha, S_E)$ is analytically $SO(2, \mathbb{R})$-reducible on an interval $I$.
(3) This and AA-duality already implies the result when $\beta = 0$.
(4) This and AA-duality also implies that Anderson Localization $\implies$ Cantor spectrum.
(5) In any case this implies that the IDS is lipschitz w.r.t. $\alpha$.

Now: two different regime
(a) Diophantine side: by Jitomirskaya’s technique localization holds for $\lambda \geq e^{(16/9)\beta}$.
(b) Liouvillian side: if $e^{-2\beta} < \lambda \leq 1$: point (5) allows to use periodic approximations which are known to give good understanding of the spectrum.

In any case Cantor spectrum. □

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\textbf{Theorem (Ten Martini problem, Avila-Jitomirskaya)}

For any $\alpha$ irrational, and $\lambda \neq 0$ the spectrum of the Almost Mathieu operator is a Cantor set.

Important quantity $\beta = \limsup_{n \to \infty} \frac{\log q_{n+1}}{q_n}$.

Important Theorem: Bourgain-Jitomirskaya theorem: $(\alpha, E) \mapsto LE(\alpha, E)$ is continuous on $(\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}$. 
Different regimes (analytic case)

- **Subcritical**: Uniform subexponential bound on the growth of $(\alpha, S_E(z))$ for $|\Im z| < \epsilon$ (implies $\text{LE}(\alpha, S_E(\cdot + i\epsilon)) = 0$ for all $|s| < \epsilon$).

- **Supercritical**: $\text{LE}(\alpha, S_E(\cdot)) > 0$.

- **Critical**: otherwise.

Conjecture

For a typical operator $H$, the spectral measures have no singular continuous component.

**Theorem**

\[ \alpha \in \mathbb{R} \setminus \mathbb{Q}, \delta > 0. \text{ The set of } (V, E) \text{ s.t. } E \text{ is critical for } H_{\alpha, V}, \text{ is contained in a countable union of codimension one analytic submanifolds of } C^\omega_0(\mathbb{R}/\mathbb{Z}, \mathbb{R}) \times \mathbb{R}. \]

**Defintion**: If $X$ is a topological space, a **stratification** of $X$ is a finite or countable strictly decreasing sequence of closed sets $X = X_0 \supset X_1 \supset \cdots$ s.t $\bigcap_i X_i = \emptyset$. $X_i \setminus X_{i+1}$ is the $i$-th stratum of the stratification. If $X$ is a subset of a real analytic manifold and $f : X \to \mathbb{R}$ a continuous function, we say that $f$ is $C^r$-stratified if there exists a stratification s.t the restriction of $f$ to each stratum is $C^r$.

**Theorem**

\[ \alpha \in \mathbb{R} \setminus \mathbb{Q}, V : \mathbb{R}/\mathbb{Z} \to \mathbb{R} \text{ real analytic. Then the Lyapunov exponent is a } C^\omega \text{-stratified analytic function of the energy} \]

- **Quantification of the acceleration**: the map $\epsilon \mapsto \text{LE}(\alpha, A(\cdot + i\epsilon))$ is piecewise linear with integer slopes.

- **Regularity**: if $\epsilon \mapsto \text{LE}(\alpha, A(\cdot + i\epsilon))$ is affine for $\epsilon$ in a neighborhood of 0.

- **Regularity theorem**: if $(\alpha, A)$ is regular and $\text{LE}(\alpha, A) > 0$ then $(\alpha, A)$ is uniformly hyperbolic.
The Almost reducibility conjecture

**Conjecture**

\[ LE(\alpha, A) = 0 + \text{regularity} \implies \text{almost reducibility} \]

Conjecture proved for
- Almost Mathieu
- Cocycles close to constants
- \( \alpha \) exponentially well approximated by rationals (\( \lim \sup \frac{\log q_{n+1}}{q_n} > 0 \))