Problem 1.

a. Find the $3 \times 3$ matrix $P$ that projects every point in $\mathbb{R}^3$ onto the line of intersection of the two planes $x + y + z = 0$ and $x + 2y + 3z = 0$.

b. What point on that line is closest to $b = (1, 1, 0)$.

Solution

The system

\[
\begin{align*}
x + y + z &= 0 \\
x + 2y + 3z &= 0
\end{align*}
\]

has the solution

\[
\mathbf{x} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} .
\]

Therefore, a basis for the line of intersection of the two planes is the vector

\[
\mathbf{a} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}
\]

and the projection matrix is

\[
P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix} .
\]

To find the closest point to $b = (1, 1, 0)$, we compute the product of the matrix $P$ and $b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. So, $(-1/6, 1/3, -1/6)$ is the closest point.

Problem 2. Decide the stability or instability of the differential equation

\[
\frac{du}{dt} = Au
\]

for each of the matrices:

\[
A_1 = \begin{bmatrix} -1 & -5 \\ -1 & -3 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix} \quad A_3 = \begin{bmatrix} -4 & 3 \\ 5 & 3 \end{bmatrix}
\]

Solution None of them is stable.
Problem 3. Consider matrix
\[ A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \]

a. Compute \( e^{At} \).

b. Find the solution to the differential equation \( \frac{du}{dt} = Au \) that satisfies the initial condition \( u(0) = u_0 = (1, 0) \).

Solution The characteristic equation is: \( (\lambda - 1)(\lambda - 5) = 0 \). A diagonalization of \( A \) is
\[ A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \]

There follows
\[ e^{At} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(e^t + e^{5t}) & \frac{1}{2}(-e^t + e^{5t}) \\ \frac{1}{2}(-e^t + e^{5t}) & \frac{1}{2}(e^t + e^{5t}) \end{bmatrix} \]

The solution to the differential equation is:
\[ u(t) = e^{At}u = \begin{bmatrix} \frac{1}{2}(e^t + e^{5t}) & \frac{1}{2}(-e^t + e^{5t}) \\ \frac{1}{2}(-e^t + e^{5t}) & \frac{1}{2}(e^t + e^{5t}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(e^t + e^{5t}) \\ \frac{1}{2}(-e^t + e^{5t}) \end{bmatrix} \]

Problem 4. Find by least squares the best straight line fit to the following measurements, and sketch your solution

\[ y = 1 \text{ at } t = -1, \quad y = 0 \text{ at } t = 1, \quad y = -3 \text{ at } t = 3. \]

Solution Let \( y = c + dt \) be the equation of the line we seek. Then, substituting the points into the equation, we get:
\[ \begin{align*}
    c - d &= 1 \\
    c + d &= 0 \\
    c + 3d &= -3
\end{align*} \]

We denote
\[ A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \]

Then, the least squares solution is
\[ \mathbf{x} = (A^T A)^{-1} A^T b = \begin{bmatrix} \frac{1}{3} \\ -1 \end{bmatrix} \]

The best straight line fit to the measurements is \( y = -t + \frac{1}{3} \).

Problems 5-11 were solved in class.