1 Outline of the model

As a motivation, think of a forest fire. To restrict its propagation, assume that a barrier can be constructed, in real time. This could be a thin strip of land which is either soaked with water poured down from a helicopter, or cleared from all vegetation using a bulldozer, or sprayed with fire extinguisher by a team of firemen. In any case, the fire will not cross that particular strip of land. Here the key point is that the barrier is being constructed at the same time as the fire front is advancing.

In this setting, a natural question is whether one can completely block the propagation of the fire, by eventually surrounding it with barriers. Of course, the answer depends on the speed at which we can construct the barrier, compared with the speed at which the fire front advances.

In cases where the fire can be blocked, one can further ask: what is the best blocking strategy? To answer this second question, one should determine the optimal location of the barriers, in order to minimize:

\[
\text{[total value of the burned area]} + \text{[total cost for constructing the barriers]} \quad (1)
\]

To study these problems, a mathematical model was introduced in [2]. For each time \( t \geq 0 \), we call \( R(t) \) the region in the plane burned by the fire up to time \( t \), while \( \gamma(t) \) is the portion of the barrier (a one-dimensional curve) constructed up to time \( t \). Clearly

\[
0 \leq t_1 \leq t_2 \implies R(t_1) \subseteq R(t_2), \quad \gamma(t_1) \subseteq \gamma(t_2).
\]

We assume that the barrier can be constructed at a maximum speed \( \sigma \). We thus say that the strategy \( \gamma(\cdot) \) is \textit{admissible} if

\[
[\text{length of } \gamma(t)] \leq \sigma t \quad \text{for all } t > 0. \quad (2)
\]

Concerning fire propagation (without barriers), a large literature is currently available [13]. The very simplest mathematical model assumes that fire propagates isotropically with unit speed in all directions. Calling \( R_0 \) the region burned at the initial time \( t = 0 \), the region burned at time \( t \) is thus

\[
R(t) = \{x \in \mathbb{R}^2 ; \ d(x,R_0) \leq t\}. \quad (3)
\]

Here \( d(x,R_0) = \inf\{d(x,y) ; \ y \in R_0\} \) is the distance of the point \( x \) to the set \( R_0 \). If the terrain has variable characteristics, or if wind is blowing, a more realistic model will allow
the fire to spread at different speeds, depending on the location and on the direction of the advancing front.

Fire propagation can equivalently be described in terms of the scalar function

$$T(x) = \inf\{t \geq 0 ; \ x \in R(t)\}.$$ 

Here $T(x)$ is the time needed for the fire to reach the point $x$. Notice that $T(x) = +\infty$ if the point $x$ is never burned by the fire. In general, the minimal time function $T$ can be computed by solving a Hamilton-Jacobi PDE of the form

$$H(x, \nabla T(x)) = 1, \quad T(x) = 0 \quad \text{if} \quad x \in R_0.$$ (4)

If the function $T$ is known, we can then recover the region $R(t)$ burned within time $t$ as

$$R(t) = \{x \in \mathbb{R}^2 ; \ T(x) < t\}.$$ 

Clearly, if a barrier $\gamma$ is being constructed, at any time $t > 0$ the fire will have destroyed a smaller region $R^\gamma(t) \subseteq R(t)$. Two mathematical problems can now be formulated.

(BP) Blocking Problem. For a given model of fire propagation, decide whether there exists an admissible strategy $\gamma(\cdot)$ as in (2) such that the entire region burned by the fire $\bigcup_{t>0} R^\gamma(t)$ is bounded.

(OP) Optimization Problem. If a blocking strategy exists, find an admissible strategy $t \mapsto \gamma(t)$ which minimizes the total cost (1).

In connection with these problems, the following issues are of interest:

- existence or non-existence of blocking strategies,
- existence of optimal strategies,
- necessary conditions for optimality,
- sufficient conditions for optimality,
- regularity of optimal barriers,
- numerical computation of optimal barriers.

2 A sample of results

2.1 Existence of blocking strategies.

Throughout the following, we always assume that at the initial time $t = 0$ the fire is burning on some bounded open set $R_0 \subset \mathbb{R}^2$.

We first consider the isotropic case, assuming that the fire propagates with unit speed in all directions, while the barrier is constructed at a constant speed $\sigma$. 

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As proved in [2], if $\sigma > 2$ there exists an admissible strategy that blocks the fire. This can be achieved with a barrier consisting of two logarithmic spirals (Fig. 1, left).

On the other hand, if $\sigma \leq 1$, no admissible strategy can block the fire. It is not known what happens if $1 < \sigma \leq 2$.

If the fire is restricted to a half plane, then it can be blocked if and only if $\sigma > 1$ (see [8]).

If the initial set $R_0$ is a disc, and $\gamma$ is a spiral-like barrier constructed along the edge of the advancing fire front, the analysis in [4] shows that the barrier closes on itself (thus blocking the fire) if and only if $\sigma > \sigma^\dagger = 2.614430844\ldots$ (Fig. 1, right).

In the non-isotropic case, where the fire propagates faster in certain directions (in the presence of wind), further results on the existence of a blocking strategy were obtained in [4].

### 2.2 Optimal blocking strategies

Here is a sample of results proved for the optimization problem (OP):

- Under very general assumptions, it was proved in [5] that if the fire can be blocked, then one can also find an optimal blocking strategy, minimizing the cost functional (1). An alternative existence proof, based on the analysis of Hamilton-Jacobi for the minimum time function, was achieved in [11].

- Necessary conditions satisfied by an optimal barrier were derived in [2, 9, 14]. These conditions take the form of ODEs describing the curves where barriers have to be constructed. They are of two very different types. The first kind of ODEs describes “boundary arcs”, constructed right along the edge of the advancing fire-front. The second type describes “free arcs”, constructed at a certain distance away from the advancing front.

- A numerical method to compute the optimal barrier was developed in [7].
• In the isotropic case, it was proved in [10] that a specific barrier consisting of an arc of circumference and two arcs logarithmic spirals (Fig. 2) is optimal (i.e. it minimizes the total area burned by the fire) among all simple closed curves.

Figure 2: In the isotropic case, in order to minimize the area of the burned region a supposedly optimal strategy is the following: First construct the arc of circumference $P^-P^+$, then construct the two arcs of logarithmic spirals $P^-Q$ and $P^+Q$. The portion $P^-P^+$ of the barrier is constructed in advance of the time when the fire front touches it, while the two portions $P^-Q$ and $P^+Q$ are constructed right at the edge of the advancing fire front.

3 Some open problems

1 - Existence of a blocking strategy. Assume that the fire initially burns on a bounded open set $R_0 \subset \mathbb{R}^2$ and propagates with unit speed in all directions. It can then be shown that there exists a critical number $\sigma^*$ such that,

- If the barrier is constructed at speed $\sigma > \sigma^*$, then the fire can be entirely encircled by the barrier.
- If the barrier is constructed at speed $\sigma < \sigma^*$, then there is no way to block the fire.

What is this number $\sigma^*$? At present, we only know that $\sigma^* \in [1,2]$.

2 - Sufficient conditions for optimality. At the present date, not one single example is known of a blocking strategy which is provably optimal.

A partial result was obtained in [10] in the the isotropic case, assuming that $R_0$ is an open disc. Namely, a specific barrier $\Gamma^*$ consisting of an arc of circumference and two arcs of logarithmic spirals (Fig. 2) is the one which encloses the minimum area, among all admissible barriers which are simple closed curves.

One conjectures that the same curve $\Gamma^*$ is the global minimizer among all admissible barriers, regardless of their topological structure. In other words, constructing a barrier $\Gamma$ which is not connected, or is not a simple closed curve, will always yield a higher cost than $\Gamma^*$. 
3 - Qualitative properties of optimal barriers.

According to the existence results proved in [5, 11], the optimal barrier $\Gamma^*$ for (OP) is always a complete rectifiable set, possibly with countably many connected components. Unfortunately, the above result does not allow us to derive any of the necessary conditions for optimality in [2, 9, 14], which require stronger regularity assumptions.

It would be of interest to close this gap, proving that any optimal barrier $\Gamma^*$ must satisfy additional regularity conditions. In particular, assuming that the initial set $R_0$ has a smooth boundary and the cost functions $\alpha(\cdot), \beta(\cdot)$ are smooth, the following questions naturally arise:

- What is the regularity of an optimal barrier? Is $\Gamma^*$ the union of finitely many $C^1$ arcs?
- If $R_0$ is connected, does this imply that the optimal barrier $\Gamma^*$ is connected?
- Can the optimal barrier contain purely delaying arcs, i.e. arcs that are built only to slow down the advancement of the fire, not to block it?

More details can be found in the review paper [3].

References


