Exercise 1: Prove that the exponential map for $GL(n, \mathbb{C})$ is surjective, but not injective.

Exercise 2: Show that $GL(n, \mathbb{C})$ is a connected Lie group.

Exercise 3: Let $\mathcal{F}_n(\mathbb{C})$ be the set of all flags in $\mathbb{C}^n$. Show that

$$\mathcal{F}_n(\mathbb{C}) = GL(n, \mathbb{C})/B(n, \mathbb{C}) = U(n)/T(n),$$

where $B(n, \mathbb{C})$ is the group of invertible complex upper triangular matrices, and $T(n)$ is the group of diagonal unitary matrices (which can be viewed as the n-torus). Deduce from this that $\mathcal{F}_n(\mathbb{C})$ is a compact complex manifold and find its dimension.

Exercise 4: Let $L_n$ be the set of all Lagrangian subspaces in $\mathbb{R}^{2n}$ with the standard symplectic form $\omega$. Show that $Sp(n, \mathbb{R})$ acts transitively on $L_n$ and use it to define on $L_n$ a structure of a smooth manifold and find its dimension.

Exercise 5: Show that $SU(2)$ is the universal cover of $SO(3, \mathbb{R})$. Hint: Use the adjoint map $Ad: SU(2) \to GL(\mathfrak{su}(2))$ and show that the image of $SU(2)$ is isomorphic to $SO(3, \mathbb{R})$. Deduce that

$$SO(3, \mathbb{R}) = SU(2)/\mathbb{Z}_2.$$