Prize offered for the solution of a problem on mixing flows

Posted by A. Bressan on February 20, 2006

Statement of the problem

Consider the two-dimensional torus $Q \doteq \mathbb{R}^2/\mathbb{Z}^2$. Points in Q will be described using the coordinates $x = (x_1, x_2) \in [0,1[\times[0,1[$. The Lebesgue measure on the unit square naturally induces a measure on the torus. Moreover, the Euclidean distance on $\mathbb{R}^2$ naturally induces a (geodesic) distance on Q. We split Q as the union of the two disjoint sets (fig. 1)

$$A \doteq \big\{(x_1, x_2)\,;\quad 0 \leq x_1 < 1/2\big\}\,,$$

$$A' \doteq \big\{(x_1, x_2)\,;\quad 1/2 \leq x_1 < 1\big\}\,.$$

Let $f : [0,T] \times Q \mapsto \mathcal{T}Q$ be a time dependent vector field on Q. Here $\mathcal{T}Q$ denotes the tangent bundle to the manifold Q. For any initial point y, denote by $t \mapsto x(t) \doteq \Psi_t(y)$ the solution of the Cauchy problem

$$\frac{d}{dt}x(t) = f\big(t, x(t)\big)\,, \qquad\qquad x(0) = y\,. \tag{1}$$

Definition. We say that the flow Ψ_T **mixes the sets** A, A' **up to scale** ε if every ball of radius ε contains at least 1/3 of its points coming from A and at least 1/3 of its points coming from A'. Otherwise stated, for every $x \in Q$, calling B_ε^x the open ball centered at x with radius ε one has

$$\text{meas}\big(B_\varepsilon^x \cap \Psi_T(A)\big) \geq \frac{\text{meas}(B_\varepsilon^x)}{3}\,,$$

$$\text{meas}\big(B_\varepsilon^x \cap \Psi_T(A')\big) \geq \frac{\text{meas}(B_\varepsilon^x)}{3}\,.$$

I am offering a prize of US $ 500 to the first person who provides a proof or a counterexample to the following

==

Conjecture: There exists a constant $\beta > 0$ such that the following holds. For any $\varepsilon \in]0, 1/4]$ and $T > 0$, if $f : [0,T] \times Q \mapsto \mathcal{T}Q$ is a smooth, divergence-free vector field on Q whose flow Ψ_T mixes the sets A, A' up to scale ε, then

$$\int_0^T \int_Q \big|\nabla_x f(t,x)\big|\,dxdt \geq \beta \cdot |\ln \varepsilon|\,. \tag{2}$$

==

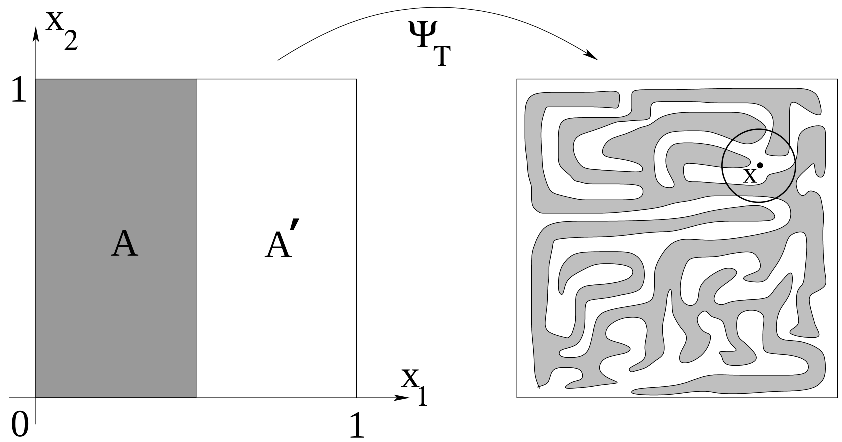

figure 1

Remarks

1. A slightly more general form of this conjecture was stated in the paper

[B] A. Bressan, A lemma and a conjecture on the cost of rearrangements, *Rendiconti del Seminario Matematico dell'Università di Padova* **110** (2003), pp.97–102.

2. A vector field on the torus can be identified to a doubly periodic vector field on the plane $I\!R^2$, i.e. a map $f : [0,T] \times I\!R^2 \mapsto I\!R^2$ such that

$$f(t,\, x_1,\, x_2) = f(t,\, x_1+1,\, x_2) = f(t,\, x_1,\, x_2+1)$$

for all $t \in [0,T]$ and $(x_1,\, x_2) \in I\!R^2$.

3. The assumption that the vector field $f = (f^1, f^2)$ is divergence-free (w.r.t. the x-variable) means that

$$\frac{\partial f^1}{\partial x_1} + \frac{\partial f^2}{\partial x_2} \equiv 0 \qquad\qquad \text{for all } t,\, x_1,\, x_2\,.$$

This condition implies that the flow is measure preserving, i.e.

$$\text{meas}\big(\Psi_t(S)\big) = \text{meas}(S) \tag{3}$$

for every $t \in [0,T]$ and every measurable subset $S \subseteq Q$.

4. Defining the rescaled vector field

$$\tilde f(t,x) \doteq T \cdot f(t/T,\, x)\,,$$

it is not restrictive to consider only the case $T = 1$.

5. Assume that there exists a measure-preserving map $\varphi : \Psi_T(A') \mapsto \Psi_T(A)$ such that

$$d\big(x,\, \varphi(x)\big) \le \rho \qquad\qquad \text{for all } \ x \in \Psi_T(A')\,,$$

where $d(\cdot,\cdot)$ denotes the (geodesic) distance on the torus. Then we claim that the flow Ψ_T mixes the sets A, A' up to scale ε, for every ε such that

$$\varepsilon \ge \frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}\,\rho\,. \tag{4}$$

Notice that the inequality (4) is equivalent to

$$\frac{\mathrm{meas}(B^x_{\varepsilon-\rho})}{2} \geq \frac{\mathrm{meas}(B^x_\varepsilon)}{3}\,.$$

To prove the claim, for any $x \in Q$ consider the smaller circle $B^x_{\varepsilon-\rho}$ centered at x with radius $\varepsilon - \rho$. Two possibilities can arise:

(i) If at least half of the points in $B^x_{\varepsilon-\rho}$ come from A, then

$$\mathrm{meas}\big(B^x_\varepsilon \cap \Psi_T(A)\big) \geq \mathrm{meas}\big(B^x_{\varepsilon-\rho} \cap \Psi_T(A)\big) \geq \frac{\mathrm{meas}(B^x_{\varepsilon-\rho})}{2} \geq \frac{\mathrm{meas}(B^x_\varepsilon)}{3}\,.$$

(ii) In the opposite case, at least half of the points in $B^x_{\varepsilon-\rho}$ come from A'. In other words, setting

$$S \doteq \Psi_T(A') \cap B^x_{\varepsilon-\rho}$$

we have

$$\mathrm{meas}(S) \geq \frac{\mathrm{meas}(B^x_{\varepsilon-\rho})}{2}\,.$$

By assumptions, $\varphi(S) \subseteq \Psi_T(A) \cap B^x_\varepsilon$. Therefore

$$\mathrm{meas}\big(B^x_\varepsilon \cap \Psi_T(A)\big) \geq \mathrm{meas}\big(\varphi(S)\big) = \mathrm{meas}(S) \geq \frac{\mathrm{meas}(B^x_{\varepsilon-\rho})}{2} \geq \frac{\mathrm{meas}(B^x_\varepsilon)}{3}\,.$$

Reversing the roles of A, A', the claim is proved.

6. As a motivation for the estimate (2), consider the spatially periodic vector field $f : [0,\infty[\, \times I\!R^2 \mapsto I\!R^2$ defined as follows (here $[[s]]$ denotes the largest integer $\leq s$).

$$f(t,x_1,x_2) = \begin{cases} (2^{-1},0) & \text{if} \quad [[2x_2]] \text{ odd}\,, \\ (0,0) & \text{if} \quad [[2x_2]] \text{ even}\,, \end{cases} \qquad t \in [0,1]\,,$$

while, for $n \geq 1$,

$$f(t,x_1,x_2) = \begin{cases} (0,\, 2^{-n}) & \text{for} \quad [[2^{n+1}x_1]] \text{ odd}\,, \\ (0,0) & \text{for} \quad [[2^{n+1}x_1]] \text{ even}\,, \end{cases} \qquad t \in]2n-1,\, 2n]\,,$$

$$f(t,x_1,x_2) = \begin{cases} (2^{-n-1},\, 0) & \text{for} \quad [[2^{-1} + 2^n x_2]] \text{ even}\,, \\ (0,0) & \text{for} \quad [[2^{-1} + 2^n x_2]] \text{ odd}\,, \end{cases} \qquad t \in [2n,\, 2n+1]\,.$$

Fix any integer $n \geq 1$. For $T = 2n - 1$, the flow Ψ_T maps A, A' into a checkerboard pattern, with squares of side length 2^{-n} (see fig. 2). Hence it it easy to construct a measure preserving map $\varphi : \Psi_T(A') \mapsto \Psi_T(A)$ such that

$$\big|x - \varphi(x)\big| \leq 2^{-n} \qquad \text{for all } x\,.$$

By the previous remark, the flow Ψ_{2n-1} mixes the sets A, A' up to scale

$$\varepsilon \doteq \frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}} \cdot 2^{-n}.$$

On the other hand, the total variation of the vector field $f(t,\cdot)$ is 2 or 4, respectively for $t \in]2n,\, 2n+1]$ or $t \in]2n-1,\, 2n]$.

By taking a mollification of f, we thus obtain a smooth vector field $\tilde{f}$ such that

$$\int_Q \big|\nabla_x \tilde{f}(t,x)\big|\, dx \leq 4 \qquad\qquad t \geq 0\,,$$

while the corresponding flow $\widetilde{\Psi}_T$ mixes the sets A, A' at a scale $\tilde{\varepsilon}$ arbitrarily close to ε. For every large integer n, this provides an example where

$$\int_0^T \int_Q \big|\nabla_x \tilde{f}(t,x)\big|\, dx < 4T = 4(2n-1) < 8n\,,$$

$$|\ln \tilde{\varepsilon}| > n \ln 2 - \ln\Big(\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}\Big) - 1\,.$$

This analysis shows that, if a constant $\beta > 0$ satisfying (2) does exists, it must satisfy

$$\beta \leq \frac{8}{\ln 2}\,.$$

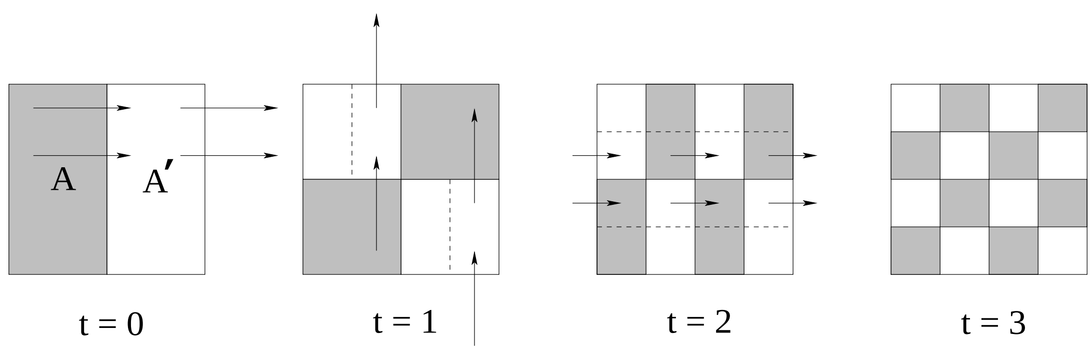

figure 2

==

Added on Jan. 15, 2011:

At the present date, the problem has not yet been solved. However, some very interesting related results have appeared. See in particular:

[CD] G. Crippa and C. De Lellis, Estimates and regularity results for the DiPerna-Lions flow. *J. Reine Angew. Math.* **616** (2008), 15-46.

==