**MATH 401 - Introduction to Real Analysis**  
**Solutions to Quiz # 3**

**Problem 1.** Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^3 + 2x$. Prove that $f$ has an inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$. Calculate the value of the derivative $Df^{-1}(y)$ at $y = f(-1) = -3$.

**Solution:** We have $f'(x) = Df(x) = 3x^2 + 2$ for all $x \in \mathbb{R}$. Thus $f$ is differentiable on $\mathbb{R}$ and strictly increasing since $f'(x) > 0$. Hence $f$ is one-to-one. Moreover,  

$$
\lim_{x \to \infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = -\infty
$$

Thus $f(\mathbb{R}) = \mathbb{R}$ and $f$ has an inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$. Their derivatives are linked by the relation:

$$
Df^{-1}(y) = \frac{1}{Df(x)}.
$$

In particular, when $y = f(-1) = -3$, one gets

$$
Df^{-1}(-3) = \frac{1}{Df(-1)} = \frac{1}{3(-1)^2 + 2} = \frac{1}{5}.
$$

**Problem 2.** Which of the following functions are convex:

- $f(x) = |x|$ defined on $\mathbb{R}$,  
- $g(x) = 1 - \sqrt{1 - x^2}$ defined on $[-1, 1]$,  
- $h(x) = x^3 - 2x$ defined on $\mathbb{R}$.

**Solution:**

(i) $f(x) = |x|$ is convex since given any $\alpha > 0$, $\beta > 0$ with $\alpha + \beta = 1$, one has

$$
|\alpha x + \beta y| \leq |\alpha x| + |\beta y|
$$

by the triangle inequality. Since $\alpha > 0$ and $\beta > 0$, this gives:

$$
|\alpha x + \beta y| \leq |x| + |\beta y|
$$

Hence $f$ is convex.

(ii) $g(x) = 1 - \sqrt{1 - x^2}$ is continuous on $[-1, 1]$. Moreover,

$$
g'(x) = \frac{2x}{2\sqrt{1 - x^2}} = \frac{x}{\sqrt{1 - x^2}} \quad \text{on } (-1, 1).
$$

Furthermore,

$$
g''(x) = \frac{\sqrt{1 - x^2} - \frac{x^2}{2\sqrt{1 - x^2}}}{1 - x^2} = \frac{1}{(1 - x^2)^{3/2}}
$$

Since $g''(x) > 0$ on the interval $(-1, 1)$, we conclude that $g$ is convex.

(iii) $h(x) = x^3 - 2x$ is twice differentiable. Indeed, $h'(x) = 3x^2 - 2$, $h''(x) = 6x$ on $\mathbb{R}$. But $h''$ is negative on $(-\infty, 0)$. Hence $h$ in not convex.
Problem 3. Consider the function $f(x) = x^2$ on the interval $[0, 1]$.

(i) Compute the upper Riemann sum $S(P)$ and the lower Riemann sum $\underline{S}(P)$ corresponding to $f$ and the partition $P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$.

(ii) Check that $\underline{S}(P) \leq \int_0^1 x^2 \, dx \leq S(P)$.

Solution: Set $y_0 = 0, y_1 = \frac{1}{4}, y_2 = \frac{1}{2}, y_3 = \frac{3}{4}$, and $y_4 = 1$. We know that $f(x) = x^2$ is continuous on $[0, 1]$, moreover the maxima $M_k$ and minima $m_k$ over the intervals $[y_k-1, y_k]$ for $k = 1, 2, 3, 4$ are:

$$m_1 = 0, \quad M_1 = \frac{1}{16}, \quad m_2 = \frac{1}{16}, \quad M_2 = \frac{1}{4}, \quad m_3 = \frac{1}{16}, \quad M_3 = \frac{9}{16}, \quad m_4 = \frac{9}{16}, \quad M_4 = 1.$$

The lower Riemann sum is:

$$\underline{S}(P) = \frac{1}{4} \left( 0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16} \right) = \frac{14}{64} = \frac{7}{32}.$$

The upper Riemann sum is:

$$\overline{S}(P) = \frac{1}{4} \left( \frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1 \right) = \frac{30}{64} = \frac{15}{32}.$$

(ii) One has

$$\int_0^1 x^2 \, dx = \frac{1}{3}$$

One gets

$$\underline{S}(P) = \frac{7}{32} < \frac{1}{3} = \int_0^1 x^2 \, dx < \frac{15}{32} = \overline{S}(P).$$