For a perfect score you should give complete solutions of two problems from each of the two sections.

SECTION 1

1.1. Prove that the set of Lebesgue density points of a measurable set $A$ is a $G_{δσ}$ set, i.e. a countable union of countable intersections of open sets.

1.2. Prove using only course material (and not any reference to Fourier analysis) that for any measurable set $A \subset [0, 1]$

\[ \int_A \sin nx \, dx \to 0 \text{ as } n \to \infty. \]

1.3. Let $ν_n$ be a sequence of signed Borel measures on $[0, 1]$, $\text{Var}(ν_n) = 1$. Is it possible that for every interval $I \subset [0, 1] \nu_n(I) \to 0$?

1.4. Suppose $ν$ is a signed measure on the unit square $I^2 = [0, 1] \times [0, 1]$ such that for any rectangle $R = [a, b] \times [c, d]$, $ν(R)$ depends only on $b - a$ and $d - c$.

Prove that $ν$ is proportional to the Lebesgue measure.

SECTION 2

2.1. Let $1 < p < \infty$. Construct a function which belongs to $L^{p-\epsilon}([0, 1], λ)$ for every $\epsilon > 0$ but not to $L^p([0, 1], λ)$.

2.2. A function $f$ on $[0, 1]$ is convex if for any $x, y, t \in [0, 1]$,

\[ f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y). \]

Prove that any convex function has derivative almost everywhere.

2.3. Prove that any sequence of functions $f_n \in L^p(X, μ)$ which converges in measure and such that $|f_n(x)| \leq f(x)$, for some function $f \in L^p(X, μ)$, converges in $L^p$.

2.4. Prove that for any function $f \in L^1(X, μ)$ the integral

\[ \int_A f \, dμ \]

is a continuous function on the metric space of the equivalence classes of measurable sets mod 0 with the distance $d(A, B) = μ(AΔB)$. 